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Stability of weighted Nash equilibrium for multiobjective population games

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Abstract

This paper studies the existence and stability of weighted Nash equilibria for multiobjective population games. By constructing a Nash's mapping, the existence of weighted Nash equilibria is established. Furthermore, via the generic continuity method, each weighted Nash equilibrium is shown to be stable for most of multiobjective population games when weight combinations and payoff functions are simultaneously perturbed. Besides, this leads to the stability of Nash equilibria for classical population games with the perturbed payoff functions. These results play cornerstone role in the research concerning multiobjective population games.

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1. Introduction and Preliminaries

Multiobjective population games (MPGs) are population games with vector-valued payoffs. The classical population games [19] originate from Nash's "mass-action" interpretation of equilibrium points in his dissertation ([16]) and his related literatures [15, 17]. Population games serve as a general model for studying strategic interactions among large numbers of agents, hence they are widely applied to modelling many economic, social and technological environment in which large collections of small agents make strategically interdependent decisions, such as network congestion, cultural integration and assimilation, etc.

Recently, population games and their applications have received increasing attention from an evolutionary point of view, see [2, 5, 7, 11, 18]. However, note that all the payoffs in the current researches still remain single-objective. In other words, the payoffs in these researches are all real-valued. It is well-known that

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population games with vector-valued payoffs can be better applied in real world, however, such a related study has not been reported until now. Hence, a generalization from the scalar case to multiple criteria is one of our main tasks in this paper, it is both theoretically and practically significant for population games.

Moreover, the additive weight method and weighted solutions or equilibria are very prominent in vector optimization and multiobjective games. Thus, the existence of weighted equilibria (or solutions) is a fundamental topic. Based on the existence of weighted equilibria, Wang and Yu [22, 28] proved the existence of Pareto equilibria for multiobjective games under different conditions, respectively. Similarly, the existence of Pareto equilibria is investigated for constrained multiobjective games by Ansari and Khan[1]. Related literatures can be referred to [6, 12, 14, 20].

Besides, the stability of weighted Nash equilibria plays a significant role in vector optimization and multiobjective games. Xiang [24] proved the stability of weighted solutions of vector optimization problems with the perturbed payoff. Grounded on this result, Xiang [25] further studied the stability of efficient solutions for vector optimization problems. Based on the stability of weighted Nash equilibria, Song and Wang [21] showed the stability of Pareto-Nash equilibria for multiobjective generalized games with the perturbation of payoff functions.

Inspired by [21, 24, 25], this paper aims to extend the single-objective population games to multiobjective ones, further investigate the existence and stability of weighted Nash equilibria for (MPGs), which is fundamental to the coming future and better applications of (MPGs). These are exactly our contributions in this paper.

In addition, it is well-known that in nonlinear science, the stability of solutions to a problem is essentially converted into the continuity of the set of solutions, and this method has always attracted considerable attention, see [8, 23]. And it is widely applied to the study on the stability of solutions to optimal theory; variational inequality, game theory, general equilibrium theory, such as [9, 13, 21, 24–27].

In order to establish the stability of weighted Nash equilibria for (MPGs), we review the continuity concerning set-valued mapping, referred to [3, 10].

Definition 1.1. If X and Y are two Hausdorff topological spaces and $T : X \to 2^Y$ is a set-valued mapping, then

(1) T is upper semi-continuous at $x_0 \in X$ if for each open set $U \subset Y$ with $U \supset T(x_0)$, there exists an open neighborhood $O(x_0)$ of x_0 such that $U \supset T(x)$ for each $x \in O(x_0)$;
(2) T is lower semi-continuous at $x_0 \in X$ if for each open set $W \subset Y$ with $W \cap T(x_0) \neq \emptyset$, there exists an open neighborhood $O(x_0)$ of x_0 such that $W \cap T(x) \neq \emptyset$ for each $x \in O(x_0)$;
(3) T is continuous at $x_0 \in X$ if T is both upper and lower semi-continuous at $x_0 \in X$; T is continuous on X if T is continuous at any point $x \in X$; and
(4) T is u.s.c.o. if T is upper semi-continuous on X and $T(x)$ is compact for every $x \in X$.

