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Menger probabilistic G-metric-like space and fixed point theorems

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Abstract

In this paper, we first introduce a concept called Menger probabilistic G-metric-like space which is a generalization of Menger probabilistic metric-like space of Hierro and Sen. Some fixed point theorems for various kinds of contractions in framework of this space are given. Our results extend some recent ones of Zhou et al., Hua et al. and Alsulami et al.. Finally, an example is given to illustrate the main result of this paper.

Keywords: Menger space, fixed point, φ-contraction, metric-like space.
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1. Introduction and preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ be the set of all natural numbers. A mapping $f : \mathbb{R} \to \mathbb{R}^+$ is called a distribution if it is non-decreasing left-continuous with $\sup_{t\in\mathbb{R}} f(t) = \lim_{t\to+\infty} f(t) = 1$ and $\inf_{t\in\mathbb{R}} f(t) = \lim_{t\to-\infty} f(t) = 0$. We shall denote by $\mathcal{D}$ the set of all distribution functions.

Let $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if for all $a, b, c, d \in [0, 1]$, $\Delta(a, 1) = a$, $\Delta(a, b) \leq \Delta(c, d)$ if $a \leq c, b \leq d$, $\Delta(a, b) = \Delta(b, a)$ and $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$. It is known that $\Delta_M(a, b) = \min\{a, b\}$ and $\Delta_P(a, b) = ab$ are the classic examples of t-norms.

A t-norm Δ is said to be of H-type [10] if the family of functions $\{\Delta(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

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$$\Delta^1(t) = \Delta(t,t),\ \Delta^2(t) = \Delta(\Delta^1(t),t),\ \Delta^m(t) = \Delta(\Delta^{m-1}(t),t),\quad m = 3,4,\cdots,\ t \in [0,1].$$

The t-norm Δ_M is a trivial example of t-norm of H-type. But Δ_P is not the t-norm of H-type. In [18], the authors pointed out that t-norm of H-type is a big class. On the examples of t-norm of H-type, also refer to [10].

In 1942, Menger [15] developed the theory of metric spaces and proposed a generalization of metric spaces called Menger probabilistic metric spaces (briefly, Menger PM-space).

Definition 1.1. A *Menger PM-space* is a triple (X, F, Δ), where X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X \to \mathcal{D}$ ($F_{x,y}$ denotes the value of F at the pair (x, y)) satisfying the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for all $x, y \in X$ and $t > 0$ if and only $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;
(PM-3) $F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. Since Menger, many authors have considered fixed point theory in Menger PM-spaces and its applications as a part of probabilistic analysis (see [2–7, 9–11, 14, 16, 17, 20, 22]).

Let (X, F, Δ) be a probabilistic metric space and $T : X \to X$ be a mapping. If there exists a gauge function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t),\ \text{for all } x, y \in X \text{ and } t > 0,$$

then the mapping T is called a probabilistic φ-contraction. The probabilistic φ-contraction is a generalization of probabilistic k-contraction given by Sehgal and Bharucha-Reid [21].

In 2010, Jachymski [13] proved a fixed point theorem for probabilistic φ-contraction which improves the result of Ćirić [4] by weakening the condition on the function φ. More precisely, the author gave the following result:

Theorem 1.2 ([13])**.** *Let (X, F, Δ) be a complete Menger probabilistic metric space with the t-norm Δ of H-type, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying conditions*

$$0 < \varphi(t) < t \quad \text{and} \quad \lim_{n\to\infty} \varphi^n(t) = 0 \ for\ all\ t > 0.$$

If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.

In order to further improve Theorem 1.1, Fang [8] considered a new condition on the gauge function φ. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying the following condition:

$$\text{for each } t > 0 \text{ there exists } r \geq t \text{ such that } \lim_{n\to\infty} \varphi^n(t) = 0. \tag{1.1}$$

Let $\mathbf{\Phi_w}$ denote the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition (1.1). By using the condition (1.1), Fang gave the following result:

Theorem 1.3 ([8])**.** *Let (X, F, Δ) be a complete Menger space with a t-norm Δ of H-type. If $T : X \to X$ is a probabilistic φ-contraction, where $\varphi \in \mathbf{\Phi_w}$, then T has a unique fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.*

Very recently, Hua et al. [12] investigated the work of Fang above and further improved Theorem 1.2 by weakening the condition on the gauge function φ in Theorem 1.2. Let $\mathbf{\Phi_{w^*}}$ denote the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:

$$\begin{gathered}\text{for each } t_1, t_2 > 0, \text{ there exists } r \geq \max\{t_1, t_2\} \text{ and } N \in \mathbb{N} \\ \text{such that } \varphi^n(r) < \min\{t_1, t_2\} \text{ for all } n > N.\end{gathered} \tag{1.2}$$

Hua et al. [12] pointed out if $\varphi \in \mathbf{\Phi_w}$, then $\varphi \in \mathbf{\Phi_{w^*}}$ and the inverse is not true. The following result improves Theorem 1.2.

Theorem 1.4 ([12])**.** *Let (X, F, Δ) be a complete Menger space with a t-norm Δ of H-type. If $T : X \to X$ is a probabilistic φ-contraction, where $\varphi \in \mathbf{\Phi_{w^*}}$, then T has a unique fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.*

Recently, Zhou et al. [23] introduced a new probabilistic space called Menger probabilistic G-metric space (shortly, Menger PGM-space) which generalizes the Menger PM-space.

Definition 1.5 ([23])**.** A *Menger probabilistic G-metric space* (shortly, *PGM-space*) is a triple (X, G^*, Δ), where X is a nonempty set, Δ is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathcal{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x, y, z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y \in X$ with $z \neq y$ and $t > 0$;
(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \cdots$ (: symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t + s) \geq \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t > 0$.

Definition 1.6 ([23])**.** Let (X, G^*, Δ) be a Menger PGM-space and $x_0 \in X$. For any $\epsilon > 0$ and δ with $0 < \delta < 1$, an (ϵ, δ)-*neighborhood* of x_0 is the set of all points y in X for which $G^*_{x_0,y,y}(\epsilon) > 1 - \delta$ and $G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\epsilon) > 1 - \delta, G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta\}.$$

This means that $N_{x_0}(\epsilon, \delta)$ is the set of all points y in X for which the probability of the distance from x_0 to y being less than ϵ is greater than $1 - \delta$.

Theorem 1.7 ([23])**.** *Let (X, G^*, Δ) be a Menger PGM-space. Then (X, G^*, Δ) is a Hausdorff space in the topology induced by the family $\{N_{x_0}(\epsilon, \delta)\}$ of (ϵ, δ)-neighborhoods.*

Definition 1.8 ([23])**.**

(1) A sequence $\{x_n\}$ in a PGM-space (X, G^*, Δ) is said to be *convergent* to a point $x \in X$ (write $x_n \to x$) if, for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_x(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$.

