Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4355–4363
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# A numerical approximation with IP/SUPG algorithm for P-T-T viscoelastic flows

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Communicated by Y. Yao

**Abstract**

A numerical approximation for Phan-Thien-Tanner(P-T-T) viscoelastic flow problems has investigated. The approximation is proposed by an interior penalty(IP) method and a Streamline Upwind Petrov-Galerkin(SUPG) method. Meanwhile, the error estimates for the above numerical approximation of the P-T-T model is derived. The numerical results support the efficiency of the algorithm. 

*Keywords:* Viscoelastic flows, P-T-T model, finite element method, stokes, constitutive equation.
*2010 MSC:* 65M60, 76A10, 76M10.

## 1. Introduction and Preliminaries

The investigation of the nonlinear material in the viscoelastic flow problems has practical significance in both engineering and medical fields, such as polymer processes, artificial organs, etc. Due to the complex material character of the fluid, numerical simulation of the impact viscoelastic flow problems is a difficult and expensive task. Some useful progress, such as [4, 11], has been made in the past decades. They give mathematical and engineering perspectives on the viscoelastic flows. The fluid properties can be characterized by modern technology, such as constitutive equations. However, the complex theological responses of fluid and the elastic effect under high Weissenberg number make the numerical simulation of the viscoelastic flows become a difficult task, see [9, 10, 13].

Many constitutive models have become available in recent years that are able to describe the dominant convective behavior and the nonlinear coupling increases. In this paper we adopt the Phan-Thien-Tanner

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*Received 2016-04-01*

(P-T-T) equation, which is the differential type constitutive equation, to calculate the viscoelastic flows. To date, many numerical methods have been developed and adopted for the simulation of the nonlinear material properties, like the finite difference method (FDM), the finite volume method(FVD) and the finite element method (FEM). As we know, to improve the stability and efficiency of numerical simulation, many numerical schemes have been established and adopted. In [5], the discrete elastic viscous split stress algorithm is proposed for improving the stability of simulation by improving the ellipticity of the momentum equation. Moreover, the elastic viscous split stress scheme, the adaptive viscoelastic stress split scheme and the discrete adaptive viscoelastic stress split algorithm also can be used to stabilize the calculation program, see [14]. A streamline upwind Petrov-Galerkin (SUPG) method was introduced as a discretization method by Baranger and Sandri [1] for viscoelastic flows. Najib and Sandri [12] studied a numerical method for oldroyd-B fluid. The main idea of the method is to decouple the oldroyd-B model into two equation systems. Bonite et al. [2] presented a face interior penalty finite element method for solving stokes equations. Hou [7, 8] considered some physical applications of P-T-T model.

In this paper, we decouple the P-T-T model into two parts: the stokes-like problem and the constitutive equation. In details, the stokes-like problem is computed by IP method, and the constitutive equation is calculated by SUPG method. Moreover, we shall obtain error estimates of an IP/SUPG finite element method for the P-T-T model.

The paper is organized as follows. In Section 2, we introduce the P-T-T model and the mathematical notation. Section 3 displays an IP/SUPG method and the discrete approximation. In Section 4, error estimates of the IP/SUPG method for P-T-T model are presented. Section 5 is the numerical results.

## 2. P-T-T model and mathematical notation

Let us first introduce some notation. For a bounded domain $\Omega$ in $\mathbb{R}^2$, with boundary $\partial\Omega$. We consider viscoelastic flow governed by P-T-T model:

$$\begin{cases} -\nabla\cdot(2\eta_p\mathbf{D}(\boldsymbol{u})+\boldsymbol{\tau})+\nabla p=\boldsymbol{f}, & \text{in } \Omega\\ \nabla\cdot\boldsymbol{u}=0, & \text{in } \Omega\\ \left(1+\dfrac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau})\right)\boldsymbol{\tau}+\lambda((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau}-\boldsymbol{\tau}\nabla\boldsymbol{u}-\nabla\boldsymbol{u}^T\boldsymbol{\tau})=2(1-\eta_p)\mathbf{D}(\boldsymbol{u}), & \text{in } \Omega\\ \boldsymbol{u}=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\boldsymbol{u}$, $p$ and $\boldsymbol{\tau}$ denote the velocity, viscoelastic stress tensor and pressure fields, respectively, $\varepsilon$, $\lambda$ and $\eta_p$ represent dimensionless material constant, Weissenberg number and viscosity constant, respectively. $\mathbf{D}(\boldsymbol{u})$ denote the rate of deformation tensor and $\mathbf{D}(\boldsymbol{u})=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$.