The generic continuity of set-valued mappings with semi-continuity as below:

Lemma 1.2 ([8]). *Let X be a Baire space and Y be a metric space, if $T : X \to 2^Y$ is u.s.c.o., then there is a dense residual set Q such that T is lower semi-continuous and then continuous on it.*

The paper is organized as follows. Section 2 introduces the model of (MPGs) and the concept of weighted Nash equilibrium for (MPGs). The existence of weighted Nash equilibria is discussed in Section 3. The main result concerning the stability of weighted Nash equilibria for (MPGs) is established in Section 4.

2. Model and Weighted Nash equilibrium

Throughout this paper, for each positive integer k,

$$R^k_+ = \{a = (a_1, \dots, a_k) \in R^k : a_j \geq 0,\ j = 1, \dots, k\},$$

$$T^k_+ = \{a = (a_1, \dots, a_k) \in R^k_+ : \sum_{j=1}^{k} a_j = 1\}.$$

Let $a * b$ be the inner product between two vectors $a, b \in R^k$.

In this paper, we consider (MPGs): let $\mathcal{P} = \{1, \ldots, P\}$ be a society consisting of $P \geq 1$ populations of agents. Each population is a unit mass of a large but finite number of agents with the same strategies set and the same payoff function. For each population $p \in \mathcal{P}$, pure strategies set is $S^p = \{1, \ldots, n^p\}$. Let $m = \sum_{p\in\mathcal{P}} n^p$ equal the total number of pure strategies in all populations.

For each $p \in \mathcal{P}$, the set of population states is denoted by $X^p = \big\{x^p = (x_1^p, \ldots, x_{n^p}^p) \in R_+^{n^p} : \sum_{i=1}^{n^p} x_i^p = 1\big\} = \triangle^{n^p-1}$, where the nonnegative scalar $x_i^p \in [0,1]$ represents the share of members playing strategy $i \in S^p$. The set of social states is denoted by $X = \prod_{p\in\mathcal{P}} X^p = \big\{x = (x^1, \ldots, x^P) \in R^m : x^p \in X^p\big\} \subset R^m$, in which the element $x = (x^1, \ldots, x^P) \in X$ describes all populations' behavior at once.

Additionally, we assume that agents in each population $p \in \mathcal{P}$ share k^p objectives whenever they play a strategy, and denoted by $G^p = \{1, \ldots, k^p\}(k^p \geq 2)$ the set of finite behavioral objectives, thus the vector-valued payoff function to each strategy $i \in S^p$ is denoted by $F_i^p = (F_{i1}^p, \ldots, F_{ik^p}^p) : X \to R^{k^p}$, where F_{ij}^p represents the jth objective real-valued payoff corresponding to a strategy $i \in S^p$; $F^p = (F_1^p, \ldots, F_{n^p}^p)^T : X \to R^{n^p k^p}$ describes population p's payoff functions for all strategies in S^p. Now let $N = \sum_{p\in\mathcal{P}} n^p k^p$, the payoff functions $F : X \to R^N$ is a map that assigns each social state a vector of payoffs, one for each criterion corresponding to each strategy in each population. Since the sets of populations and strategies are generally taken as fixed, (MPGs) is identified with its payoff functions F in the remainder of this paper.

From the above model, it is shown that the difference between (MPGs) and the classical population games mainly lies in the dimensions of payoff functions to one strategy in each population. Precisely, agents in each population all have at least a bi-objective payoff function to every strategy in (MPGs), while in the classical population games, agents in a population merely have a single-objective payoff to a strategy. Thus, (MPGs) include the classical population games as a special case where $k^p = 1$ for all $p \in \mathcal{P}$. Then (MPGs) are more general than the single-objective ones, and we can imagine that the much wider applications of (MPGs) will emerge in the near future.

For (MPG) F and for any $\lambda^p \in R_+^{k^p}, p \in \mathcal{P}$, define

$$(F_\lambda^p)_i(x) = \lambda^p * F_i^p(x), \quad \forall x \in X,$$

it is said to be weighted payoff function at $x \in X$ to a strategy $i \in S^p$ corresponding to the weight $\lambda^p \in R_+^{k^p}$. Further, denoted by $F_\lambda^p(x) = ((F_\lambda^p)_1(x), \cdots, (F_\lambda^p)_{n^p}(x))^T$ weighted payoff vector of population p; thus, the weighted payoff functions of the whole society are written as $F_\lambda(x) = (F_\lambda^1(x), \cdots, F_\lambda^P(x))$.

we propose the notion of weighted Nash equilibrium of F below.