(2) A sequence $\{x_n\}$ in a PGM-space (X, G^*, Δ) is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ whenever $m, n, l > M_{\epsilon,\delta}$.

(3) A PGM-space (X, G^*, Δ) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

In [23], the author proved the following fixed point theorem:

Theorem 1.9 ([23])**.** *Let (X, G^*, Δ) be a complete Menger PGM-space with Δ of H-type and $T : X \to X$ be a mapping. If there exists a constant $\lambda \in (0, 1)$ such that*

$$G^*_{Tx,Ty,Tz}(\lambda t) \geq G^*_{x,y,z}(t)$$

for all $x, y, z \in X$ and $t > 0$, then, for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to a unique fixed point of T.

Another recent work on Menger PGM-space is from Alsulami et al. [1]. They proved the following fixed point theorem:

Theorem 1.10 ([1])**.** *Let (X, G^*, Δ) be a complete Menger PGM-space with Δ of H-type and $T : X \to X$ be a mapping. If there exists a function $\varphi \in \mathbf{\Phi_w}$ such that*

$$G^*_{Tx,Ty,Tz}(\varphi(t)) \geq G^*_{x,y,z}(t)$$

for all $x, y, z \in X$ and $t > 0$, then, the mapping T has a unique fixed point in X.

In this paper, we first introduce a concept of Menger probabilistic G-metric-like space (shortly, Menger PGML-space) and prove several necessary lemmas which will be used in the main results of this paper. Some fixed point theorems for φ-contractions in Menger PGML-space are proved. The conditions on the gauge function φ are different with the known ones in the present results. Our results extend and improve the ones of Zhou et al. [23], Fang [8], Hua at al. [12] and Alsuami et al. [1]. Finally, an example is given to illustrate the main result of this paper.

2. Menger probabilistic G-metric-like space

In [19], Hierro and Sen introduced a new concept called Menger probabilistic metric-like space as follows.

Definition 2.1 ([19])**.** A Menger probabilistic metric-like space is a triple (X, F, Δ), where X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X \to \mathcal{D}$ ($F_{x,y}$ denotes the value of F at the pair (x, y)) satisfying following conditions:

(PM-1) if $F_{x,y}(t) = 1$ for all $t > 0$, then $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;
(PM-3) $F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Inspired by the idea of Hierro and Sen, for our purpose we give the following concept called Menger probabilistic G-metric-like space by modifying (PGM-1) in Definition 1.5. We still denote by G^* Menger probabilistic G-metric-like without confusion.

Definition 2.2. Let X be a nonempty set, Δ be a continuous t-norm and G^* be a mapping from $X \times X \times X$ into $\mathcal{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x, y, z)). Assume that G^* satisfies the following conditions:

(PGML-1) if $G^*_{x,y,z}(t) = 1$ for all $t > 0$, then $x = y = z$;
(PGML-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y \in X$ with $z \neq y$ and $t > 0$;
(PGML-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \cdots$ (: symmetry in all three variables);
(PGML-4) $G^*_{x,y,z}(t + s) \geq \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t \geq 0$.

We call G^* a Menger probabilistic G-metric-like and (X, G^*, Δ) a Menger probabilistic G-metric-like space (shortly, Menger PGML-space).

Obviously, every Menger PGM-space is a Menger PGML-space and the inverse is not true.

Example 2.3. Let (X, F, Δ) be a Menger probabilistic metric-like space. Let $G^* : X^3 \to \mathcal{D}$ be defined by

$$G^*_{x,y,z}(t) = \min\{F_{x,y}(t), F_{x,z}(t), F_{y,z}(t)\}.$$

Then (X, G^*, Δ) is a Menger PGML-space. The proof is directly from Definition 2.2 and [23, Example 1.7].

Example 2.4. Let $X = \mathbb{R}^+$ and define the mapping $G^* : X^3 \to \mathcal{D}$ by

$$G^*_{x,y,z}(t) = \begin{cases} 1, & t > \max\{x, y, z\}, \\ \dfrac{t}{t + \max\{x, y, z\}}, & t \leq \max\{x, y, z\}. \end{cases} \tag{2.1}$$

First, if $G^*_{x,y,z}(t) = 1$ for all $t > 0$, then $x = y = z = 0$. Hence G^* satisfies (PGML-1). Since $t > \max\{x, y, z\}$ implies $t > \max\{x, y\}$, if $G^*_{x,y,z}(t) = 1$, then $G^*_{x,x,y}(t) = 1$. On the other hand, if $t \leq \{x, y\}$, then $t \leq \max\{x, y, z\}$ and hence $G^*_{x,x,y}(t) = \frac{t}{t+\max\{x,y\}} \geq \frac{t}{t+\max\{x,y,z\}} = G^*_{x,y,z}(t)$. Thus G^* satisfies (PGML-2). It is easy to see that G^* satisfies (PGML-3). Now we show that the following holds:

$$G^*_{x,y,z}(s + t) \geq \min\left\{G^*_{x,a,a}(s), G^*_{a,y,z}(t)\right\}, \quad \text{for all } x, y, z, a \in X \text{ and } s, t > 0. \tag{2.2}$$

If $s+t > \max\{x,y,z\}$, then $G^*_{x,y,z}(s+t) = 1$ and (2.2) holds. If $s+t \le \max\{x,y,z\}$, it is impossible that $s > \max\{x,a\}$ and $t > \max\{a,y,z\}$. We first assume that $s \le \max\{x,a\}$ and $t \le \max\{a,y,z\}$. If $G^*_{x,a,a}(s) > G^*_{x,y,z}(s+t)$ and $G^*_{a,y,z}(t) > G^*_{x,y,z}(s+t)$. Then

$$\frac{s}{s+\max\{x,a\}} > \frac{s+t}{s+t+\max\{x,y,z\}} \quad \text{and} \quad \frac{t}{t+\max\{a,y,z\}} > \frac{s+t}{s+t+\max\{x,y,z\}}.$$

That is

$$s\max\{x,y,z\} > (s+t)\max\{x,a\} \quad \text{and} \quad t\max\{x,y,z\} > (s+t)\max\{a,y,z\}.$$

It follows that

$$(s+t)\max\{x,y,z\} > (s+t)[\max\{x,a\} + \max\{a,y,z\}], \text{ i.e., } \max\{x,y,z\} > \max\{x,a\} + \max\{a,y,z\}.$$

It is a contradiction.