Throughout the paper, We denote by$\|\cdot\|_s$ and $(\cdot,\cdot)_s$ the norm and inner product on the Sobolev spaces $\mathrm{H}^s(\Omega), \mathrm{s}\geq 0$. In what follows, velocity $\boldsymbol{u}$, pressure $p$ and viscoelastic stress tensor $\boldsymbol{\tau}$ belong to their respective spaces $V$, $Q$ and $S$ given by

$$\begin{aligned} &V=\{\boldsymbol{u}\in H^1(\Omega); u=0, \text{on } \partial\Omega\},\\ &Q=\{p\in \mathrm{L}^2(\Omega), \int_\Omega q\mathrm{d}x=0\},\\ &S=\{\boldsymbol{T}\in \mathrm{L}^2(\Omega), \boldsymbol{T}=(T_{ij}), T_{ij}=T_{ji}, i,j=1,2\} \end{aligned}$$

and let $X=V\times Q\times S$.

Let $\Gamma_h=\{K\}$ denotes a partition of $\Omega$ and $K$ can be a triangle or a quadrilateral in two dimensions. The parameter $h$ denotes the mesh size of $\Gamma_h$ given by $h=\max_{K\in\Gamma_h} h_K$, where $h_K$ is the diameter of $K$.

We shall use the following finite element space:

$$V_h=\{\boldsymbol{v}\in H^1(\Omega):\ \boldsymbol{v}|_k\in p^l(K)^2\ \forall K\in\Gamma_h\},$$

$$Q_h = \{q \in L_0^2(\Omega) : \ q|_k \in p^l(K) \ \forall K \in \Gamma_h\},$$
$$S_h = \{\boldsymbol{\sigma} \in H^1(\Omega) : \ \boldsymbol{\sigma}|_k \in p^l(K)^4 \ \forall K \in \Gamma_h\},$$

here $p^l(K)$ denotes the space of polynomials of total degree at most $l$ on $K$, $l \geq 1$. Let $X_h = V_h \times Q_h \times S_h$.

## 3. Formulation of finite element method and error bounds

*3.1. IP method and error estimate*

We consider the stokes-like problem

$$\begin{cases} -\nabla \cdot \boldsymbol{T} + \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{T} = 2\eta_p \mathbf{D}(\boldsymbol{u}) + \boldsymbol{\tau}, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\boldsymbol{T}$ denotes extra stress tensor.

We define

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \mathrm{d}x,$$

and

$$< \boldsymbol{u}, \boldsymbol{v} >_\Gamma = \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v} \mathrm{d}s.$$

As for the stokes-like equation (3.1), we define the bilinear forms by

$$a_h(\boldsymbol{T}_h, \boldsymbol{v}_h) = (\boldsymbol{T}_h, \mathbf{D}(\boldsymbol{v}_h)) - \langle \boldsymbol{T}_h \cdot \boldsymbol{n}, \boldsymbol{v}_h \rangle_{\partial\Omega}, \tag{3.2}$$

$$b_h(p_h, \boldsymbol{v}_h) = -(p_h, \nabla \cdot \boldsymbol{v}_h) + \langle p_h, \boldsymbol{v}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega} \tag{3.3}$$

and the jump operators

$$j_1(\boldsymbol{u}_h, \boldsymbol{v}_h) = 2\eta_p \sum_{K \in \varepsilon_h^0} h \int_{\partial K} [\nabla \boldsymbol{u}_h] \cdot [\nabla \boldsymbol{v}_h] \mathrm{d}s + \frac{\eta_p \beta}{h} \int_{\partial\Omega} \boldsymbol{u}_h \cdot \boldsymbol{v}_h \mathrm{d}s, \tag{3.4}$$

$$j_2(p_h, q_h) = \sum_{K \in \varepsilon_h^0} \frac{\gamma h^3}{\eta_p} \int_{\partial K} [\nabla p_h][\nabla q_h] \mathrm{d}s, \tag{3.5}$$

where $\alpha, \beta, \gamma$ are positive constants. $[\cdot]$ is the interior penalty term.

Then, the IP method for solving the stokes-like problem (3.1) is to find $(\boldsymbol{u}_h, p_h, \boldsymbol{T}_h) \in X_h$ such that

$$\begin{aligned} &a_h(\boldsymbol{T}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u}_h) - a_h(\boldsymbol{s}_h, \boldsymbol{u}_h) + \left(\frac{1}{2\eta_p}\boldsymbol{T}_h, \boldsymbol{s}_h\right) + j_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + j_2(p_h, q_h) \\ &\quad = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{\tau}, \boldsymbol{s}_h) \end{aligned} \tag{3.6}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{s}_h) \in X_h$.

Furthermore, we state an approximation property.

