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Caustics of de Sitter spacelike curves in Minkowski 3-space

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Abstract

In this paper, we consider evolutes of spacelike curves in de Sitter 2-space. Applying the theory of singularity theory, we find that these evolutes can be seen as one dimensional caustics which are locally diffeomorphic to lines or ordinary cusps. We establish the relationships between singularities of caustics and geometric invariants of curves under the action of the Lorentz group.

Keywords: De Sitter 2-space, caustics, singularity theory.
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1. Introduction

The Lorentzian space form with the positive constant curvature is called de Sitter space. It is an important subject in the theory of relativity and the astrophysics [5, 7]. De Sitter 4-space is know as a vacuum solution of the Einstein equation. In this paper, we consider, however, spacelike curves in de Sitter 2-space as the most elementary case for the study of higher codimensional spacelike submanifolds in non-flat Lorentzian space forms. We consider spacelike curves in de Sitter 2-space and the caustic curves associated to these curves. Remark, similarly to geometrical optics in Euclidean 3-space, a submanifold of positive codimension in Euclidean space $\mathbb{R}^{m+1}$ may be considered as a source of light (or as an initial wave front). The normal lines to this source submanifold are called normal light rays and its focal set (on which the light intensity is much more concentrated than in the other points of the space) is called the caustic of that submanifold. If we consider the caustic from the viewpoint of singularity theory, it is defined to be the set of critical values of a Lagrangian map [1]. We know that evolute of a regular plane curve in Euclidean space

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can be seen as caustic. As a consequence, it has only Lagrangian singularities. Inspired by these results, we hope to investigate caustics of spacelike curves in de sitter 2-space.

In this paper, we adopt a special pseudo-orthogonal frame in $\mathbb{R}^3_1$ and show the de Sitter Frenet-Serret formula (cf. Proposition 2.1). We also define de Sitter height functions on these curves which can be seen as generating families of some Lagrangian maps. With the aid of a bit of singularity theory of de Sitter height functions, we study singularities of caustics and we establish the relation between these singularities and de Sitter invariants of the original curve.

The rest of this paper is organized as follows. In Section 2, we investigate the Frenet-Serret-type formula for spacelike curves in de Sitter 2-space. Then, we introduce two different families of functions on spacelike curves $\boldsymbol{\gamma}$ that will be useful to study the singularities of the caustics and investigate the geometric meaning of the de sitter invariants in Section 3 and Section 4. Afterwards, some general results on the singularity theory are used for families of function germs and the main results (Theorem 5.3 and Theorem 6.2) are proved in Section 5 and Section 6.

All maps considered here are of class C^∞ unless otherwise stated.

2. The basic concepts

In this section, we use the basic notions and results in Lorentzian geometry. For more detail descriptions, see [3, 4, 5, 6, 7, 8]. Let $\mathbb{R}^3$ be a 3-dimensional vector space, for any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, their pseudo scalar product is defined by $\langle \mathbf{x}, \mathbf{y}\rangle = -x_1y_1 + x_2y_2 + x_3y_3$. The pair $(\mathbb{R}^3, \langle , \rangle)$ is called Minkowski 3-space. We write $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle , \rangle)$.

For any $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the pseudo vector product of $\mathbf{x}$ and $\mathbf{y}$ is defined as follows:

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

We remark that $\langle \mathbf{x} \wedge \mathbf{y}, \mathbf{z}\rangle = det(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Hence, $\mathbf{x} \wedge \mathbf{y}$ is pseudo-orthogonal to $\mathbf{x}, \mathbf{y}$. A non-zero vector $\mathbf{x}$ in $\mathbb{R}^3_1$ is called spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x}\rangle > 0$, $\langle \mathbf{x}, \mathbf{x}\rangle = 0$, $\langle \mathbf{x}, \mathbf{x}\rangle < 0$, respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}^3_1$ is defined by $\parallel \mathbf{x} \parallel = \sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}$.

We now define spheres in $\mathbb{R}^3_1$ as follows:

$$\begin{aligned} H^2_+ &= \{\mathbf{x} \in \mathbb{R}^3_1 | -x_1^2 + x_2^2 + x_3^2 = -1, x_1 \geq 1\}, \\ H^2_- &= \{\mathbf{x} \in \mathbb{R}^3_1 | -x_1^2 + x_2^2 + x_3^2 = -1, x_1 \leq 1\}, \\ S^2_1 &= \{\mathbf{x} \in \mathbb{R}^3_1 | -x_1^2 + x_2^2 + x_3^2 = 1\}. \end{aligned}$$

We call $H^2_\pm$ hyperbola and S^2_1 de Sitter 2-space.