Next we assume that $s \le \max\{x,a\}$, but $t > \max\{a,y,z\}$. It follows that

$$\varDelta_M(G^*_{x,a,a}(s), G^*_{a,y,z}(t)) = G^*_{x,a,a}(s).$$

If $G^*_{x,a,a}(s) > G^*_{x,y,z}(s+t)$, i.e., $\frac{s}{s+\max\{x,a\}} > \frac{s+t}{s+t+\max\{x,y,z\}}$, then $s\max\{x,y,z\} > (s+t)\max\{x,a\}$. Further, we have $s\max\{x,y,z,a\} \ge s\max\{x,y,z\} > (s+t)\max\{x,a\}$. If $\max\{x,a\} \ge \max\{a,y,z\}$, then $s\max\{x,a\} = s\max\{x,y,z,a\} > (s+t)\max\{x,a\}$. It is a contradiction. Hence it must be $\max\{x,a\} < \max\{y,z,a\}$. Then we have $s\max\{y,z,a\} = s\max\{x,y,z,a\} > (s+t)\max\{x,a\}$. Note that $s \le \max\{x,a\}$ and $t > \max\{a,y,z\}$, we have $st > s\max\{y,z,a\} > (s+t)\max\{x,a\} \ge (s+t)s$. It is a contradiction. Hence it must be $G^*_{x,a,a}(s) \le G^*_{x,y,z}(s+t)$. Then (2.2) holds.

Similarly, under the assumption that $s > \max\{x,a\}$, but $t \le \max\{a,y,z\}$, we also can conclude that (2.2) holds. Therefore, $(X, G^*, \varDelta_M)$ is a Menger PGML-space.

Definition 2.5.

(1) A sequence $\{x_n\}$ in a PGML-space $(X, G^*, \varDelta)$ is said to be *convergent* to a point $x \in X$ (write $x_n \to x$) if, for any $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer $N_{t,\epsilon}$ such that $G^*_{x,x_n,x_n(t)} > 1-\epsilon$ and $G^*_{x,x,x_n(t)} > 1-\epsilon$ for all $n > N_{t,\epsilon}$.

(2) A sequence $\{x_n\}$ in a PGML-space $(X, G^*, \varDelta)$ is called a *Cauchy sequence* if, for any $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer $N_{t,\epsilon}$ such that $G^*_{x_n,x_m,x_l}(t) > 1-\epsilon$ whenever $m,n,l > N_{t,\epsilon}$.

(3) A PGML-space $(X, G^*, \varDelta)$ is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Notice that although the statement of concepts of convergence and completeness in Definition 2.5 are same with the ones in [23], the content of these concepts are different. In other words, if the sequence $\{x_n\}$ converges to some point x in Menger PGM-space, but it does not necessarily converge to x in menger PGML-space. However, the inverse is true. See the following example.

Example 2.6. Let $X = \mathbb{R}^+$ and let $G^* : X^3 \to \mathcal{D}$ be a mapping defined by $G^*_{x,x,x}(t) = 1$ for all $x \in X$ and $t > 0$, for all $x,y,z \in X$ without $x = y = z$, G^* is defined as (2.1). Then $(X, G^*, \min)$ is a Menger PGM-space. Now let $x_{1,n} = 1 - \frac{1}{n}$ for each $n \in \mathbb{N}$. As the concept of convergence in [23], the sequence $\{x_{1,n}\}$ converges to $x = 1$. However $\{x_{1,n}\}$ does not converge to 1 in the sense of Definition 2.5. Let $x_{2,n} = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then $\{x_{2,n}\}$ converges to $x = 0$ as both Definition 2.5 and the concept of convergence of [23]. That is, $\{x_{2,n}\}$ converges to $x = 0$ in both Menger PGML-space and Menger PGM-space.

Proposition 2.7. *Let $(X, G^*, \varDelta)$ be a Menger PGML-space. Let $\{x_n\} \subset X$ and $x, y \in X$. If $x_n \to x$ and $x_n \to y$, then $x = y$.*

Proof. For any $t > 0$ and $\epsilon \in (0,1)$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$G^*_{x_n,x,x}(t/2) > 1-\epsilon \text{ and } G_{y,x_n,x_n}(t/2) > 1-\epsilon$$

for all $n > \max\{N_1, N_2\}$. By (PGML-4), we have

$$G^*_{y,x,x}(t) \geq \Delta(G^*_{y,x_n,x_n}(t/2), G^*_{x_n,x,x}(t/2)) > \Delta(1-\epsilon, 1-\epsilon)$$

for all $n > \max\{N_1, N_2\}$. Since ϵ is arbitrary and Δ is continuous, it follows that $G^*_{y,x,x}(t) = 1$. By (PGML-1) we get $x = y$. This completes the proof. □

3. Some lemmas

Lemma 3.1. *Let (X, G^*, Δ) be a Menger PGML-space. For each $\lambda \in (0,1]$, define a function $d_\lambda(x,y) : X^2 \to \mathbb{R}^+$ by*

$$d_\lambda(x,y) = \inf\{t > 0 : G^*_{x,y,y}(t) > 1-\lambda\}. \tag{3.1}$$

Then the following hold:

(1) *$d_\lambda(x,y) < r$ if and only if $G^*_{x,y,y}(r) > 1-\lambda$;*

(2) *if $d_\lambda(x,y) = 0$ for all $\lambda \in (0,1]$, then $x = y$;*

(3) *if $\Delta = \Delta_M$, then for each $\lambda \in (0,1]$,*

$$d_\lambda(x,z) \leq d_\lambda(x,y) + d_\lambda(y,z) \ for\ all\ x,y,z \in X.$$

Proof.

(1) Assume that $d_\lambda(x,y) < r$. If $G^*_{x,y,y}(r) \leq 1-\lambda$, then for all $t > 0$ satisfying that $G^*_{x,y,y}(t) > 1-\lambda$, one must have $r < t$ since $G^*_{x,y,y}(t)$ is non-decreasing in t. Hence $r \leq \inf\{t > 0 : G^*_{x,y,y}(t) > 1-\lambda\} = d_\lambda(x,y)$. It is a contradiction. So $G^*_{x,y,y}(r) > 1-\lambda$. Now assume that $G^*_{x,y,y}(r) > 1-\lambda$. From (3.1) it follows that $d_\lambda(x,y) \leq r$. Since $G^*_{x,y,y}(t)$ is left continuous in t, we have $\lim_{n\to\infty} G^*_{x,y,y}(r - \frac{1}{n}) = G^*_{x,y,y}(r)$. Thus there exists $N \in \mathbb{N}$ such that $G^*_{x,y,y}(r - \frac{1}{n}) > 1-\lambda$ for all $n > N$. Therefore, by (3.1) we conclude that $d_\lambda(x,y) \leq r - \frac{1}{n} < r$ for all $n > N$.

(2) Assume that $d_\lambda(x,y) = 0$ for all $\lambda \in (0,1)$. By (1) we see that for any $t > 0$, $G^*_{x,y,y}(t) > 1-\lambda$ for all $\lambda \in (0,1]$. From (PGML-1) it follows that $x = y$.