Definition 2.1. A social state $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^P) \in X$ is called a weighted Nash equilibrium of F with respect to a weight combination $\lambda = (\lambda^1, \cdots, \lambda^P)$ satisfying $\lambda^p \in T_+^{k^p}(p \in \mathcal{P})$ if, for each $p \in \mathcal{P}$ and $i \in S^p$,

$$\bar{x}_i^p > 0 \Rightarrow (F_\lambda^p)_i(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \quad \forall l \in S^p.$$

And $E(\lambda, F)$ denotes the set of all the weighted Nash equilibria of F with respect to a weight combination λ.

Remark 2.2.

(1) For $p \in \mathcal{P}$, the element λ_j^p of $\lambda^p \in T_+^{k^p}$ is interpreted as the share distribution of members choosing the jth objective; furthermore, the product $\lambda_j^p x_i^p$ represents the share of members who take the jth objective as their main evaluation criterion when they choose a strategy $i \in S^p$.

(2) In particular, if $k^p = 1$ for each $p \in \mathcal{P}$, a weighted Nash equilibrium of (MPGs) reduces to a Nash equilibrium of the classical population games.

3. Existence

In this section, we mainly study the existence of weighted Nash equilibria. For this purpose, we firstly establish an equivalent description on weighted Nash equilibria for (MPGs).

Lemma 3.1. *Given a weight combination* $\lambda = (\lambda^1, \cdots, \lambda^P)$ *satisfying* $\lambda^p \in T_+^{k^p}$ *for all* $p \in \mathcal{P}$. *A social state* $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^P) \in E(\lambda, F)$ *if and only if for each* $p \in \mathcal{P}$, $\bar{x}^p$ *is an optimal solution of the following optimization problem* $(\mathbf{P}_\lambda)$*:*

$$(\mathbf{P}_\lambda): \qquad \max_{y^p \in X^p} y^p * F_\lambda^p(\bar{x}).$$

Proof. For a given weight combination $\lambda = (\lambda^1, \cdots, \lambda^P)$ with $\lambda^p \in T_+^{k^p}(p \in \mathcal{P})$, suppose that $\bar{x} \in E(\lambda, F)$, then for each $p \in \mathcal{P}$ and $i \in S^p$,

$$\bar{x}_i^p > 0 \Rightarrow (F_\lambda^p)_i(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \quad \forall l \in S^p.$$

Then it holds that

$$\bar{x}_i^p[(F_\lambda^p)_i(\bar{x})] \geq \bar{x}_i^p[(F_\lambda^p)_l(\bar{x})], \quad \forall l \in S^p.$$

Summed with respect to $i \in S^p$ on both sides of the above inequality, note that $\sum_{i \in S^p} \bar{x}_i^p = 1$, hence

$$\bar{x}^p * (F_\lambda^p)(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \;\; \forall l \in S^p.$$

For any $y^p = (y_1^p, \cdots, y_{n^p}^p) \in X^p$, multiplied both sides of the above inequality by y_l^p and summed with respect to $l \in S^p$, note that $\sum_{l \in S^p} y_l^p = 1$, then

$$\bar{x}^p * (F_\lambda^p)(\bar{x}) \geq y^p * (F_\lambda^p)(\bar{x}), \;\; \forall p \in \mathcal{P}.$$

Thus for each $p \in \mathcal{P}$, $\bar{x}^p$ is an optimal solution to $(\mathbf{P}_\lambda)$.

Conversely, for each $p \in \mathcal{P}$, if $\bar{x}^p$ is an optimal solution to $(\mathbf{P}_\lambda)$, that is,

$$y^p * F_\lambda^p(\bar{x}) \leq \bar{x}^p * F_\lambda^p(\bar{x}), \quad \forall y^p \in X^p.$$

Since $\bar{x}^p \in X^p = \triangle^{n^p - 1}$, without loss of generality, assume that $\bar{x}_i^p > 0$ for a certain $i \in S^p$, then for any $l \in S^p$, we set

$$\hat{y}^p = (\bar{x}_1^p, \cdots, \underbrace{0}_{i}, \cdots, \underbrace{\bar{x}_i^p + \bar{x}_l^p}_{l}, \cdots, \bar{x}_{n^p}^p).$$

Thus $\hat{y}^p \in X^p$, and the above inequalities further becomes

$$\bar{x}_i^p[(F_\lambda^p)_i(\bar{x}) - (F_\lambda^p)_l(\bar{x})] \geq 0, \quad \forall l \in S^p.$$