Let $\boldsymbol{\gamma} : I \to S^2_1 \subset \mathbb{R}^3_1$ be a smooth regular curve in S^2_1 (i.e.,$\dot{\boldsymbol{\gamma}}(t) \neq 0$ for any $t \in I$), where I is an open interval. If $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle > 0$ for any $t \in I$, we call such a curve a spacelike curve. The arc-length of a spacelike curve $\boldsymbol{\gamma}(t)$, measured from $\boldsymbol{\gamma}(t_0), t_0 \in I$, is $s(t) = \int_{t_0}^t \parallel \dot{\boldsymbol{\gamma}}(t) \parallel dt$. Then the parameter s is determined such that $\parallel \boldsymbol{\gamma}'(s) \parallel = 1$, where $\boldsymbol{\gamma}'(s) = \frac{d\boldsymbol{\gamma}}{ds}$. So we say that a spacelike curve $\boldsymbol{\gamma}$ is parameterized by arc-length if it satisfies that $\parallel \boldsymbol{\gamma}'(s) \parallel = 1$. Throughout the remainder in this paper, we denote the parameter s of $\boldsymbol{\gamma}$ as the arc-length parameter. We denote $\mathbf{t}(s) = \boldsymbol{\gamma}'(s)$ and we call $\mathbf{t}(s)$ a unit tangent vector of $\boldsymbol{\gamma}$ at s. One can construct a unit vector $\mathbf{e}(s) = \boldsymbol{\gamma}(s) \wedge \mathbf{t}(s)$. By definition, we can calculate that $\langle \mathbf{e}(s), \mathbf{e}(s)\rangle = -1$. Then, we have a pseudo-orthonormal frame $\{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{e}(s)\}$ along $\boldsymbol{\gamma}(s)$. By the standard arguments, we can show the following de Sitter Frenet-Serret formula of spacelike curves:

Proposition 2.1. *Under the above notations, we have the following de Sitter Frenet-Serret formula of spacelike curves:*

$$\begin{cases} \boldsymbol{\gamma}'(s) = \mathbf{t}(s), \\ \mathbf{t}'(s) = -\boldsymbol{\gamma}(s) + k_g(s)\mathbf{e}(s), \\ \mathbf{e}'(s) = k_g(s)\mathbf{t}(s), \end{cases}$$

where $k_g(s)$ is the geodesic curvature of the curve $\boldsymbol{\gamma}$ in S_1^2, which is given by $k_g(s) = -\langle \mathbf{t}(s), \mathbf{e}'(s)\rangle$.

Under the assumption that $k_g(s) \neq 1$, we define a curve in $\mathbb{R}_1^3$ by

$$SE_\gamma(s) = \frac{1}{\sqrt{\mid k_g^2(s) - 1 \mid}} \left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right).$$

We remark that $SE_\gamma(s)$ is located in S_1^2 if and only if $\mid k_g(s) \mid > 1$, otherwise it is in $H_\pm^2$. We call $SE_\gamma(s)$ the caustic of $\boldsymbol{\gamma}$. The geometric meanings of caustic will be discussed in Section 4.

3. Height functions on spacelike curves in de Sitter 2-space

In this section, we introduce two different families of functions on a regular curve $\boldsymbol{\gamma} : I \to S_1^2$. We now define a function $H^S : I \times S_1^2 \longrightarrow \mathbb{R}$ by $H^S(s, \mathbf{v}) = \langle \boldsymbol{\gamma}(s), \mathbf{v}\rangle$. We call H^S the de Sitter spacelike height function on a curve $\boldsymbol{\gamma}$. We also define a function $H^T : I \times H_+^2 \longrightarrow \mathbb{R}$ by $H^T(s, \mathbf{v}) = \langle \boldsymbol{\gamma}(s), \mathbf{v}\rangle$. We call H^T the de Sitter timelike height function on a curve $\boldsymbol{\gamma}$. We denote $(h_v^S)(s) = H^S(s, \mathbf{v})$ and $(h_v^T)(s) = H^T(s, \mathbf{v})$. We have the following proposition.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \to S_1^2$ be a unit speed space curve.*
(A) *For any $(s, \mathbf{v}) \in I \times S_1^2$, we have the following claims.*

(1) $(h_v^S)'(s) = 0$ *if and only if there are real numbers λ and μ such that* $\mathbf{v} = \lambda\boldsymbol{\gamma}(s) + \mu\mathbf{e}(s)$ *and* $\lambda^2 - \mu^2 = 1$.