(3) For any given $\epsilon > 0$, by the definitions of $d_\lambda(x,y)$ and $d_\lambda(y,z)$ there exist $t_1 > 0$ and $t_2 > 0$ such that

$$d_\lambda(x,y) + \epsilon/2 > t_1 \text{ and } d_\lambda(y,z) + \epsilon/2 > t_2,$$

where $G^*_{x,y,y}(t_1) > 1-\lambda$ and $G^*_{y,z,z}(t_2) > 1-\lambda$. By (PGML-4) we get

$$\begin{aligned} G^*_{x,z,z}(t_1+t_2) &\geq \Delta_M(G^*_{x,y,y}(t_1), G^*_{y,z,z}(t_2)) \\ &\geq \Delta_M(1-\lambda, 1-\lambda) \\ &= 1-\lambda. \end{aligned}$$

By the definition of $d_\lambda(x,z)$, we get

$$d_\lambda(x,z) \leq t_1 + t_2 < d_\lambda(x,y) + d_\lambda(y,z) + \epsilon.$$

From the arbitrariness of ϵ it follows that

$$d_\lambda(x,z) \leq d_\lambda(x,y) + d_\lambda(y,z).$$

This completes the proof. □

Remark 3.2. In Lemma 3.1, $d_\lambda(x,y) = d_\lambda(y,x)$ and $d_\lambda(x,x) = 0$ do not necessarily hold. Thus (X, d_λ) is not a pseudo-metric space.

Lemma 3.3. *Let (X, G^*, Δ) be a Menger PGML-space with the t-norm Δ of H-type. Let $\{d_\lambda\}_{\lambda\in(0,1]}$ be defined as (3.1). Then for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$ such that for each $n \in \mathbb{N}$,*

$$d_\lambda(x_0, x_n) \leq \sum_{i=0}^{n-1} d_\mu(x_i, x_{i+1}) \text{ for all } x_0, x_1, \cdots, x_n \in X.$$

Proof. Since Δ is a t-norm of H-type, for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$ such that

$$\Delta^n(1-\mu) > 1-\lambda \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

For any given $n \in \mathbb{N}$ and $x_0, x_1, \cdots, x_n \in X$, put $d_\mu(x_i, x_{i+1}) = t_i$ for each $i = 0, 1, \cdots, n-1$. For any ϵ, by Lemma 3.1 (1) we have

$$G^*_{x_i,x_{i+1},x_{i+1}}(t_1 + \epsilon/n) > 1-\mu, \ i = 0, 1, \cdots, n-1. \tag{3.3}$$

Now, from (PGML-4), (3.2) and (3.3) we obtain

$$\begin{aligned} G^*_{x_0,x_n,x_n}\Big(\sum_{i=0}^{n-1} t_i + \epsilon\Big) &\geq \Delta\Big(G^*_{x_0,x_1,x_1}(t_0+\epsilon/n), G^*_{x_1,x_n,x_n}\Big(\sum_{i=1}^{n-1} t_i + (n-1)\epsilon/n\Big)\Big) \\ &\geq \Delta\Big(G^*_{x_0,x_1,x_1}(t_0+\epsilon/n), \Delta(G^*_{x_1,x_2,x_2}(t_1+\epsilon/n), \cdots, \\ &\quad \Delta(G^*_{x_{n-2},x_{n-1},x_{n-1}}(t_{n-2}+\epsilon/n), G^*_{x_{n-1},x_n,x_n}(t_{n-1}+\epsilon/n)))\Big) \\ &\geq \Delta^n(1-\mu) > 1-\lambda. \end{aligned}$$

By Lemma 3.1 (1) we see that $d_\lambda(x_0, x_n) < \sum_{i=0}^{n-1} t_i + \epsilon$. Since ϵ is arbitrary, we get

$$d_\lambda(x_0, x_1) \leq \sum_{i=0}^{n-1} t_i = \sum_{i=0}^{n-1} d_\mu(x_i, x_{i+1}).$$

This completes the proof. □

Let Φ denote the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(i) $\phi^{-1}(0) = \{0\}$ and $\phi(a) = \inf_{t>a} \phi(t)$ for all $a > 0$;

(ii) $\sum_{n=1}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$.

It is easy to see that if $\phi \in \Phi$, then ϕ is non-decreasing and $\phi(t) < t$ for all $t > 0$.

Lemma 3.4. *Let (X, G^*, Δ) be a Menger PGML-space with the continuous t-norm Δ of H-type and let $\{x_n\}$ be a sequence in X. If there exists a function $\phi \in \Phi$ such that*

$$G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)) \geq G^*_{x_{n-1},x_n,x_n}(t) \tag{3.4}$$

for all $n \in \mathbb{N}$ and $t > 0$ and $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $\{d_\lambda\}_{\lambda\in(0,1]}$ be defined by (3.1). For each $\lambda \in (0,1]$ and $n \in \mathbb{N}$, set $a_n = d_\lambda(x_{n-1}, x_n)$. Since $\phi \in \Phi$, $\phi(a_n) = \inf_{t>a_n} \phi(t)$. Hence for any given $\epsilon > 0$, there exists $t_n > a_n$ such that $\phi(a_n) + \epsilon > \phi(t_n)$. By Lemma 3.1 (1) we have $G^*_{x_{n-1},x_n}(t_n) > 1-\lambda$ since $t_n > a_n = d_\lambda(x_{n-1}, x_n)$. From (3.1) it follows that

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t_n)) &\geq G^*_{x_{n-1},x_n,x_n}(t_n) \\ &\geq 1-\lambda. \end{aligned}$$

By Lemma 3.1 (1) we have

$$a_{n+1} = d_\lambda(x_n, x_{n+1}) \le \phi(t_n) < \phi(a_n) + \epsilon.$$

Since ϵ is arbitrary, one has

$$a_{n+1} \le \phi(a_n), \ \forall n \in \mathbb{N}. \tag{3.5}$$

Since ϕ is non-decreasing, by (3.5) and (3.1) we get

$$d_\lambda(x_n, x_{n+1}) \le \phi(d_\lambda(x_{n-1}, x_n)) \le \cdots \le \phi^n(d_\lambda(x_0, x_1)), \ \forall n \in \mathbb{N}. \tag{3.6}$$

By (3.6) and Lemma 3.3 there exists $\mu \in (0, \lambda]$ such that

$$d_\lambda(x_n, x_m) \le \sum_{i=n}^{m-1} d_\mu(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} \phi^i(d_\mu(x_0, x_1)) \text{ for all } m, n \in \mathbb{N} \text{ with } m > n.$$