**Lemma 3.1.** *Let $(\boldsymbol{u}, p, \boldsymbol{T})$ be the exact solution of* (3.1)*, and let $(\boldsymbol{u}_h, p_h, \boldsymbol{T}_h) \in X_h$ be the numerical solution of* (3.6)*. Then*

$$\begin{aligned} &a_h(\boldsymbol{T} - \boldsymbol{T}_h, \boldsymbol{v}_h) + b_h(p - p_h, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u} - \boldsymbol{u}_h) - a_h(\boldsymbol{s}_h, \boldsymbol{u} - \boldsymbol{u}_h) \\ &\quad + \left(\frac{1}{2\eta_p}(\boldsymbol{T} - \boldsymbol{T}_h), \boldsymbol{s}_h\right) + j_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + j_2(p - p_h, q_h) = 0 \quad \forall \ N_h \in X_h. \end{aligned} \tag{3.7}$$

Then, we need to define the triple norm

$$||| \boldsymbol{u}, p, \boldsymbol{T} |||^2 = \frac{1}{2\eta_p} \parallel \boldsymbol{T} \parallel^2_{0,\Omega} + 2\eta_p \parallel \mathbf{D}(\boldsymbol{u}) \parallel^2_{0,\Omega} + \frac{1}{2\eta_p} \parallel p \parallel^2_{0,\Omega} \tag{3.8}$$

and the discrete triple norm

$$||| \boldsymbol{u}_h, p_h, \boldsymbol{T}_h |||^2_h = ||| \boldsymbol{u}, p, \boldsymbol{T} |||^2 + j_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + j_2(p - p_h, q_h), \tag{3.9}$$

where $(\boldsymbol{u}, p, \boldsymbol{T}) \in X$.

As in [3], we can obtain the optimal convergence rate in the triple norm if the exact solution $\boldsymbol{u}, p, \boldsymbol{T}$ satisfies the assumptions stated in the following theorem

**Theorem 3.2.** *Suppose that the mesh satisfies the quasiuniformity of the mesh and that $\boldsymbol{u}, p, \boldsymbol{T}$ be the solution of* (3.1)*, then the solution $\boldsymbol{u}_h, p_h, \boldsymbol{T}_h$ by the interior penalty method satisfies the error estimate*

$$||| (\boldsymbol{u}, p, \boldsymbol{T}) - (\boldsymbol{u}_h, p_h, \boldsymbol{T}_h) ||| \leq C \parallel \boldsymbol{\tau} - \boldsymbol{\tau}_h \parallel + o(h), \tag{3.10}$$

*where $C$ is a constant independent of $h$.*

*3.2. SUPG method and error analysis*

we shall present a SUPG algorithm to solve the equation

$$\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau})\right)\boldsymbol{\tau} + \lambda((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau} - \boldsymbol{\tau}\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^T\boldsymbol{\tau}) = 2(1-\eta_p)\mathbf{D}(\boldsymbol{u}), \text{in } \Omega. \tag{3.11}$$

The SUPG method is given as follows. An operator B is defined by

$$B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\tau}, \boldsymbol{\omega}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau}, \boldsymbol{\omega} + h(\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}) + \frac{1}{2}((\nabla\cdot\boldsymbol{u})\boldsymbol{\tau}, \boldsymbol{\omega}) \tag{3.12}$$

for all $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\tau}, \boldsymbol{\omega}) \in V_h \times V_h \times S_h \times S_h$.

Moreover, setting $\boldsymbol{u} = \boldsymbol{v}, \boldsymbol{\tau} = \boldsymbol{\omega}$, we have

$$B(\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{\tau}) = h((\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau}, (\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau}) = h \parallel (\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau} \parallel^2 . \tag{3.13}$$

For $\boldsymbol{\omega}_u = \boldsymbol{\omega} + \upsilon h\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}$, we obtain

$$B(\lambda\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{\omega}) = ((\lambda\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau}, \boldsymbol{\omega} + \upsilon h(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}) + \frac{1}{2}((\nabla\cdot\lambda\boldsymbol{u})\boldsymbol{\tau}, \boldsymbol{\omega}). \tag{3.14}$$

Taking the inner product of (3.11) with a test function $\boldsymbol{\omega}_u$, we have

$$\begin{aligned} &\left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau})\right)\boldsymbol{\tau}, \boldsymbol{\omega}_u\right) + B(\lambda\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{\omega}) - \lambda((\boldsymbol{\tau}\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\boldsymbol{\tau}), \boldsymbol{\omega}_u) \\ &\quad = 2(1-\eta_p)(\mathbf{D}(\boldsymbol{u}), \boldsymbol{\omega}_u) \quad \forall\boldsymbol{\omega} \in S. \end{aligned} \tag{3.15}$$

Now we define the discrete approximation of (3.15) as, find $\boldsymbol{\tau} \in S_h$ such that

$$\begin{aligned} &\left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau}_h)\right)\boldsymbol{\tau}_h, \boldsymbol{\omega}_{u_h}\right) + B(\lambda\boldsymbol{u}_h, \boldsymbol{\tau}_h, \boldsymbol{\omega}) - \lambda((\boldsymbol{\tau}_h\nabla\boldsymbol{u}_h + \nabla\boldsymbol{u}_h^T\boldsymbol{\tau}_h), \boldsymbol{\omega}_{u_h}) \\ &\quad = 2(1-\eta_p)(\mathbf{D}(\boldsymbol{u}_h), \boldsymbol{\omega}_{u_h}) \quad \forall\boldsymbol{\omega} \in S_h. \end{aligned} \tag{3.16}$$

In order to consider the error estimate of the finite element solution related to the IP formulation (3.6), we proved the following result.