Since $\bar{x}_i^p > 0$, it clearly implies that

$$(F_\lambda^p)_i(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \quad \forall l \in S^p.$$

That is,

$$\bar{x}_i^p > 0 \Rightarrow (F_\lambda^p)_i(\bar{x}) \geq (F_\lambda^p)_l(\bar{x}), \quad \forall l \in S^p.$$

Due to the arbitrariness of $p \in \mathcal{P}$ and $i \in S^p$, $\bar{x} \in E(\lambda, F)$. The proof is complete. □

Referred to the idea of Nash's constructive proof on the existence of equilibrium points for n-person noncooperative games [17], the existence of weighted Nash equilibria is established for (MPGs) below.

Theorem 3.2. *If* F *is continuous on* X, *then (MPG)* F *has at least one weighted Nash equilibrium for a given weight combination* $\lambda = (\lambda^1, \cdots, \lambda^P)$ *with* $\lambda^p \in T_+^{k^p}$ *for all* $p \in \mathcal{P}$.

Proof. For each population $p \in \mathcal{P}$, define a mapping $f^p(x) = \tilde{x}^p = (\tilde{x}_1^p, \ldots, \tilde{x}_{n^p}^p)$ as follows:

$$\tilde{x}_i^p = \frac{x_i^p + \phi_i^p(x)}{1 + \sum_{i=1}^{n^p} \phi_i^p(x)}, \; \forall i \in S^p,$$

where $\phi_i^p(x) = \max\{0, \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(x) - \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} x_i^p \lambda_j^p F_{ij}^p(x)\}$. Clearly, $\tilde{x}_i^p \geq 0$ and $\sum_{i=1}^{n} \tilde{x}_i^p = 1$, thus $\tilde{x}^p = (\tilde{x}_1^p, \ldots, \tilde{x}_{n^p}^p) \in X^p$ and f^p is a continuous mapping from X to X^p by the continuity of F on X. Further define

$$f(x) = \prod_{p \in \mathcal{P}} f^p(x),$$

obviously, $f : X \to X$ is continuous. In addition, since X is compact and convex, followed from Brouwer's fixed point theorem, there is $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^P) \in X$ such that $\bar{x} = f(\bar{x}) = \prod_{p \in \mathcal{P}} f^p(\bar{x})$, thus for all $p \in \mathcal{P}, \bar{x}^p = f^p(\bar{x})$, that is,

$$\bar{x}_i^p = \frac{\bar{x}_i^p + \phi_i^p(\bar{x})}{1 + \sum_{i=1}^{n^p} \phi_i^p(\bar{x})}, \ \forall i \in S^p.$$

We shall verify that such a fixed point $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^P) \in X$ is exactly a weighed Nash equilibrium of F with respect to λ. For all $p \in \mathcal{P}$, denote $I^p(\bar{x}) = \{i \in S^p : \bar{x}_i^p > 0\}$, clearly $I^p(\bar{x}) \neq \emptyset$. For each $i \in I^p(\bar{x})$, if $\phi_i^p(\bar{x}) = \max\{0, \sum_{j=1}^{k} \lambda_j^p F_{ij}^p(\bar{x}) - \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x})\} > 0$, then

$$\sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) > \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}),$$

simultaneously multiplied by $\bar{x}_i^p$ on both sides of the above inequality, then

$$\bar{x}_i^p \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) > \bar{x}_i^p \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}).$$

Notice that if $i \notin I^p(\bar{x})$, that is, $\bar{x}_i^p = 0$, then it also holds

$$\bar{x}_i^p \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) = \bar{x}_i^p \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}).$$

Then summing together with respect to $i \in S^p$ and noting that $\sum_{i=1}^{n^p} \bar{x}_i^p = 1$, we get

$$\sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}) > \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}),$$

which is a contradiction, hence there must exist some $i \in I^p(\bar{x})$ such that

$$\max\{0, \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) - \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x})\} = 0,$$

thus

$$\sum_{i=1}^{n^p} \max\{0, \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) - \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x})\} = 0.$$

Therefore, for all $i \in S^p$, it holds

$$\sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) \leq \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}).$$