Since $\phi \in \Phi$, we have $\sum_{i=n}^{m-1} \phi^i(d_\mu(x_0, x_1)) \to 0$ as $n \to \infty$ and hence

$$\lim_{m,n\to\infty} d_\lambda(x_n, x_m) = 0.$$

Thus for any $t > 0$, there exists $N \in \mathbb{N}$ such that $d_\lambda(x_n, x_m) < t$ for all $m > n > N$. By Lemma 3.1 (1) we can conclude that $G^*_{x_n,x_m,x_m}(t) > 1 - \lambda$ for all $m > n > N$. That is, for any $\epsilon > 0$,

$$\lim_{m,n\to\infty} G^*_{x_m,x_n,x_n}(\epsilon) = 1. \tag{3.7}$$

On the other hand, by (PGML-4) we have

$$\begin{aligned} G^*_{x_m,x_n,x_l}(\epsilon) &\ge \Delta(G^*_{x_m,x_n,x_n}(\epsilon/2), G^*_{x_n,x_n,x_l}(\epsilon/2)), \\ G^*_{x_m,x_n,x_n}(\epsilon/2) &\ge \Delta(G^*_{x_n,x_m,x_m}(\epsilon/4), G^*_{x_n,x_m,x_m}(\epsilon/4)), \\ G^*_{x_n,x_n,x_l}(\epsilon/2) &\ge \Delta(G^*_{x_n,x_l,x_l}(\epsilon/4), G^*_{x_n,x_l,x_l}(\epsilon/4)). \end{aligned}$$

Hence

$$G^*_{x_m,x_n,x_l}(\epsilon) \ge \Delta\Big(\Delta^2(G^*_{x_n,x_m,x_m}(\epsilon/4)), \Delta^2(G^*_{x_n,x_l,x_l}(\epsilon/4))\Big). \tag{3.8}$$

Since the Δ is continuous, from (3.6)-(3.8) it follows that, for any $\epsilon > 0$,

$$\lim_{n,m,l\to\infty} G^*_{x_m,x_n,x_l}(\epsilon) = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. This completes the proof. □

4. Fixed Point Theorems

Theorem 4.1. *Let (X, G^*, Δ) be a complete Menger PGML-space with the t-norm Δ of H-type and $T : X \to X$ be a mapping. If there exists a function $\phi \in \Phi$ such that*

$$G^*_{Tx,Ty,Tz}(\phi(t)) \ge G^*_{x,y,z}(t) \tag{4.1}$$

for all $x, y, z \in X$ and $t > 0$, then, the mapping T has a unique fixed point in X.

Proof. Take $x_0 \in X$ arbitrarily and let $x_n = T^n x_0$ for each $n \in \mathbb{N}$. By (4.1) we get

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)) &= G^*_{Tx_{n-1},Tx_n,Tx_n}(\phi(t)) \\ &\ge G^*_{x_{n-1},x_n,x_n}(t) \end{aligned}$$

for all $n \in \mathbb{N}$ and $t > 0$. By Lemma 3.4 we conclude that $\{x_n\}$ is a Cauchy sequence. Hence there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. For any $t > 0$, $\phi(t) < t$ since $\phi \in \Phi$. Hence by (PGML-4) and (4.1) it follows that for all $t > 0$,

$$\begin{aligned}
G^*_{Tx^*,x^*,x^*}(t) &\geq \Delta(G^*_{Tx^*,x_{n+1},x_{n+1}}(t/2), G^*_{x_{n+1},x^*,x^*}(t/2)) \\
&= \Delta(G^*_{Tx^*,Tx_n,Tx_n}(t/2), G^*_{x_{n+1},x^*,x^*}(t/2)) \\
&\geq \Delta(G^*_{Tx^*,Tx_n,Tx_n}(\phi(t/2)), G^*_{x_{n+1},x^*,x^*}(t/2)) \\
&\geq \Delta(G^*_{x^*,x_n,x_n}(t/2), G^*_{x_{n+1},x^*,x^*}(t/2)) \\
&\to 1, \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $x^* = Tx^*$.

Finally we show the uniqueness of fixed point of T. Suppose that x' is another fixed point of T. Let $\lambda \in (0, 1]$ and $a = d_\lambda(x^*, x')$. Since $\phi \in \Phi$, for any given $\epsilon > 0$, there exists $t > a$ such that $\phi(a) + \epsilon > \phi(t)$. From Lemma 3.1 (1) it follows that $G^*_{x^*,x',x'}(t) > 1 - \lambda$. Further by (4.1) one has

$$G^*_{x^*,x',x'}(\phi(t)) = G^*_{Tx^*,Tx',Tx'}(\phi(t)) \geq G^*_{x^*,x',x'}(t) > 1 - \lambda,$$

which implies that $d_\lambda(x^*, x') < \phi(t) < \phi(a) + \epsilon$. Since ϵ is arbitrary, $a = d_\lambda(x^*, x') \leq \phi(a)$. Since $\phi \in \Phi$, $\phi(a) < a$ if $a > 0$. Thus $a = 0$. That is, for any $\lambda \in (0, 1]$, $d_\lambda(x^*, x') = 0$. By Lemma 3.1 (2) we have $x^* = x'$. This completes the proof. □

Corollary 4.2. *Let (X, F, Δ) be a complete Menger probabilistic metric-like space with the t-norm Δ of H-type and $T : X \to X$ be a mapping. If there exists a function $\phi \in \Phi$ such that*

$$F_{Tx,Ty}(\phi(t)) \geq F_{x,y}(t) \tag{4.2}$$

for all $x, y, z \in X$ and $t > 0$, then, the mapping T has a unique fixed point in X.

Proof. Let $G^*_{x,y,z}(t) = \min\{F_{x,y}(t), F_{x,z}(t), F_{y,z}(t)\}$ for all $x, y, z \in X$ and $t > 0$, then (X, G^*, Δ) is a Menger PGML-space. Then from (4.2) we have

$$\begin{aligned}
G^*_{Tx,Ty,Tz}(\phi(t)) &= \min\{F_{Tx,Ty}(t), F_{Tx,Tz}(t), F_{Ty,Tz}(\phi(t))\} \\
&\geq \min\{F_{x,y}(t), F_{x,z}(t), F_{y,z}(t)\} \\
&= G^*_{x,y,z}(t)
\end{aligned}$$

for all $x, y, z \in X$ and $t > 0$. By Theorem 4.1 we conclude that T has a unique fixed point in X. This completes the proof. □

By setting $\phi(t) = kt$ for all $t > 0$, where $k \in (0, 1)$, we get the following corollary which extends Theorem 1.9 [23] from Menger PGM-space to PGML-space.

Corollary 4.3. *Let (X, G^*, Δ) be a complete Menger PGML-space with the t-norm Δ of H-type and $T : X \to X$ be a mapping. If there exists a constant $k \in (0, 1)$ such that*

$$G^*_{Tx,Ty,Tz}(kt) \geq G^*_{x,y,z}(t)$$

for all $x, y, z \in X$ and $t > 0$, then, the mapping T has a unique fixed point in X.