(2) $(h_v^S)'(s) = (h_v^S)''(s) = 0$ *if and only if* $\mathbf{v} = \pm\frac{1}{\sqrt{k_g^2(s)-1}}\left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$ *and* $k_g^2(s) > 1$.

(3) $(h_v^S)'(s) = (h_v^S)''(s) = (h_v^S)^{(3)}(s) = 0$ *if and only if*

$$\mathbf{v} = \pm\frac{1}{\sqrt{k_g^2(s) - 1}}\left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right), k_g^2(s) > 1 \textit{ and } k_g'(s) = 0.$$

(4) $(h_v^S)'(s) = (h_v^S)''(s) = (h_v^S)^{(3)}(s) = (h_v^S)^{(4)}(s) = 0$ *if and only if* $\mathbf{v} = \pm\frac{1}{\sqrt{k_g^2(s)-1}}\left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$, $k_g^2(s) > 1$ *and* $k_g'(s) = k_g''(s) = 0$.

(B) *For any $(s, \mathbf{v}) \in I \times H_+^2$, we have we have the following claims.*

(1) $(h_v^T)'(s) = 0$ *if and only if there are real numbers λ and μ such that* $\mathbf{v} = \lambda\boldsymbol{\gamma}(s) + \mu\mathbf{e}(s)$ *and* $\lambda^2 - \mu^2 = -1$.

(2) $(h_v^T)'(s) = (h_v^T)''(s) = 0$ *if and only if* $\mathbf{v} = \pm\frac{1}{\sqrt{1-k_g^2(s)}}\left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$ *and* $k_g^2(s) < 1$.

(3) $(h_v^T)'(s) = (h_v^T)''(s) = (h_v^T)^{(3)}(s) = 0$ *if and only if*

$$\mathbf{v} = \pm\frac{1}{\sqrt{1 - k_g^2(s)}}\left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right), k_g^2(s) < 1 \textit{ and } k_g'(s) = 0.$$

(4) $(h_v^T)'(s) = (h_v^T)''(s) = (h_v^T)^{(3)}(s) = (h_v^T)^{(4)}(s) = 0$ *if and only if* $\mathbf{v} = \pm\frac{1}{\sqrt{1-k_g^2(s)}}\left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$, $k_g^2(s) < 1$ *and* $k_g'(s) = k_g''(s) = 0$.

Proof. **(A)**

(1) Since $(h_v^S)'(s) = \langle \mathbf{t}(s), \mathbf{v}\rangle$, by the condition that $(h_v^S)'(s) = 0$, we have that there are real numbers λ and μ such that $\mathbf{v} = \lambda\boldsymbol{\gamma}(s) + \mu\mathbf{e}(s)$. By the condition that $\mathbf{v} \in S_1^2$, we get $\lambda^2 - \mu^2 = 1$. The converse direction also holds.

(2) By the Frenet formula, we get

$$(h_v^S)''(s) = \langle -\boldsymbol{\gamma}(s) + k_g(s)\mathbf{e}(s), \mathbf{v} \rangle .$$

Since $(h_v^S)'(s) = (h_v^S)''(s) = 0$, we have $\lambda = -\mu k_g(s)$. It follows from the fact that $\lambda^2 - \mu^2 = 1$, we have $\mu = \pm \frac{1}{\sqrt{k_g^2(s)-1}}$ and $k_g^2(s) > 1$. Therefore, we have $\mathbf{v} = \pm \frac{1}{\sqrt{k_g^2(s)-1}} \left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$ and $k_g^2(s) > 1$.

(3) We can get

$$(h_v^S)^{(3)}(s) = \left\langle (k_g^2(s) - 1)\mathbf{t}(s) + k_g'(s)\mathbf{e}(s), \mathbf{v} \right\rangle .$$

Since $(h_v^S)'(s) = (h_v^S)''(s) = (h_v^S)^{(3)}(s) = 0$, we have $\pm \frac{1}{\sqrt{k_g^2(s)-1}}(-k_g'(s)) = 0$. This is equivalent to the condition $k_g'(s) = 0$. Therefore, we have $\mathbf{v} = \pm \frac{1}{\sqrt{k_g^2(s)-1}}(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s))$, $k_g^2(s) > 1$ and $k_g'(s) = 0$.