For any $y^p = (y_1^p, \ldots, y_{n^p}^p) \in X^p$, since

$$y_i^p \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(\bar{x}) \leq y_i^p \sum_{i=1}^{n^p} \sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}),$$

summing with respect to $i \in S^p$ and due to $\sum_{i=1}^{n^p} y_i^p = 1$, we further have

$$\sum_{i=1}^{n^p}\sum_{j=1}^{k^p} y_i^p \lambda_j^p F_{ij}^p(\bar{x}) \leq \sum_{i=1}^{n^p}\sum_{j=1}^{k^p} \bar{x}_i^p \lambda_j^p F_{ij}^p(\bar{x}),$$

this inequality is equivalent to

$$y^p * F_\lambda^p(\bar{x}) \leq \bar{x}^p * F_\lambda^p(\bar{x}),\ \forall y^p \in X^p.$$

This shows that $\bar{x}^p$ is an optimal solution to $(\mathbf{P}_\lambda)$. From Lemma 3.1, then such an $\bar{x} \in X$ is a weighted Nash equilibrium of F with respect to a given weight combination λ. The proof is complete. □

In particular, by Remark 2.2 (2), the existence result of Nash equilibrium for classical population games (Theorem 2.1.1 of [19]) is immediately derived from Theorem 3.2.

4. Stability

In this section, we investigate the stability of weighted Nash equilibria for (MPGs).

In this part, we assume that each objective is bound to be chosen as agents are generally bounded rationality. Precisely, for a sufficiently small real constant $\delta > 0$, each objective share satisfies $\lambda_j^p \geq \delta$ $(j = 1, \cdots, k^p)$ for any weight vector $\lambda^p = (\lambda_1^p, \cdots, \lambda_{k^p}^p) \in T_+^{k^p}$ $(p \in \mathcal{P})$.

Denoted by $\mathcal{F} = \{c = (\lambda, F) :\ F = (F^1, \cdots, F^P) : X \to R^N$ is continuous and X is nonempty compact, $\lambda = (\lambda^1, \cdots, \lambda^{\mathrm{P}}) : \lambda^{\mathrm{p}} \in \mathrm{T}_+^{\mathrm{k^p}}, \lambda_{\mathrm{j}}^{\mathrm{p}} \geq \delta\ (\delta > 0, \mathrm{k^p}\delta < 1)\}$.

For any $c = (\lambda, F), \tilde{c} = (\tilde{\lambda}, \tilde{F}) \in \mathcal{F}$, define

$$\rho(c, \tilde{c}) = \max_{x \in X} \sum_{p \in \mathcal{P}} \|F^p(x) - \tilde{F}^p(x)\| + \|\lambda - \tilde{\lambda}\|_1,$$

where $\|\lambda - \tilde{\lambda}\|_1 = \sum_{p=1}^{P}\sum_{j=1}^{k^p} |\lambda_j^{k^p} - \tilde{\lambda}_j^{k^p}|$. Clearly, ρ is a metric on $\mathcal{F}$ and $(\mathcal{F}, \rho)$ is a complete metric space.

For any $c = (\lambda, F) \in \mathcal{F}$, the collection of all weighted Nash equilibria $E(c) \neq \emptyset$ from Theorem 3.2, thus $E : \mathcal{F} \to 2^X$ is a nonempty set-valued mapping. We show that E is upper-continuous and compact-valued below.

Lemma 4.1. *$E : \mathcal{F} \to 2^X$ is u.s.c.o..*

Proof. Firstly, we show that for each $c = (\lambda, F) \in \mathcal{F}, E(c)$ is compact. Since $E(c) \subset X$ and X is compact, it suffices to show that $E(c)$ is closed for each $c = (\lambda, F) \in \mathcal{F}$. Let $\{x^n\} \subset E(c)$ be a sequence with $x^n \to x$, we need to show that $x \in E(c)$. Suppose that $x \notin E(c) = E(\lambda, F)$, then there is some $p \in \mathcal{P}$ and $i \in S^p$ with $x_i^p > 0$ but $(F_\lambda)_i^p(x) < \max_{l \in S^p}(F_\lambda)_l^p(x)$. Since $x^n \to x$, we have $(x^n)_i^p \to x_i^p$ and $(x^n)_i^p > 0$ for sufficiently large positive integer n due to $x_i^p > 0$. As F_λ is continuous at x, we meanwhile have $(F_\lambda)_i^p(x^n) \to (F_\lambda)_i^p(x)$ and $\max_{l \in S^p}(F_\lambda)_l^p(x^n) \to \max_{l \in S^p}(F_\lambda)_l^p(x)$. As $(F_\lambda)_i^p(x) < \max_{l \in S^p}(F_\lambda)_l^p(x)$, then $(F_\lambda)_i^p(x^n) < \max_{l \in S^p}(F_\lambda)_l^p(x^n)$ for sufficiently large positive integer n. Therefore, $x^n \notin E(c)$, however, this contradicts the assumption $\{x^n\} \subset E(c)$. Hence $E(c)$ is closed and further compact since $E(c) \subset X$ and X is compact.