**Theorem 3.3.** *Assume that $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_h$ be the solutions of* (3.11) *and* (3.16), *respectively. Then the following inequality holds*

$$|| \boldsymbol{\tau} - \boldsymbol{\tau}_h || \leq Ch^{3/2} + (2(1-\eta_p) + C\lambda L + C\lambda M h) || \boldsymbol{u} - \boldsymbol{u}_h ||_1 /(7/8 - 2\lambda M - C\rho L) \tag{3.17}$$

*for sufficiently small $\lambda > 0$ and $\rho = \frac{\varepsilon\lambda}{1-\eta_p}$. $C$ is a positive constant independent of $h$,*

*Proof.* Let $\tilde{\boldsymbol{\tau}}$ be the $L^2$ projection of $\boldsymbol{\tau}$ in $S_h$. We have

$$(\boldsymbol{\tau} - \tilde{\boldsymbol{\tau}}, \boldsymbol{\omega}) = 0 \quad \forall \boldsymbol{\omega} \in S_h. \tag{3.18}$$

We apply the error estimate in [12] for $\boldsymbol{\tau} \in H^2(\Omega)$

$$|| \boldsymbol{\tau} - \tilde{\boldsymbol{\tau}} || + h || \boldsymbol{\tau} - \tilde{\boldsymbol{\tau}} ||_1 \leq Ch^2 || \boldsymbol{\tau} ||_2, \tag{3.19}$$

$$|| \boldsymbol{\tau} - \tilde{\boldsymbol{\tau}} ||_{0,\Gamma_h} \leq Ch^{3/2} || \boldsymbol{\tau} ||_2 . \tag{3.20}$$

The standard weak formulation of the constitutive equation in $S$ is given by

$$\begin{aligned} &\left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau})\right)\boldsymbol{\tau}, \boldsymbol{\omega}_{u_h}\right) + B(\lambda\boldsymbol{u}, \lambda\boldsymbol{u}_h, \boldsymbol{\tau}_h, \boldsymbol{\omega}) - \lambda((\boldsymbol{\tau}\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\boldsymbol{\tau}), \boldsymbol{\omega}_{u_h}) \\ &\quad = 2(1-\eta_p)(\mathbf{D}(\boldsymbol{u}), \boldsymbol{\omega}_{u_h}) \quad \forall \boldsymbol{\omega} \in S. \end{aligned} \tag{3.21}$$

Subtracting (3.21) from (3.16), and inserting the tensor $\tilde{\boldsymbol{\tau}}$ and setting $\boldsymbol{\omega} = \boldsymbol{\sigma} = \boldsymbol{\tau}_h - \tilde{\boldsymbol{\tau}}$, we obtain

$$\begin{aligned} &\left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau}_h)\right)\boldsymbol{\tau}_h, \boldsymbol{\sigma}_{u_h}\right) + \frac{\varepsilon\lambda}{\alpha}(\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{\tau}, \boldsymbol{\sigma}_{u_h}) + B(\lambda\boldsymbol{u}_h, \boldsymbol{\sigma}, \boldsymbol{\sigma}) - \lambda((\boldsymbol{\sigma}\nabla\boldsymbol{u}_h + \nabla\boldsymbol{u}_h^T\boldsymbol{\sigma}), \boldsymbol{\sigma}_{u_h}) \\ &= (\boldsymbol{\tau} - \tilde{\boldsymbol{\tau}}, \boldsymbol{\sigma}_{u_h}) - \frac{\varepsilon\lambda}{\alpha}(\mathrm{tr}(\boldsymbol{\tau}_h)(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}), \boldsymbol{\sigma}_{u_h}) - \frac{\varepsilon\lambda}{\alpha}(\mathrm{tr}(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau})\boldsymbol{\tau}, \boldsymbol{\sigma}_{u_h}) \\ &\quad - B(\lambda\boldsymbol{u}_h, \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}, \boldsymbol{\sigma}) - B(\lambda(\boldsymbol{u}_h - \boldsymbol{u}), \lambda\boldsymbol{u}_h, \boldsymbol{\tau}, \boldsymbol{\sigma}) + \lambda(((\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau})\nabla\boldsymbol{u}_h + \nabla\boldsymbol{u}_h^T(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau})), \boldsymbol{\sigma}_{u_h}) \\ &\quad - \lambda((\boldsymbol{\tau}\nabla(\boldsymbol{u} - \boldsymbol{u}_h) + \nabla(\boldsymbol{u} - \boldsymbol{u}_h)^T\boldsymbol{\tau}), \boldsymbol{\sigma}_{u_h}) + 2(1-\eta_p)(\mathbf{D}(\boldsymbol{u}_h - \boldsymbol{u}), \boldsymbol{\sigma}_{u_h}). \end{aligned} \tag{3.22}$$