(4) We can get

$$(h_v^S)^{(4)}(s) = \left\langle 3k_g(s)k_g'(s)\mathbf{t}(s) + (-k_g'(s) + k_g^3(s) + k_g''(s))\mathbf{e}(s) + (1 - k_g^2(s))\boldsymbol{\gamma}(s), \mathbf{v} \right\rangle .$$

Since

$$(h_v^S)'(s) = (h_v^S)''(s) = (h_v^S)^{(3)}(s) = (h_v^S)^{(4)}(s) = 0,$$

we have

$$\pm \frac{1}{\sqrt{k_g^2(s) - 1}}(-k_g''(s)) = 0.$$

This is equivalent to the condition $k_g''(s) = 0$. Therefore, we have $\mathbf{v} = \pm \frac{1}{\sqrt{k_g^2(s)-1}} \left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$, $k_g^2(s) > 1$ and $k_g'(s) = k_g''(s) = 0$.

(B)

(1) Since $(h_v^T)'(s) = \langle \mathbf{t}(s), \mathbf{v} \rangle$, by the condition that $(h_v^T)'(s) = 0$, we have that there are real numbers λ and μ such that $\mathbf{v} = \lambda\boldsymbol{\gamma}(s) + \mu\mathbf{e}(s)$. By the condition that $\mathbf{v} \in H_+^2$, we get $\lambda^2 - \mu^2 = -1$. The converse direction also holds.

(2) By the Frenet formula, we get

$$(h_v^T)''(s) = \langle -\boldsymbol{\gamma}(s) + k_g(s)\mathbf{e}(s), \mathbf{v} \rangle .$$

Since $(h_v^T)'(s) = (h_v^T)''(s) = 0$, we have $\lambda = -\mu k_g(s)$. It follows from the fact that $\lambda^2 - \mu^2 = -1$, we have $\mu = \pm \frac{1}{\sqrt{1-k_g^2(s)}}$ and $k_g^2(s) < 1$. Therefore, we have $\mathbf{v} = \pm \frac{1}{\sqrt{1-k_g^2(s)}} \left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$ and $k_g^2(s) < 1$.

(3) We can get

$$(h_v^T)^{(3)}(s) = \left\langle (k_g^2(s) - 1)\mathbf{t}(s) + k_g'(s)\mathbf{e}(s), \mathbf{v} \right\rangle .$$

Since $(h_v^T)'(s) = (h_v^T)''(s) = (h_v^T)^{(3)}(s) = 0$, we have $\pm \frac{1}{\sqrt{1-k_g^2(s)}}(-k_g'(s)) = 0$. This is equivalent to the condition $k_g'(s) = 0$. Therefore, we have $\mathbf{v} = \pm \frac{1}{\sqrt{1-k_g^2(s)}} \left(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s)\right)$, $k_g^2(s) < 1$ and $k_g'(s) = 0$.

(4) We can get

$$(h_v^T)^{(4)}(s) = \left\langle 3k_g(s)k_g'(s)\mathbf{t}(s) + (-k_g'(s) + k_g^3(s) + k_g''(s))\mathbf{e}(s) + (1 - k_g^2(s))\boldsymbol{\gamma}(s), \mathbf{v} \right\rangle .$$

Since

$$(h_v^T)'(s) = (h_v^T)''(s) = (h_v^T)^{(3)}(s) = (h_v^T)^{(4)}(s) = 0,$$

we have

$$\pm\frac{1}{\sqrt{1-k_g^2(s)}}\left(-k_g''(s)\right)=0.$$

This is equivalent to the condition $k_g''(s)=0$. Therefore, we have $\mathbf{v}=\pm\frac{1}{\sqrt{1-k_g^2(s)}}(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s))$, $k_g^2(s)<1$ and $k_g'(s)=k_g''(s)=0$.

□

4. De Sitter invariants of spacelike curves

In this section, we study the geometric properties of the caustics of spacelike curves in S_1^2. For any $r\in\mathbb{R}$ and $\mathbf{v}_0\in S_1^2$ or $\mathbf{v}_0\in H_+^2$, we denote

$$PS^1(\mathbf{v}_0,r)=\left\{\mathbf{v}\in S_1^2 \mid \langle\mathbf{v},\mathbf{v}_0\rangle=r\right\}.$$

We call $PS^1(\mathbf{v}_0,r)$ a pseudo-circle in S_1^2 with the center $\mathbf{v}_0$. Then we have the following proposition.