Subsequently, we show that E is upper semi-continuous on $\mathcal{F}$. Suppose that E is not upper-continuous at $c = (\lambda, F) \in \mathcal{F}$, then there exists an open set $U \subset X$ with $U \supset E(c)$, and there is a sequence $c^n = (\lambda^n, F^n)$ satisfying $c^n = (\lambda^n, F^n) \to c = (\lambda, F)$ such that there exists one sequence $x^n \in E(\lambda^n, F^n)$ for any positive integer n, yet $x^n \notin U$. Because of $x^n \in E(\lambda^n, F^n) \subset X$ and the compactness of X, without loss of generality, we assume that $x^n \to x$. Following from $x^n \notin U$, we obtain $x \notin U$ and thus $x \notin E(c) = E(\lambda, F)$, namely, even $x_i^p > 0$ for some $i \in S^p$ and $p \in \mathcal{P}$, but $(F_\lambda)_i^p(x) < \max_{l \in S^p}(F_\lambda)_l^p(x)$.

In fact, according to the continuity of F on X, then there is a constant $M > 0$ such that $|F_{ij}^p(x)| \leq M$, hence $\sum_{j=1}^{k^p} |F_{ij}^p(x)| \leq Mk^p$. As $(\lambda^n, F^n) \to (\lambda, F)$, then it simultaneously holds that $\lambda^n \to \lambda$ and $F^n \to F$.

Thus, for sufficiently small real number $\varepsilon > 0$, there is sufficiently large positive integer L such that we simultaneously obtain $\|(F^n)^p(x^n) - F^p(x^n)\| < \varepsilon/(3k^p)$ and $\|(\lambda^n)^p - \lambda^p\|_1 < \varepsilon/(3Mk^p)$ for any $n > L$.

Therefore, for the above $i \in S^p$ and $p \in \mathcal{P}$ with $x_i^p > 0$,

$$\begin{aligned}
|(F^n_{\lambda^n})^p_i(x^n) - (F_\lambda)^p_i(x)| &= |(\lambda^n)^p * (F^n)^p_i(x^n) - \lambda^p * F^p_i(x)| = |\sum_{j=1}^{k^p}(\lambda^n)^p_j(F^n)^p_{ij}(x^n) - \sum_{j=1}^{k^p}\lambda^p_j F^p_{ij}(x)| \\
&\le |\sum_{j=1}^{k^p}(\lambda^n)^p_j(F^n)^p_{ij}(x^n) - \sum_{j=1}^{k^p}(\lambda^n)^p_j F^p_{ij}(x^n)| \\
&\quad + |\sum_{j=1}^{k^p}(\lambda^n)^p_j F^p_{ij}(x^n) - \sum_{j=1}^{k^p}(\lambda^n)^p_j F^p_{ij}(x)| + |\sum_{j=1}^{k^p}(\lambda^n)^p_j F^p_{ij}(x) - \sum_{j=1}^{k^p}\lambda^p_j F^p_{ij}(x)| \\
&\le \|(F^n)^p(x^n) - F^p(x^n)\| \sum_{j=1}^{k^p} |(\lambda^n)|^p_j + \|F^p(x^n) - F^p(x)\| \sum_{j=1}^{k^p} |\lambda^n|^p_j \\
&\quad + \|(\lambda^n)^p - \lambda^p\|_1 \sum_{j=1}^{k^p} |F^p_{ij}(x)| \\
&\le k^p\varepsilon/(3k^p) + k^p\varepsilon/(3k^p) + Mk^p\varepsilon/(3Mk^p) = \varepsilon.
\end{aligned}$$

This shows that $(F^n_{\lambda^n})^p_i(x^n) \to (F_\lambda)^p_i(x)$ as $n \to \infty$.

Similarly, we have $\max_{l\in S^p}(F^n_{\lambda^n})^p_l(x^n) \to \max_{l\in S^p}(F_\lambda)^p_l(x)$ as $n \to \infty$.