Assuming that the solution $(\boldsymbol{u}, p, \boldsymbol{T}, \boldsymbol{\tau})$ is smooth enough,

$$L = \max\{|| \boldsymbol{u} ||_3, || p ||_2, || \boldsymbol{T} ||_2, || \boldsymbol{\tau} ||_2\},$$

and

$$\max\{|| \nabla\boldsymbol{u}_h ||_{0,\infty}, || \boldsymbol{\tau}_h ||_{0,\infty}\} \leq M.$$

where $\nabla\boldsymbol{u}_h, \boldsymbol{\tau}_h \in L^\infty(\Omega)$. we estimate the first part on the right-hand side of (3.22) by

$$\begin{aligned} &(\boldsymbol{\tau} - \tilde{\boldsymbol{\tau}}, \boldsymbol{\sigma}_{u_h}) - \frac{\varepsilon\lambda}{\alpha}(\mathrm{tr}(\boldsymbol{\tau}_h)(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}), \boldsymbol{\sigma}_{u_h}) - \frac{\varepsilon\lambda}{\alpha}(\mathrm{tr}(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau})\boldsymbol{\tau}, \boldsymbol{\sigma}_{u_h}) \\ &\quad \leq || \boldsymbol{\tau} - \tilde{\boldsymbol{\tau}} || \, || \tilde{\boldsymbol{\tau}} ||_{u_h} + \frac{2\varepsilon\lambda}{\alpha} || \boldsymbol{\tau}_h ||_{0,\infty} || \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau} || \, || \boldsymbol{\sigma} ||_{u_h} + || \boldsymbol{\tau} ||_{0,\infty} || \mathrm{tr}(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}) || \, || \boldsymbol{\sigma} ||_{u_h} \\ &\quad \leq Ch^2 || \boldsymbol{\tau} ||_2 || \boldsymbol{\sigma} ||_{u_h} + CLMh^2 || \boldsymbol{\sigma} ||_{u_h} + \sqrt{2}C || \boldsymbol{\tau} ||_2 || \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau} || \, || \boldsymbol{\sigma} ||_{u_h} \\ &\quad \leq CLh^2(1 + M + L) || \boldsymbol{\sigma} ||_{u_h} . \end{aligned} \tag{3.23}$$

For the first $B$ term on the right-hand side of (3.22), using (3.14),

$$\begin{aligned} &B(\lambda\boldsymbol{u}_h, \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}, \boldsymbol{\sigma}) \\ &\quad = -((\lambda\boldsymbol{u}_h \cdot \nabla)\boldsymbol{\sigma}, \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}) - ((\nabla \cdot \lambda\boldsymbol{u}_h)(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}), \boldsymbol{\sigma})/2 + ((\lambda\boldsymbol{u}_h \cdot \nabla)(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}), (\lambda\boldsymbol{u}_h \cdot \nabla)\tilde{\boldsymbol{\tau}}). \end{aligned} \tag{3.24}$$

Note that $\nabla \cdot \boldsymbol{u} = 0$ and using lemma from [15], we obtain an estimate for (3.24)

$$
\begin{aligned}
B(\lambda \boldsymbol{u}_h, \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}, \boldsymbol{\sigma}) \leq & \| \lambda(\boldsymbol{u}_h \cdot \nabla)\boldsymbol{\sigma}) \| \| \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau} \| + \lambda \| \boldsymbol{u}_h - \boldsymbol{u} \|_1 \| \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau} \| \| \boldsymbol{\sigma} \|_{0,\infty} /2 \\
& + \lambda h^{1/2} \| \boldsymbol{u}_h \|_{0,\infty} \| \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau} \|_1 \| \lambda h^{1/2} (\boldsymbol{u}_h \cdot \nabla)\boldsymbol{\sigma} \| \\
\leq & CLh^{3/2} \| \lambda h^{1/2}(\boldsymbol{u}_h \cdot \nabla)\boldsymbol{\sigma} \| + \lambda CLh \| \boldsymbol{u}_h - \boldsymbol{u} \|_1 \| \boldsymbol{\sigma} \| /2 \\
& + CLMh^{3/2} \| \lambda h^{1/2}(\boldsymbol{u}_h \cdot \nabla)\boldsymbol{\sigma} \| \\
\leq & CL((1+\lambda M)h^{3/2} + \lambda h \| \boldsymbol{u}_h - \boldsymbol{u} \|_1 /2) \| \boldsymbol{\sigma} \|_{u_h} .
\end{aligned} \tag{3.25}
$$