Proposition 4.1. *Let $\boldsymbol{\gamma}:I\to S_1^2$ be a unit speed space curve with $k_g(s)\neq 1$. Then $k_g'(s)\equiv 0$ if and only if $\mathbf{v}_0=\pm\frac{1}{\sqrt{|k_g^2(s)-1|}}(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s))$ are constant vectors. Under this condition, $\boldsymbol{\gamma}$ is a part of a pseudo-circle in S_1^2 whose center is $\mathbf{v}_0$.*

Proof. We denote

$$P_\pm(s)=\pm\mathbf{v}_0=\pm\frac{1}{\sqrt{|\ k_g^2(s)-1\ |}}\left(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s)\right).$$

Then we have

$$\begin{aligned}
P_\pm'(s)&=\pm\frac{k_gk_g'}{|\ k_g^2(s)-1\ |^{\frac{3}{2}}}\left(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s)\right)\pm\frac{1}{|\ k_g^2(s)-1\ |}\left(-k_g'(s)\boldsymbol{\gamma}(s)-k_g(s)\boldsymbol{\gamma}'(s)+\mathbf{e}(s)\right)\\
&=\pm\frac{k_gk_g'}{|\ k_g^2(s)-1\ |^{\frac{3}{2}}}\left(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s)\right)\pm\frac{k_g^2(s)-1}{(k_g^2(s)-1)^{\frac{3}{2}}}\left(-k_g'(s)\boldsymbol{\gamma}(s)-k_g(s)\mathbf{t}(s)+k_g(s)\mathbf{t}(s)\right)\\
&=\pm\frac{k_g'}{|\ k_g^2(s)-1\ |^{\frac{3}{2}}}\boldsymbol{\gamma}(s)\pm\frac{k_g(s)k_g'(s)}{|\ k_g^2(s)-1\ |^{\frac{3}{2}}}\mathbf{e}(s).
\end{aligned}$$

Then $P_\pm'(s)\equiv 0$ if and only if $k_g'(s)\equiv 0$. Under this condition, we put $r=\pm\frac{k_g(s)}{\sqrt{|k_g^2(s)-1|}}$ and

$$\mathbf{v}_0=\pm\frac{1}{\sqrt{|\ k_g^2(s)-1\ |}}\left(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s)\right).$$

Then it is easy to show that $\boldsymbol{\gamma}(s)$ is a part of the pseudo-circle $PS^1(\mathbf{v}_0,r)$. □

Let $\boldsymbol{\gamma}:I\to S_1^2$ be a unit speed space curve with $k_g(s)\neq 1$. Then, for any $s_0\in I$, we consider the pseudo-circle $PS^1(\mathbf{v}_0,r_0^\pm)$, where $\mathbf{v}_0=SE\boldsymbol{\gamma}(s_0)$ and $r_0=-\frac{k_g(s_0)}{\sqrt{|k_g^2(s_0)-1|}}$. Then we have the following proposition.

Proposition 4.2. *Under the above notations, $\boldsymbol{\gamma}(s)$ and $PS^1(\mathbf{v}_0,r_0)$ have at least a 3-point contact at $\boldsymbol{\gamma}(s_0)$.*

Proof. We assume that $PS^1(\mathbf{v}_0,r_0)\subset S_1^2$. In this case, we consider the de Sitter spacelike height function H^S. By definition, we have $PS^1(\mathbf{v}_0,r_0)=(h^S_{\mathbf{v_0}})^{-1}(r_0)$. Proposition 3.1 (A) (2) means that $\boldsymbol{\gamma}$ and $PS^1(\mathbf{v}_0,r_0)$ have at least a 3-point contact at $\boldsymbol{\gamma}(s_0)$. If $PS^1(\mathbf{v}_0,r_0)\subset H_+^2$, we adopt the hyperbolic spacelike height function H^T, and the assertion follows from exactly the same arguments as those of the previous case. □

We call $PS^1(\mathbf{v}_0, r_0)$ in Proposition 4.2 the osculating pseudo-circle (or the pseudo-circle of geodesic curvature); its center $\mathbf{v}_0$ is called the center of geodesic curvature. So the caustic is the locus of the center of geodesic curvature. Moreover, we have the following Corollary of Propositions 3.1 and 4.2.