On the other hand, taking into account $x^n \to x$ and $x^p_i > 0$ but $(F_\lambda)^p_i(x) < \max_{l\in S^p}(F_\lambda)^p_l(x)$, we then have $(x^n)^p_i > 0$ and $(F^n_{\lambda^n})^p_i(x^n) < \max_{l\in S^p}(F^n_{\lambda^n})^p_l(x^n)$ for sufficiently large positive integer n, this shows $x^n \notin E(\lambda^n, F^n)$, which contradicts the fact $x^n \in E(\lambda^n, F^n) = E(c^n)$ for any positive integer n. Consequently, E is upper semi-continuous on $\mathcal{F}$. The proof is complete. □

Notice that, in general, the set of weighted Nash equilibria $E(c) = E(\lambda, F)$ need not be lower semi-continuous on $\mathcal{F}$. The following example shows this point.

Example 4.2. Consider a bi-objective single population game F with two strategies: let its state space be $X = [0,1] \times [0,1]$ and for any $x \in X$, population payoff vector be $F(x) = (F_1(x), F_2(x))^T$, where $F_1(x) = \big(F_{11}(x), F_{12}(x)\big) = (1,2)$, $F_2(x) = \big(F_{21}(x), F_{22}(x)\big) = (3,0)$.

For a given $c = (\lambda_0, F) \in \mathcal{F}$, where the combination $\lambda_0 = (\frac{1}{2}, \frac{1}{2}) \in \bar{\Lambda}$, its weighted payoff

$$F_\lambda(x) = \big((F_\lambda)_1(x), (F_\lambda)_2(x)\big)^T = (\frac{3}{2}, \frac{3}{2})^T,$$

clearly, the set of weighted Nash equilibria $E(c) = [0,1] \times [0,1]$.

However, there exists a perturbed sequence $c^m = (\lambda^m, F^m) \in \mathcal{F}$, where the perturbed weight combination is given as $\lambda^m = (\frac{1}{2} + \frac{1}{m}, \frac{1}{2} - \frac{1}{m})$, and the perturbed payoff vector is $F^m(x) = \big(F^m_1(x), F^m_2(x)\big)^T$, concretely, $F^m_1(x) = \big(F^m_{11}(x), F^m_{12}(x)\big) = (1 + \frac{1}{m}, 2 + \frac{2}{m})$, $F^m_2(x) = \big(F^m_{21}(x), F^m_{22}(x)\big) = (3 + \frac{1}{m}, \frac{2}{m})$. Clearly, $c^m = (\lambda^m, F^m) \to c = (\lambda_0, F)$ as $m \to \infty$. And the corresponding weighted payoff functions equal

$$(F^m_{\lambda^m})(x) = \big((F^m_{\lambda^m})_1(x), (F^m_{\lambda^m})_2(x)\big)^T,$$

where $(F^m_{\lambda^m})_1(x) = (\frac{1}{2} + \frac{1}{m})(1 + \frac{1}{m}) + (\frac{1}{2} - \frac{1}{m})(2 + \frac{2}{m})$, and $(F^m_{\lambda^m})_2(x) = (\frac{1}{2} + \frac{1}{m})(3 + \frac{1}{m}) + \frac{2}{m}(\frac{1}{2} - \frac{1}{m})$. Then the resulting set of weighted Nash equilibria is $E(c^m) = E(\lambda^m, F^m) = \{(0,1)\}$ as $(F^m_{\lambda^m})_2(x) > (F^m_{\lambda^m})_1(x)$.

Nevertheless, for a special weighted Nash equilibrium $\{(1,0)\} \in E(c)$, we can choose a small enough neighborhood $\mathcal{N}(1,0)$; no matter how close $c^m = (\lambda^m, F^m)$ is to $c = (\lambda_0, F)$, $\{(0,1)\} \cap \mathcal{N}(1,0) = \emptyset$. Therefore, E is not lower semi-continuous at $c = (\lambda_0, F)$.