In view of expression $B$ in (3.12) and imbedding theorem in [6], we consider the second $B$ term on the right-hand side of (3.22)

$$
\begin{aligned}
B(\lambda(\boldsymbol{u}_h - \boldsymbol{u}), \lambda \boldsymbol{u}_h, \boldsymbol{\tau}, \boldsymbol{\sigma}) &= (\lambda(\boldsymbol{u}_h - \boldsymbol{u}) \cdot \nabla \boldsymbol{\tau}, \boldsymbol{\sigma}_{u_h}) + \lambda(\nabla \cdot (\boldsymbol{u}_h - \boldsymbol{u})\boldsymbol{\tau}, \boldsymbol{\sigma})/2 \\
&\leq C\lambda(\| \boldsymbol{u}_h - \boldsymbol{u} \|_1 \| \boldsymbol{\tau} \|_2 \| \boldsymbol{\sigma} \|_{u_h} + \| \boldsymbol{u}_h - \boldsymbol{u} \|_1 \| \boldsymbol{\tau} \|_2 \| \boldsymbol{\sigma} \|) \\
&\leq C\lambda L \| \boldsymbol{u}_h - \boldsymbol{u} \|_1 \| \boldsymbol{\sigma} \|_{u_h} .
\end{aligned} \tag{3.26}
$$

Using (3.19), we easily see that

$$
\begin{aligned}
&\lambda(((\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau})\nabla \boldsymbol{u}_h + \nabla \boldsymbol{u}_h^T(\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau})), \boldsymbol{\sigma}_{u_h}) - \lambda((\boldsymbol{\tau}\nabla(\boldsymbol{u} - \boldsymbol{u}_h) + \nabla(\boldsymbol{u} - \boldsymbol{u}_h)^T\boldsymbol{\tau}), \boldsymbol{\sigma}_{u_h}) \\
&\quad \leq 2\lambda \| \nabla \boldsymbol{u}_h \|_{0,\infty} \| \tilde{\boldsymbol{\tau}} - \boldsymbol{\tau} \| \| \boldsymbol{\sigma} \|_{u_h} + 2\lambda \| \boldsymbol{u} - \boldsymbol{u}_h \|_1 \| \boldsymbol{\tau} \|_{0,\infty} \| \boldsymbol{\sigma} \|_{u_h} \\
&\quad \leq C\lambda L(Mh^2 + \| \boldsymbol{u} - \boldsymbol{u}_h \|_1) \| \boldsymbol{\sigma} \|_{u_h} .
\end{aligned} \tag{3.27}
$$

For the last term on the right-hand side of (3.22), we obtain

$$
2(1-\eta_p)(\mathbf{D}(\boldsymbol{u}_h - \boldsymbol{u}), \boldsymbol{\sigma}_{u_h}) \leq 2(1-\eta_p) \| \boldsymbol{u} - \boldsymbol{u}_h \|_1 \| \boldsymbol{\sigma} \|_{u_h} . \tag{3.28}
$$

Hence, combining (3.23) and (3.25)-(3.28), we get upper bound of the right-hand side of (3.22)

$$
\begin{aligned}
RH \leq & CL(h^2 + \rho M h^2 + \rho L h^2 + \lambda M h^2 + h^{3/2} + \lambda M h^{3/2}) \| \boldsymbol{\sigma} \|_{u_h} \\
& + (2(1-\eta_p) + C\lambda L + C\lambda M h) \| \boldsymbol{u} - \boldsymbol{u}_h \|_1 \| \boldsymbol{\sigma} \|_{u_h},
\end{aligned} \tag{3.29}
$$

where $\rho = \varepsilon\lambda/1-\eta_p$.