Next we give another fixed point theorem for φ-contraction in Menger PGML-space in which the gauge function $\varphi \in \Phi_{w^*}$.

Lemma 4.4 ([12]). *Let $\varphi \in \Phi_{w^*}$. Then for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) < t$.*

Lemma 4.5. *Let (X, G^*, Δ) be a Menger PGML-space with the t-norm Δ of H-type and let $\{x_n\}$ be a sequence in X. If there exists a function $\varphi \in \Phi_{w^*}$ such that*

$$G^*_{x_n,x_m,x_m}(\varphi(t)) \geq G^*_{x_{n-1},x_{m-1},x_{m-1}}(t) \tag{4.3}$$

for all $m, n \in \mathbb{N}$ and $t > 0$, then $\{x_n\}$ is a Cauchy sequence.

Proof. Since $\varphi \in \Phi_{w^*}$, $\varphi^n(t) > 0$ for all $n \in \mathbb{N}$ and $t > 0$. We show that

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t) = 1, \quad \forall t > 0. \tag{4.4}$$

First, from (4.3) it follows that

$$G^*_{x_n,x_{n+1},x_{n+1}}(\varphi^n(t)) \geq G^*_{x_0,x-1,x_1}(t), \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{4.5}$$

On the other hand, since $\lim_{t\to\infty} G^*_{x_0,x_1,x_1}(t) = 1$, for any $\epsilon \in (0, 1)$, there exists $t_0 > 0$ such that $G^*_{x_0,x_1,x_1}(t_0) > 1 - \epsilon$. For each $t > 0$, since $\varphi \in \Phi_{w^*}$, there exists $t_1 \geq \max\{t, t_0\}$ and $N \in \mathbb{N}$ such that $\varphi^n(t_1) < \min\{t_0, t\}$ for all $n \geq N$. By the monotonicity of $G^*_{x,y,z}(\cdot)$ and (4.5) we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(t) &\geq G^*_{x_n,x_{n+1},x_{n+1}}(\varphi^n(t_1)) \\ &\geq G^*_{x_0,x_1,x_1}(t_1) \geq G^*_{x_0,x_1,x_1}(t_0) \\ &> 1 - \epsilon. \end{aligned}$$

It follows that (4.4) holds.

Assume that $\lim_{n\to\infty} G^*_{x_n,x_{n+k},x_{n+k}}(t) = 1$ for some $k \in \mathbb{N}$ and all $t > 0$. Since Δ is continuous, we have

$$G^*_{x_n,x_{n+k},x_{n+k}}(t) \geq \Delta\Big(G^*_{x_n,x_{n+k},x_{n+k}}(t/2), G^*_{x_{n+k},x_{n+k+1},x_{n+k+1}}(t/2)\Big) \to \Delta(1, 1) = 1, \text{ as } n \to \infty.$$

By induction we conclude that

$$\lim_{n\to\infty} G^*_{x_n,x_{n+k},x_{n+k}}(t) = 1, \quad \forall k \in \mathbb{N} \text{ and } t > 0.$$

Let $t > 0$. By Lemma 4.4 there exists $r \geq t$ such that $\varphi(r) < t$. Next we show by induction that

$$G^*_{x_n,x_{n+k},x_{n+k}}(t) \geq \Delta^k\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r))\Big), \ \forall k \in \mathbb{N}. \tag{4.6}$$

It is easy to see that (4.6) holds for $k = 1$. Assume that (4.6) holds for some $k \in \mathbb{N}$. By (PGML-4) and (4.3) we have

$$\begin{aligned} G^*_{x_n,x_{n+k+1},x_{n+k+1}}(t) &\geq \Delta\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), G^*_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\varphi(r))\Big) \\ &\geq \Delta\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), G^*_{x_n,x_{n+k},x_{n+k}}(r)\Big) \\ &\geq \Delta\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), G^*_{x_n,x_{n+k},x_{n+k}}(t)\Big) \\ &\geq \Delta\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), \Delta^k\big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r))\big)\Big) \\ &= \Delta^{k+1}\big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r))\big). \end{aligned}$$

It follows that (4.6) holds for $k + 1$. Thus (4.6) holds by induction.

For each $t > 0$, set $a_n = \inf_{k\in\mathbb{N}} G^*_{x_n,x_{n+k},x_{n+k}}(t)$ for each $n \in \mathbb{N}$. Since $\varphi \in \Phi_{w^*}$, Lemma 4.4 shows that there exists $r \geq t$ such that $\varphi(r) < t$. By (4.3) we have

$$a_n = \inf_{k\in\mathbb{N}} G^*_{x_n,x_{n+k},x_{n+k}}(t)$$

$$
\begin{aligned}
&\geq \inf_{k\in\mathbb{N}} G^*_{x_n,x_{n+k},x_{n+k}}(\varphi(r))\\
&\geq \inf_{k\in\mathbb{N}} G^*_{x_{n-1},x_{n+k-1},x_{n+k-1}}(r)\\
&\geq \inf_{k\in\mathbb{N}} G^*_{x_n,x_{n+k-1},x_{n+k-1}}(t)\\
&= a_{n-1}, \quad \forall n\in\mathbb{N},
\end{aligned}
$$

which implies that $\{a_n\}$ is non-decreasing. So there exists $a\in[0,1]$ such that $a_n\to a$ as $n\to\infty$. Assume that $a<1$. For any given $\epsilon\in(0,1)$, by the definition of a_n, there exists $k_0=k_0(\epsilon,n)\in\mathbb{N}$ such that

$$a_n > G^*_{x_n,x_{n+k_0},x_{n+k_0}}(t)-\epsilon/2. \tag{4.7}$$

Note that $\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t_0))=1$ by (4.4). Therefore, there exists $\delta\in(0,1)$ and $N\in\mathbb{N}$ such that $G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t_0))\in(1-\delta,1)$ for all $n>N$. Since $\varDelta$ is of H-type, $\varDelta^k(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t_0)))>1-\epsilon$ for all $n>N$ and all $k\in\mathbb{N}$. Now, combining (4.6) with (4.7) we get

$$1>a\geq a_n>1-\epsilon$$

for all $n\in\mathbb{N}$. Since ϵ is arbitrary, one has

$$1>a\geq a_n\geq 1.$$

It is a contradiction. So $a=1$, i.e, $a_n\to 1$ as $n\to\infty$. For any $\tau\in(0,1)$, there exists $N\in\mathbb{N}$ such that $a_n>1-\tau$ for all $n>N$. By the definition of a_n, we have

$$G^*_{x_n,x_{n+k},x_{n+k}}(t)>1-\tau$$

for all $n\in\mathbb{N}$ with $n>N$ and all $k\in\mathbb{N}$. This follows that

$$\lim_{m,n\to\infty} G^*_{x_n,x_m,x_m}(t)=1, \quad \forall t>0.$$

By (PGML-3) and (PGML-4) we have

$$
\begin{aligned}
G^*_{x_n,x_m,x_l}(t) &\geq \varDelta\Big(G^*_{x_n,x_m,x_m}(t/2), G^*_{x_m,x_m,x_l}(t/2)\Big)\\
&\to \varDelta(1,1)=1, \quad \forall t>0, \quad \text{as } m,n,l\to\infty.
\end{aligned}
$$

It follows that $\{x_n\}$ is a Cauchy sequence. This completes the proof. □

The following fixed point theorem extends Theorem 1.10 of Alsulami et al. [1] from Menger PGM-space to PGML-space and the class of functions from Φ_w to Φ_{w^*}.