Corollary 4.3. *The osculating pseudo-circle and $\boldsymbol{\gamma}$ have a 4-point contact at $\boldsymbol{\gamma}(s_0)$ if and only if $k_g'(s_0) = 0$ and $k_g''(s_0) \neq 0$.*

5. Cusps of caustics of spacelike curves in de sitter 2-space

In this section we use some general results on the singularity theory for families of function germs to classify the singularities of the caustics. Detailed descriptions can be found in the book [2]. Let function germ $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$, where $f(s) = F(s, \mathbf{x}_0)$. We say that f has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let $F(s, \mathbf{x})$ be an unfolding of $f(s)$ and $f(s)$ has A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}(s, \mathbf{x})$ at s_0 by $j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s_0, \mathbf{x})\right)(s_0) = \sum_{j=1}^{k-1} a_{ji} s^j$ for $i = 1, \cdots, r$. Then $F(s, \mathbf{x})$ is called a (p)-versal unfolding, if the $(k-1) \times r$ matrix of coefficients (a_{ji}) has rank $k-1$ $(k-1 \leq r)$. The bifurcation set of F is defined by

$$\mathfrak{B}_F = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \frac{\partial F}{\partial s}(s, \mathbf{x}) = \frac{\partial^2 F}{\partial s^2}(s, \mathbf{x}) = 0 \right\}.$$

Then we have the following well-known result [2].

Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0. Suppose that F is a (p)-versal unfolding, then we have the following:*

(a) *If $k = 2$, then $\mathfrak{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*

(b) *If $k = 3$, then $\mathfrak{B}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

Where the ordinary cusp is $C = \left\{(x_1, x_2) \mid x_1^2 = x_2^3\right\}$.

We consider that $H^S(s, \mathbf{v})$ (resp., $H^T(s, \mathbf{v})$) is a unfolding of $h_{v_0}^S(s)$ (resp., $h_{v_0}^T(s)$). Then we have the following proposition.

Proposition 5.2. *Let $\boldsymbol{\gamma} : I \to S_1^2$ be a unit speed space curve with $k_g(s_0) \neq 1$.*

(1) *If $h_{v_0}^S(s)$ has A_3-singularity at s_0, then H^S is a (p)-versal unfolding of $h_{v_0}^S(s)$.*

(2) *If $h_{v_0}^T(s)$ has A_3-singularity at s_0, then H^T is a (p)-versal unfolding of $h_{v_0}^T(s)$.*

Proof. **(1)** We denote that

$$\boldsymbol{\gamma}(s) = (x_1(s), x_2(s), x_3(s)) \quad \text{and} \quad \mathbf{v} = \left(v_1, v_2, \pm\sqrt{1 + v_1^2 - v_2^2}\right).$$

Under this notation, we have

$$H^S(s, \mathbf{v}) = -x_1(s)v_1 + x_2(s)v_2 \pm x_3(s)\sqrt{1 + v_1^2 - v_2^2}.$$

Thus we have

$$\begin{aligned} \frac{\partial H^S}{\partial v_1} &= -x_1(s) \pm \frac{v_1 x_3(s)}{\sqrt{1 + v_1^2 - v_2^2}}, \\ \frac{\partial H^S}{\partial v_2} &= x_2(s) \pm \frac{v_2 x_3(s)}{\sqrt{1 + v_1^2 - v_2^2}}. \end{aligned}$$

We also have

$$\frac{\partial}{\partial s}\frac{\partial H^S}{\partial v_1}=-x_1'(s)\pm\frac{v_1x_3'(s)}{\sqrt{1+v_1^2-v_2^2}},$$
$$\frac{\partial}{\partial s}\frac{\partial H^S}{\partial v_2}=x_2'(s)\pm\frac{v_2x_3'(s)}{\sqrt{1+v_1^2-v_2^2}},$$

and

$$\frac{\partial^2}{\partial s^2}\frac{\partial H^S}{\partial v_1}=-x_1''(s)\pm\frac{v_1x_3''(s)}{\sqrt{1+v_1^2-v_2^2}},$$
$$\frac{\partial^2}{\partial s^2}\frac{\partial H^S}{\partial v_2}=x_2''(s)\pm\frac{v_1x_3''(s)}{\sqrt{1+v_1^2-v_2^2}}.$$

Therefore the 2-jet of $\frac{\partial H^S}{\partial v_i}(s,v)(i=1,2)$at s_0 is given by

$$\begin{aligned}j^2\left(\frac{\partial H^S}{\partial v_i}(s,\mathbf{v}_0)\right)(s_0)&=\frac{\partial}{\partial s}\frac{\partial H^S}{\partial v_i}(s-s_0)+\frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H^S}{\partial v_i}(s-s_0)^2\\&=a_{1i}(s-s_0)+\frac{1}{2}a_{2i}(s-s_0)^2.\end{aligned}$$

It is enough to show that the rank of the matrix A is 2, where

$$A=\begin{pmatrix}-x_1'(s)\pm\frac{v_1x_3'(s)}{\sqrt{1+v_1^2-v_2^2}} & x_2'(s)\pm\frac{v_2x_3'(s)}{\sqrt{1+v_1^2-v_2^2}}\\ -x_1''(s)\pm\frac{v_1x_3''(s)}{\sqrt{1+v_1^2-v_2^2}} & x_2''(s)\pm\frac{v_2x_3''(s)}{\sqrt{1+v_1^2-v_2^2}}\end{pmatrix}.$$