We will derive a lower bound of the left-hand side of (3.22). By using Young's inequality and the equality (3.13), we have

$$
\begin{aligned}
&\left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau}_h)\right)\boldsymbol{\tau}_h, \boldsymbol{\sigma}_{u_h}\right) + B(\lambda \boldsymbol{u}_h, \boldsymbol{\sigma}, \boldsymbol{\sigma}) \\
&= \left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau}_h)\right)\boldsymbol{\tau}_h, \boldsymbol{\sigma}\right) + \left(\left(1 + \frac{\varepsilon\lambda}{1-\eta_p}\mathrm{tr}(\boldsymbol{\tau}_h)\right)\boldsymbol{\tau}_h, \upsilon h(\boldsymbol{\sigma}_{u_h} \cdot \nabla)\boldsymbol{\sigma}\right) + B(\lambda \boldsymbol{u}_h, \boldsymbol{\sigma}, \boldsymbol{\sigma}) \\
&\geq (1 - \frac{2\varepsilon h}{1-\eta_p} \| \boldsymbol{\tau}_h \|_{0,\infty}) \| \boldsymbol{\sigma} \|^2 - h(1 + \frac{2\varepsilon h}{1-\eta_p} \| \boldsymbol{\tau}_h \|_{0,\infty}) \| \boldsymbol{\sigma} \| \| \upsilon(\boldsymbol{\sigma}_{u_h} \cdot \nabla)\boldsymbol{\sigma} \| + B(\lambda \boldsymbol{u}_h, \boldsymbol{\sigma}, \boldsymbol{\sigma}) \\
&\geq 15 \| \boldsymbol{\sigma} \|^2 /16 - h((1 + \frac{2\varepsilon h}{1-\eta_p} \| \boldsymbol{\tau}_h \|_{0,\infty})^2 \| \boldsymbol{\sigma} \|^2 + \| \upsilon(\boldsymbol{\sigma}_{u_h} \cdot \nabla)\boldsymbol{\sigma} \|^2 /4) + B(\lambda \boldsymbol{u}_h, \boldsymbol{\sigma}, \boldsymbol{\sigma}) \\
&\geq 7 \| \boldsymbol{\sigma} \|_{u_h}^2 /8,
\end{aligned} \tag{3.30}
$$

where $h \leq 16/289$ and $\| \boldsymbol{\tau}_h \|_{0,\infty} \leq (1-\eta_p)/32\varepsilon\lambda \leq M$. For the rest of term on the left-hand side of (3.22), we get

$$
\begin{aligned}
&\frac{\varepsilon\lambda}{1-\eta_p}(\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{\tau}, \boldsymbol{\sigma}_{u_h}) - \lambda((\boldsymbol{\sigma}\nabla \boldsymbol{u}_h + \nabla \boldsymbol{u}_h^T \boldsymbol{\sigma}), \boldsymbol{\sigma}_{u_h}) \\
&\quad \leq \frac{\varepsilon\lambda}{1-\eta_p} \| \mathrm{tr}(\boldsymbol{\sigma}) \| \| \boldsymbol{\tau} \|_{0,\infty} \| \boldsymbol{\sigma} \|_{u_h} + 2\lambda \| \nabla \boldsymbol{u}_h \|_{0,\infty} \| \boldsymbol{\sigma} \| \| \boldsymbol{\sigma} \|_{u_h} \\
&\quad \leq (CL\frac{\varepsilon\lambda}{1-\eta_p} + 2\lambda M) \| \boldsymbol{\sigma} \|_{u_h}^2 .
\end{aligned} \tag{3.31}
$$

Using the bounds (3.30) and (3.31), we get a lower bound of the left-side of (3.22)

$$LH \geq (7/8 - 2\lambda M - C\rho L) \mid\mid \boldsymbol{\sigma} \mid\mid_{u_h}^2 . \tag{3.32}$$

Therefore, combining (3.29) and (3.32), we obtain

$$\begin{aligned}(7/8 - 2\lambda M - C\rho L) \mid\mid \boldsymbol{\sigma} \mid\mid_{u_h} &\leq CL(h^2 + \rho M h^2 + \rho L h^2 + \lambda M h^2 + h^{3/2} + \lambda M h^{3/2}) \mid\mid \boldsymbol{\sigma} \mid\mid_{u_h} \\ &+ (2(1-\eta_p) + C\lambda L + C\lambda M h) \mid\mid \boldsymbol{u} - \boldsymbol{u}_h \mid\mid_1 \mid\mid \boldsymbol{\sigma} \mid\mid_{u_h} .\end{aligned} \tag{3.33}$$

Using the triangle inequality, we proved the error estimate (3.17). □

**Theorem 3.4.** *Suppose that the exact solution* $(\boldsymbol{u}, p, \boldsymbol{T}, \boldsymbol{\tau})$ *of* (2.1 *is smooth enough, then the finite element solution* $(\boldsymbol{u}_h, p_h, \boldsymbol{T}_h, \boldsymbol{\tau}_h)$ *by the IP/SUPG method satisfies the following error estimate:*

$$||| \, (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, \boldsymbol{T} - \boldsymbol{T}_h, ) \, ||| + \mid\mid \boldsymbol{\tau} - \boldsymbol{\tau}_h \mid\mid \leq Ch. \tag{3.34}$$

*Proof.* Combining Theorem 3.2 with Theorem 3.3, we obtain the bound (3.34), which completes the proof. □

## 4. Numerical experiments

Here, we consider the creeping flow in a planar channel problem with P-T-T model (2.1). We shall solve the problem using the proposed IP/SUPG method to illustrate the theoretical analysis. Figure 1 shows the test domain and Dirichlet boundary conditions. $u_y$ represents the velocity in $y$-direction.