Theorem 4.6. *Let $(X,G^*,\varDelta)$ be a complete Menger PGML-space with the t-norm $\varDelta$ of H-type and $T: X\to X$ be a mapping. If there exists a function $\varphi\in\Phi_{w^*}$ such that*

$$G^*_{Tx,Ty,Tz}(\varphi(t))\geq G^*_{x,y,z}(t) \tag{4.8}$$

for all $x,y,z\in X$ and $t>0$, then, the mapping T has a unique fixed point in X.

Proof. Take $x_0\in X$ arbitrarily and let $x_n=T^nx_0$ for each $n\in\mathbb{N}$. By (4.8) we have

$$
\begin{aligned}
G^*_{x_n,x_m,x_m}(\varphi(t))G^*_{Tx_{n-1},Tx_{m-1},Tx_{m-1}}(\varphi(t))\\
\geq G^*_{x_{n-1},x_{m-1},x_{m-1}}(t)
\end{aligned}
$$

for all $m,n\in\mathbb{N}$ and $t>0$. By Lemma 4.5 we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Finally, by a similar process with the proof of Theorem 4.1 we can show that x^* is the unique fixed point of T. This completes the proof. □

By a similar method with the proof of Corollary 4.2, we get the following result which extends Theorem 1.4 of [12, Theorem 3.1] from Menger PM-space to PML-space.

Corollary 4.7. *Let (X, F, Δ) be a complete Menger probabilistic metric-like space with the t-norm Δ of H-type and $T : X \to X$ be a mapping. If there exists a function $\varphi \in \Phi_{w^*}$ such that*

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t)$$

for all $x, y, \in X$ and $t > 0$, then, the mapping T has a unique fixed point in X.

Corollary 4.8. *Let (X, G^*, Δ) be a complete Menger PGML-space with the t-norm Δ of H-type. Let $S_1, S_2 : X \to X$ be two mappings satisfying*

$$G^*_{S_1x,S_1y,S_1z}(\varphi(t)) \geq G^*_{x,y,z}(t) \quad \text{and} \quad G^*_{S_2x,S_2y,S_2z}(\varphi(t)) \geq G^*_{x,y,z}(t)$$

for all $x, y, z \in X$ and $t > 0$, where $\varphi \in \Phi_{w^*}$. If S_1 commutes with S_2, then S_1 and S_2 have a unique common fixed point in X.

Proof. Let $T = S_1S_2$. Then (4.8) implies that T satisfies the condition (4.7). By Theorem 4.2 it follows that T has a unique fixed point $x^* \in X$. Since S_1 commutes with S_2, we have $S_1S_2x^* = S_2S_1x^*$. Further we have $TS_1x^* = S_1S_2S_1x^* = S_1Tx^* = S_1x^*$, which shows that S_1x^* is a fixed point of T. Hence $x^* = S_1x^*$ since the fixed point of T is unique. Similarly, we have $x^* = S_2x^*$. That is, x^* is the unique fixed point of S_1 and S_2. Assume that x' is another common fixed point of S_1 and S_2. Since S_1 commutes with S_2, we have $TS_1x' = S_1S_2S_1x' = S_1S_1S_2x' = S_1x'$, which shows that S_1x' is the fixed point of T. Hence $x' = S_1x' = x^*$ since x^* is the unique fixed point of T. Thus x^* is the unique common fixed point of S_1 and S_2. This completes the proof. □

Finally, we give an example to illustrate Theorem 4.6 as follows.

Example 4.9. Let $X = \{2^{n+1} : n \in \mathbb{N}\} \cup \{0, 2\}$ and define the mapping $G^* : X^3 \to \mathcal{D}$ by $G^*x, y, z(t) = 0$ for all $x, y, z \in X$ and all $t \leq 0$, $G_{0,0,0}(t) = 1$ for all $t > 0$, and

$$G^*_{x,y,z}(t) = \begin{cases} \frac{1}{2}, & 0 < t \leq \max\{x, y, z\}, \\ 1, & t > \max\{x, y, z\} \end{cases}$$

for all $x, y, z \in X$. We show that $(X, G^*, \min)$ is a Menger PGML-space. It is easy to see that G^* satisfies (PGML-1) and (PGML-3). Obviously, $G^*_{x,x,y}(t) = 1$ if $G^*_{x,y,z}(t) = 1$, i.e, $t > \max\{x, y, z\}$. If $t \leq \max\{x, y, z\}$ and $t > \max\{x, y\}$, then $G^*_{x,x,y}(t) > G^*_{x,y,z}(t)$. If $t \leq \max\{x, y\}$ and hence $t \leq \max\{x, y, z\}$, then $G^*_{x,x,y}(t) = G^*_{x,y,z}(t)$. It follows that G^* satisfies (PGML-2). Now we show that for all $s, t > 0$ and all $x, y, z, a \in X$, the following holds:

$$G^*_{x,y,z}(s + t) \geq \min\{G^*_{x,a,a}(s), G^*_{a,y,z}(t)\}. \tag{4.9}$$

If $x = y = z = 0$ or $s + t > \max\{x, y, z\}$, then (4.9) holds. Assume that $s + t \leq \max\{x, y, z\}$ and hence $G^*_{x,y,z}(s + t) = \frac{1}{2}$. Then $s + t \leq \max\{x, y, z, a\}$. It follows that $s \leq \max\{x, a\}$ or $t \leq \max\{y, z, a\}$. Thus we have $G^*_{x,a,a}(s) = \frac{1}{2}$ or $G^*_{a,y,z}(t) = \frac{1}{2}$. Then (4.9) holds. Hence G^* satisfies (PGML-4). It follows that $(X, G^*, \min)$ is a Menger PGML-space. Moreover, $(X, G^*, \min)$ is complete and $G^*_{x,y,z}(t)$ is non-decreasing in $t > 0$. But notice that $(X, G^*, \min)$ is not a Menger PGM-space.