We have

$$\begin{aligned}detA&=\pm\frac{v_1}{v_3}(x_3''x_2'-x_3'x_2'')\pm\frac{v_2}{v_3}(x_1''x_3'-x_1'x_3'')+(x_2''x_1'-x_2'x_1'')\\&=\pm\frac{1}{v_3}\left\langle\mathbf{v_0},\mathbf{t}(s)\wedge\mathbf{t}'(s)\right\rangle\\&=\pm\frac{1}{v_3}\left\langle\pm\frac{1}{\sqrt{k_g^2(s)-1}}(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s)),\mathbf{t}\wedge(-\boldsymbol{\gamma}(s)+k_g(s)\mathbf{e}(s))\right\rangle\\&=\pm\frac{1}{v_3}\frac{1}{\sqrt{k_g^2(s)-1}}\left\langle-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s),k_g(s)\boldsymbol{\gamma}(s)-\mathbf{e}(s)\right\rangle\\&=\pm\frac{1}{v_3}\frac{1}{\sqrt{k_g^2(s)-1}}(k_g^2(s)-1)\\&=\pm\frac{\sqrt{k_g^2(s)-1}}{v_3}\neq 0.\end{aligned}$$

(2) For H^T, we denote that

$$\boldsymbol{\gamma}(s)=(x_1(s),x_2(s),x_3(s))\text{ and }\mathbf{v}=\left(v_1,v_2,\pm\sqrt{-1+v_1^2-v_2^2}\right).$$

Under this notation we have

$$H^T(s,\mathbf{v})=-x_1(s)v_1+x_2(s)v_2\pm x_3(s)\sqrt{-1+v_1^2-v_2^2}.$$

By the same computation as above , we have the 2-jet of $\frac{\partial H^T}{\partial v_i}(s,v)(i=1,2)$at s_0 is given by

$$\begin{aligned} j^2\left(\frac{\partial H^T}{\partial v_i}(s,v_0)\right)(s_0) =& \frac{\partial}{\partial s}\frac{\partial H^T}{\partial v_i}(s-s_0)+\frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H^T}{\partial v_i}(s-s_0)^2 \\ =& b_{1i}(s-s_0)+\frac{1}{2}b_{2i}(s-s_0)^2. \end{aligned}$$

It is enough to show that the rank of the matrix B is 2, where

$$B=\begin{pmatrix} -x_1'(s)\pm\dfrac{v_1x_3'(s)}{\sqrt{-1+v_1^2-v_2^2}} & x_2'(s)\pm\dfrac{v_2x_3'(s)}{\sqrt{-1+v_1^2-v_2^2}} \\ -x_1''(s)\pm\dfrac{v_1x_3''(s)}{\sqrt{-1+v_1^2-v_2^2}} & x_2''(s)\pm\dfrac{v_2x_3''(s)}{\sqrt{-1+v_1^2-v_2^2}} \end{pmatrix}.$$

We have

$$\begin{aligned} detB &= \pm\frac{v_1}{v_3}(x_3''x_2'-x_3'x_2'')\pm\frac{v_2}{v_3}(x_1''x_3'-x_1'x_3'')+(x_2''x_1'-x_2'x_1'') \\ &= \pm\frac{1}{v_3}\left\langle \mathbf{v_0},\mathbf{t}(s)\wedge\mathbf{t}'(s)\right\rangle \\ &= \pm\frac{1}{v_3}\left\langle \pm\frac{1}{\sqrt{1-k_g^2(s)}}(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s)),\mathbf{t}(s)\wedge(-\boldsymbol{\gamma}(s)+k_g(s)\mathbf{e}(s))\right\rangle \\ &= \pm\frac{1}{v_3}\frac{1}{\sqrt{1-k_g^2(s)}}\left\langle -k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s),k_g(s)\boldsymbol{\gamma}(s)-\mathbf{e}(s)\right\rangle \\ &= \pm\frac{1}{v_3}\frac{1}{\sqrt{1-k_g^2(s)}}(k_g^2(s)-1) \\ &= \pm\frac{\sqrt{1-k_g^2(s)}}{v_3}\neq 0. \end{aligned}$$

This completes the proof. □

Theorem 5.3. *Let $\boldsymbol{\gamma}:I\to S_1^2$ be a unit speed space curve with $k_g(s_0)\neq 1$, then we have the following claims.*