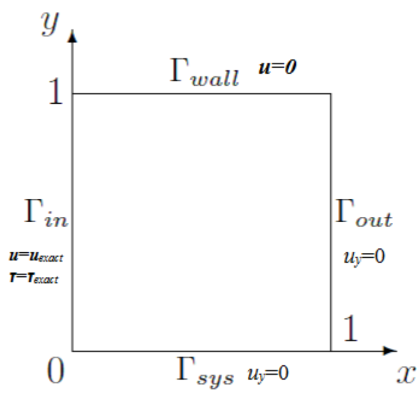


Figure 1: The geometry and boundary conditions.

The exact solution $(\boldsymbol{u}, p, \boldsymbol{\tau})$ is given by

$$\begin{aligned}&\boldsymbol{u} = \begin{pmatrix} 1 - y^4 \\ 0 \end{pmatrix}, \\ &p = -x^2, \\ &\boldsymbol{\tau} = \begin{pmatrix} 32\lambda\alpha y^6 & -4\alpha y^3 \\ -4\alpha y^3 & 0 \end{pmatrix}.\end{aligned} \tag{4.1}$$

Therefore, the right hand side terms of the momentum and constitutive equations modified by substituting (4.1) into (3.1) and (3.11) are given by

$$\boldsymbol{f} = \begin{pmatrix} 12y^2 - 2x \\ 0 \end{pmatrix}, \tag{4.2}$$

$$\boldsymbol{f}_{stress} = \begin{pmatrix} 32\lambda y^6/9 + 1024\varepsilon\lambda^3 y^{12}/9 & 128\lambda^2(1-\varepsilon)y^9/9 \\ 128\lambda^2(1-\varepsilon)y^9/9 & -32\lambda y^6/9 \end{pmatrix}. \tag{4.3}$$

The error is defined as follows

$$\begin{aligned} E_{L_2}(\boldsymbol{u}) &= (\sum_{j=0}^{m} \| \boldsymbol{u}(j) - \boldsymbol{u}_{exact}(j) \|)^{1/2}, \\ E_{L_2}(p) &= (\sum_{j=0}^{m} \| p(j) - p_{exact}(j) \|)^{1/2}, \\ E_{L_2}(\boldsymbol{\tau}) &= (\sum_{j=0}^{m} \| \boldsymbol{\tau}(j) - \boldsymbol{\tau}_{exact}(j) \|)^{1/2}, \\ E_{L_2}(\boldsymbol{T}) &= (\sum_{j=0}^{m} \| \boldsymbol{T}(j) - \boldsymbol{T}_{exact}(j) \|)^{1/2}. \end{aligned} \tag{4.4}$$

The effects of material parameter $\lambda$ on the error estimates of velocity, pressure and stress are displayed in Figure 2 and Figure 3.

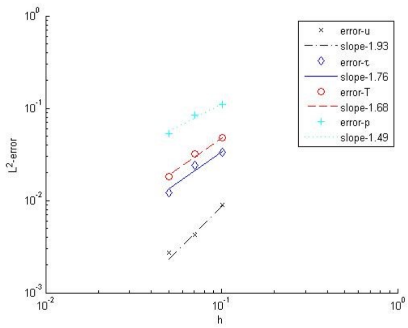


Figure 2: Error estimates in $\boldsymbol{u}$, p, $\boldsymbol{\tau}$ and $\boldsymbol{T}$ with $\lambda = 0.5$ by IP/SUPG algorithm.

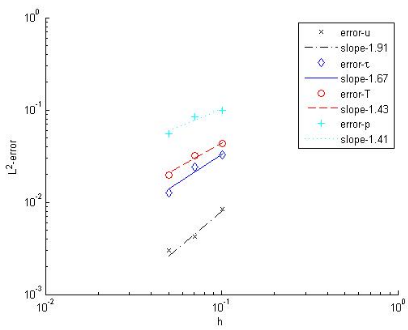


Figure 3: Error estimates in $\boldsymbol{u}$, p, $\boldsymbol{\tau}$ and $\boldsymbol{T}$ with $\lambda = 2.5$ by IP/SUPG algorithm.

## 5. Conclusions

In this paper, we construct the IP/SUPG finite element schemes and present the error estimates for the IP/SUPG algorithm of the P-T-T viscoelastic flow problems. It is proved that the algorithm is stable and convergent.

## Acknowledgment

The first author is supported by the National Natural Science Foundation of China (No. 11271247).

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