Let $\varphi(t) = t$ for $t \in [0, 1]$ and $\varphi(t) = t - 1$ for $t > 1$. Then $\varphi \in \Phi_{w^*}$. Let $T : X \to X$ be a mapping defined by $T0 = T2 = 0$, $T2^{n+1} = 2^n$ for each $n \in \mathbb{N}$. We show that T satisfies

$$G^*_{Tx,Ty,Tz}(\varphi(t)) \geq G^*_{x,y,z}(t), \text{ for all } x, y, z \in X \text{ and } t > 0. \tag{4.10}$$

We discuss (4.10) by the following cases:

(i) $x, y, z \in \{0, 2\}$, then $G^*_{Tx,Ty,Tz}(\varphi(t)) = G^*_{0,0,0}(\varphi(t)) = 1$. Thus (4.10) holds.

(ii) $x, y \in \{0, 2\}$, $z = 2^{n+1}, n \in \mathbb{N}$. Then $G^*_{Tx,Ty,Tz}(\varphi(t)) = G^*_{0,0,2^n}(\varphi(t))$. If $\varphi(t) > 2^n$, then $G^*_{0,0,2^n}(\varphi(t)) = 1$ and hence (4.10) holds. If $\varphi(t) < 2^n$, then we have $t < z = 2^{n+1}$ whenever $\varphi(t) = t$ or $t - 1$. It follows that $G^*_{Tx,Ty,Tz}(\varphi(t)) = \frac{1}{2} = G^*_{x,y,z}(t)$. Thus (4.10) holds.

(iii) $x = 2^{m+1}, y = 2^{n+1}, z = 2^{l+1}$, $m, l, n \in \mathbb{N}$. Assume that $l = \max\{m, n, l\}$. Obviously (4.9) holds if $\varphi(t) > 2^l$. Assume that $\varphi(t) \leq 2^l$. Then $G^*_{Tx,Ty,Tz}(\varphi(t)) = G^*_{2^m,2^n,2^l}(\varphi(t)) = \frac{1}{2}$. Since $\varphi(t) \leq 2^l$, one has $t \leq 2^{l+1}$ whenever $\varphi(t) = t$ or $t - 1$. Thus $G^*_{x,y,z}(t) = \frac{1}{2}$. It follows that (4.10) holds.

By the discussion above, we see that all conditions in Theorem 4.6 are satisfied. Hence by Theorem 4.6 we conclude that T has a unique fixed point in X. In fact, the unique fixed point of T is $x^* = 0$. By the proof of Theorem 4.6 we know that taking $x_0 = 2^{k+1} \in X$ ($k \in \mathbb{N}$) arbitrarily, the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ ($n \in \mathbb{N}$) converges to the unique fixed point of T. Obviously, $x_n = 0$ for all $n \geq k + 1$, which is the unique fixed point of T.

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References

[1] S. Alsulami, B. S. Choudhury, P. Das, *φ-contraction in generalized probabilistic metric spaces*, Fixed Point Theory Appl., **2015** (2015), 9 pages. 1, 1.10, 1, 4

[2] S. Z. Chang, *Fixed point theory of mappings in probabilistic metric spaces with applications*, Sci. Sinica Ser. A, **26** (1983), 1144–1155. 1

[3] S. S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nonva Science Publishers, Inc., New York, (2001).

[4] Lj. Ciric, *Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces*, Nonlinear Anal., **72** (2010), 2009–2018. 1

[5] Lj. Ciric, D. Mihet, R. Saadati, *Monotone generzliaed contractions in partially ordered probabilistic metric spaces*, Topol. Appl., **156** (2009), 2838–2844.

[6] J. X. Fang, *Fixed point theorems of local contraction mappings on Menger spaces*, Appl. Math. Mech., **12** (1991), 363–372.

[7] J. X. Fang, *On fixed degree theorems for fuzzy mappings in Menger PM-spaces*, Fuzzy Sets Syst., **157** (2006), 270–285. 1

[8] J. X. Fang, *On φ-contractions in probabilistic and fuzzy metric spaces*, Fuzzy Sets Syst., **267** (2015), 86–99. 1, 1.3, 1

[9] O. Hadzic, *Fixed point theorem for multivalued mappings in probabilistic metric spaces*, Fuzzy Sets Syst., **98** (1997), 219–226. 1

[10] O. Hadzic, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht, (2001). 1

[11] O. Hadzic, E. Pap, M. Budincevic, *A generalization of Tardiff's fixed point theorem in probabilistic metric spaces and applications to random equations*, Fuzzy Sets Syst., **156** (2005), 124–134. 1

[12] H. C. Hua, M. J. Chen, S. H. Wang, *New result on fixed point theorems for φ-contractions in Menger spaces*, Fixed Point Theory Appl., **2015** (2015), 9 pages. 1, 1, 1.4, 1, 4.4, 4

[13] J. Jachymski, *On probabilistic φ-contractions on Menger spaces*, Nonlinear Anal., **73** (2010), 2199–2203. 1, 1.2

[14] Y. C. Liu, Z. X. Li, *Coincidence point theorems in probabilistic and fuzzy metric spaces*, Fuzzy Sets Syst., **158** (2007), 58–70. 1

[15] K. Menger, *Statistical metrics*, Proc. Nat. Acad. of Sci. U.S.A., **28** (1942), 535–537. 1

[16] D. Mihet, *Multivalued generalizations of probabilistic contractions*, J. Math. Anal. Appl., **304** (2005), 462–472. 1

[17] A. Razani, K. Fouladgar, *Extension of contractive maps in the Menger probabilistic metric space*, Chaos. Solit. Frac., **34** (2007), 1724–1731. 1

[18] A. Roldan, J. Martinez-Moreno, C. Roldan, *Tripled fixed point theorems in fuzzy metric spaces and applications*, Fixed Point Theory Appl., **2013** (2013), 13 pages. 1

[19] A. F. R. L. Hierro, M. Sen, *Some fixed point theorems in Menger probabilistic metric-like spaces*, Fixed Point Theory Appl., **2015** (2015), 16 pages. 2, 2.1

[20] B. Schweizer, A. Sklar, *Probabilisitc Metric Spaces*, Elsevier/North-Holland, New York, (1983). 1

[21] V. M. Sehgal, A. T. Bharucha-Reid, *Fixed points of contraction mappings on PM-spaces*, Math. Syst. Theory, **6** (1972), 97–102. 1

[22] S. H. Wang, A. A. N. Abdou, Y. J. Cho, *Coupled common fixed point theorems for φ-contractions in probabilistic metric spaces and applications*, Iranian J. Fuzzy Syst., **12** (2015), 95–108. 1

[23] C. L. Zhou, S. H. Wang, Lj. Ciric, S. M. Alsulami, *Generalized probabilistic metric spaces and fixed point thoerems*, Fixed Point Theory Appl., **2014** (2014), 15 pages. 1, 1.5, 1.6, 1.7, 1.8, 1, 1.9, 1, 2.3, 2, 2.6, 4