Combining the fact that E is u.s.c.o. on $\mathcal{F}$ with Fort's Theorem (Lemma 1.2), the stability result of weighted Nash equilibria is obtained under both weight combinations and perturbed payoff functions simultaneously as follows:

Theorem 4.3. *There is a dense residual set $Q \subset \mathcal{F}$ such that $E : \mathcal{F} \to 2^X$ is lower semi-continuous and then continuous on it.*

Proof. Since $(\mathcal{F}, \rho)$ is complete and $E : \mathcal{F} \to 2^X$ is u.s.c.o. from Lemma 4.1, then by Fort's Theorem (Lemma 1.2), there is a dense residual set $Q \subset \mathcal{F}$ such that E is lower semi-continuous and then continuous on it. The proof is complete. □

Remark 4.4. Theorem 4.3 indicates that in the sense of Baire's category, each weighted Nash equilibrium most of (MPG) F is stable against the perturbation of both weight combinations and payoff functions.

Moreover, Theorem 4.3 includes a special case in which only payoff functions are perturbed while weight combinations are fixed. Let

$$\mathfrak{F} = \{F : \ F = (F^1, \cdots, F^P) : X \to R^N \text{ is continuous and X is nonempty compact}\},$$

and it is equipped with the maximum norm.

Corollary 4.5. *Given a weight combination $\lambda = (\lambda^1, \cdots, \lambda^P)$ with $\lambda^p \in T_+^{k^p} (p \in \mathcal{P})$, each weighted Nash equilibrium of most of (MPG) $F \in \mathfrak{F}$ is stable under perturbed payoff functions.*

Furthermore, suppose that $k^p = 1$ and $\lambda^p = 1$ for each $p \in \mathcal{P}$, let

$$\mathbb{F} = \{F : \ F = (F^1, \cdots, F^P) : X \to R^m \text{ is continuous and X is nonempty compact}\},$$

and it is still equipped with the maximum norm.

By Corollary 4.5 and Remark 2.2 (2), we further obtain the stability of Nash equilibria of classical population games below:

Corollary 4.6. *Each Nash equilibrium of most of population games $F \in \mathbb{F}$ is stable under perturbed payoff functions F.*

In addition, notice that weighted Nash equilibria of (MPGs) are distinguished from the weighted solutions to multiobjective optimization problems [24] even when (MPGs) contain only one population with a single strategy. The following example shows this point.

Example 4.7. Consider a bi-objective single population game F with only one strategy: that is, $\mathcal{P} = \{1\}, s^1 = \{1\}, k = 2$. Then its resulting state space is $\tilde{X} = [0, 1]$, and for any $x \in \tilde{X}$, let population payoff vector be $F(x) = (F_1(x), F_2(x))^T = \big(\frac{x^2}{2} + 1, 2 - x\big)^T$.

Given a weight vector $\lambda = (\frac{1}{3}, \frac{2}{3}) \in \bar{\Lambda}$, its associated weighted payoff is

$$F_\lambda(x) = \frac{1}{3}(\frac{x^2}{2} + 1) + \frac{2}{3}(2 - x).$$

By Definition 2.1, each $\bar{x} \in (0, 1]$ is a weighted Nash equilibrium with respect to the weight vector $\lambda = (\frac{1}{3}, \frac{2}{3})$.

However, corresponding to this population game F and $\tilde{X}$, the bi-objective optimization problem is

$$\max_{x \in \tilde{X}} F(x) = (F_1(x), F_2(x))^T = \big(\frac{x^2}{2} + 1, 2 - x\big)^T;$$

furthermore, the resulting weighted optimization problem with respect to the given weight vector $\lambda = (\frac{1}{3}, \frac{2}{3})$ is

$$\max_{x \in \tilde{X}} F_\lambda(x) = \frac{1}{3}(\frac{x^2}{2} + 1) + \frac{2}{3}(2 - x),$$

it is easy to check that there is a unique weighted solution $\bar{x} = 0$ to this problem. Nevertheless, it is not a weighted Nash equilibrium of F.

To a certain extent, Example 4.7 further reveals that the existence and stability of weighted Nash equilibria of (MPGs) are new and distinguished from [24].

5. Conclusion

In this paper, the stability of weighted Nash equilibria for (MPGs) are established. As a fundamental topic for (MPGs), the existence of weighted Nash equilibria is first proven by constructing a Nash's mapping. Moreover, by the method of generic continuity, the stability of weighted Nash equilibria is obtained when weight combinations and payoff functions are simultaneously perturbed. These results lead to the existence and stability property of Nash equilibria of classical population games with respect to the perturbed payoff functions. All the obtained results are essential to the future research for (MPGs), which are new and different from the literature by means of an example (see Example 4.7).

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