(1) *The caustic at $SE_\gamma(s_0)$ is regular if $k_g'(s_0)\neq 0$.*

(2) *The caustic at $SE_\gamma(s_0)$ is locally diffeomorphic to the ordinary cusp if $k_g'(s_0)=0$ and $k_g''(s_0)\neq 0$.*

Proof. For the proof of assertion (1), we can calculate the derivative of the caustics as follow:

$$SE_\gamma'(s)=\pm\frac{k_g'(s)}{(k_g^2(s)-1)^{\frac{3}{2}}}\left(\boldsymbol{\gamma}(s)-k_g(s)\mathbf{e}(s)\right).$$

Therefore assertion (1) follows.

For the proof of assertions (2), we consider the de Sitter spacelike height functions H^S on the curves. By Proposition 3.1, the bifurcation set of H^S is

$$\mathfrak{B}_H^S=\left\{\mathbf{v}=\pm\frac{1}{\sqrt{k_g^2(s)-1}}(-k_g(s)\boldsymbol{\gamma}(s)+\mathbf{e}(s))\mid s\in I\right\}$$

and $k_g^2(s)>1$.

The caustic of $\boldsymbol{\gamma}$ in S_1^2 is a part of this set. By Theorem 5.1 and Proposition 5.2, the bifurcation set $\mathfrak{B}_H^S$ at $\mathbf{v}_0 = \frac{1}{\sqrt{k_g^2(s)-1}}(-k_g(s)\boldsymbol{\gamma}(s) + \mathbf{e}(s))$ is locally diffeomorphic to the ordinary cusp if $k_g'(s_0) = 0$ and $k_g''(s_0) \neq 0$. If we consider the hyperbolic timelike height function H^T, we can prove the remaining assertions of the theorem. □

Corollary 4.3 and Theorem 5.3 assert that the cusp point of the caustic corresponds to the point $\boldsymbol{\gamma}(s_0)$ where the osculating pseudo-circle and $\boldsymbol{\gamma}$ have a 4-point contact. Such a point is called the ordinary vertex of the curve $\boldsymbol{\gamma}$. We also call the point $\boldsymbol{\gamma}(s_0)$ with $k_g'(s_0) = k_g''(s_0) = 0$ the heigher vertex of the curve $\boldsymbol{\gamma}$.

6. Generic properties of spacelike curves

In this section we consider generic properties of spacelike curves in S_1^2. Let $Emb_{sp}(I, S_1^2)$ be the space of spacelike embeddings $\boldsymbol{\gamma} : I \to S_1^2$ with $\langle \mathbf{t}', \mathbf{t}' \rangle \neq 1$ equipped with Whitney C^∞-topology. We also consider the function $\mathcal{H} : S_1^2 \times S_1^2 \to \mathbb{R}$ defined by $\mathcal{H}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$. We claim that $\mathcal{H}_v$ is a submersion for any $\mathbf{v} \in S_1^2$, where $\mathcal{H}_v(\mathbf{u}) = \mathcal{H}(\mathbf{u}, \mathbf{v})$. For any $\boldsymbol{\gamma} \in Emb_{sp}(I, S_1^2)$, we have $H = \mathcal{H} \circ (\boldsymbol{\gamma} \times id_{S_1^2})$. We also have the ℓ-jet extension $j_1^\ell H : I \times S_1^2 \to J^\ell(I, \mathbb{R})$ defined by $j_1^\ell H(s, \mathbf{v}) = j^\ell h_v(s)$. We consider the trivialization $J^\ell(I, \mathbb{R}) \equiv I \times \mathbb{R} \times J^\ell(1, 1)$. For any submanifold $Q \subset J^\ell(1, 1)$, we denote that $\widetilde{Q} = I \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [9].

Proposition 6.1. *Let Q be a submanifold of $J^\ell(1, 1)$. Then the set*

$$T_Q = \{\boldsymbol{\gamma} \in Emb_{sp}(I, S_1^2) \mid j^\ell H \textit{is transversal to } Q\}$$

is a residual subset of $Emb_{sp}(I, S_1^2)$. If Q is a closed subset, then T_Q is open.

We can prove the following generic classification theorem.

Theorem 6.2. *There exists an open and dense subset $\mathcal{O} \subset Emb_{sp}(I, S_1^2)$ such that for any $\boldsymbol{\gamma} \in \mathcal{O}$, the caustics $SE_\gamma(s)$ of $\gamma(s)$ is locally diffeomorphic to the ordinariy cusp at any singular point.*

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