Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4364–4370
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

The multi-$\mathscr{F}$-sensitivity and $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity for product systems

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Communicated by C. Vetro

Abstract

In this paper, it is proved that the product system $(X \times Y, T \times S)$ is multi-$\mathscr{F}$-sensitive (resp., $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive) if and only if (X, T) or (Y, S) is multi-$\mathscr{F}$-sensitive (resp., $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive) when Furstehberg families $\mathscr{F}$ and $\mathscr{F}_2$ have the Ramsey property, improving the main results in [N. Değirmenci, Ş. Koçak, Turk. J. Math., **34** (2010), 593–600] and [R. Li, X. Zhou, Turk. J. Math., **37** (2013), 665–675]. Moreover, some analogical results for semi-flows are obtained.

Keywords: Multi-$\mathscr{F}$-sensitivity, $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity, Li-Yorke sensitivity, product system.
2010 MSC: 54H20, 54B20, 37B99, 37D45.

1. Introduction

A *dynamical system* is a pair (X, T), where X is a nontrivial compact metric space with a metric d and $T : X \longrightarrow X$ is a continuous map. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

The complexity of a dynamical system is a central topic of research since the term of chaos was introduced by Li and Yorke [17] in 1975, known as *Li-Yorke chaos* today. Another important feature of chaoticity is that orbits from nearby points start to diverge after finite steps. This notion, the 'butterfly effect', has been widely studied and is termed as *sensitive dependence on initial conditions* (briefly, *sensitivity*), introduced by Auslander and Yorke [4] and popularized by Devaney [6]. More precisely, a dynamical system (X, T) is *sensitive* if there exists $\varepsilon > 0$ such that for any $x \in X$ and any $\delta > 0$, there exist

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Received 2015-12-01

$y \in B(x, \delta) := \{y \in X : d(x, y) < \delta\}$ and $n \in \mathbb{Z}_+$ satisfying $d(T^n(x), T^n(y)) > \varepsilon$. When is a dynamical system sensitive? This question has gained some attention in more recent papers (see [1, 5, 8, 9, 11–14, 16, 19–24, 28]). We know that there are several ways to extend the notion of sensitivity. Here, we only list the following three ways:

(1) one may define n-sensitivity as it was done by Xiong in [27], and Ye and Zhang in [28];

(2) one may require that in any open subset U there is a pair (x, y) which is a Li-Yorke pair as Akin and Kolyada did in [3];

(3) the third way is what we now consider in the present paper, that is, study the 'size' of the set of all times where sensitivity emerges can be regarded as a measure of how sensitive a dynamical system is.

Previously, the third way was considered by several scholars. More recently, Moothathu [19] initiated a preliminary study of various forms of sensitivity and proposed three stronger forms of sensitivity: syndetic sensitivity, cofinite sensitivity, and multi-sensitivity. Then, Li [14, 15] introduced the concept of ergodic sensitivity, which is a stronger form of sensitivity, and presented some sufficient conditions for ergodical sensitivity. Akin and Kolyada [3] introduced the concept of Li-Yorke sensitivity which links the Li-Yorke chaos with the notion of sensitivity and proved that any weakly mixing dynamical system is Li-Yorke sensitive. A dynamical system (X, T) is Li-Yorke sensitive, if there exists some $\varepsilon > 0$ such that any neighbourhood of any $x \in X$ contains a point y satisfying

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0,$$

and

$$\limsup_{n\to\infty} d(T^n(x), T^n(y)) > \varepsilon.$$

So far, there have been many results on sensitivity for the product systems. Let (X, T) and (Y, S) be two dynamical systems. Değirmenci and Koçak [7] proved that the product system $(X \times Y, T \times S)$ is sensitive if and only if so is at least one of (X, T) or (Y, S). Li and Zhou [18] showed that the product system $(X \times Y, T \times S)$ (resp., the product semi-flow $T \times S$) on the product space $X \times Y$ is ergodically sensitive if and only if so is at least one of (X, T) or (Y, S) (resp., at least one of the semi-flow T or the semi-flow S). Recently, Wu et al. [24] obtained that the product system $(X \times Y, T \times S)$ is multi-sensitive if and only if so is at least one of (X, T) or (Y, S). It is well known that the theory of Furstenberg families is a very important tool in studying topological dynamical systems. In this paper, by using Furstenberg families we study the multi-$\mathscr{F}$-sensitivity and the $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity of the product dynamical systems. In particular, we proved that a product system $(X \times Y, T \times S)$ is multi-$\mathscr{F}$-sensitive (resp., $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive) if and only if (X, T) or (Y, S) is multi-$\mathscr{F}$-sensitive (resp., $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive) when Furstenberg families $\mathscr{F}$ and $\mathscr{F}_2$ have the Ramsey property, improving the main results in [7, 18]. Moreover, some analogical results for semi-flows are established.

2. Preliminaries

2.1. Furstenberg family

First, recall some basic concepts related to Furstenberg families (see [2] for more details).

Let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$. We say a collection $\mathscr{F} \subset \mathcal{P}$ is a *Furstenberg family* if it is hereditary upwards, that is, $F_1 \subset F_2$ and $F_1 \in \mathscr{F}$ imply $F_2 \in \mathscr{F}$; and is *proper* if it is a proper subset of $\mathcal{P}$, that is neither empty nor the whole $\mathcal{P}$. In this paper all Furstenberg families considered are proper. It is not hard to see that a family $\mathscr{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathscr{F}$ and $\emptyset \notin \mathscr{F}$. Given a Furstenberg family $\mathscr{F}$, define its *dual family* as

$$\kappa\mathscr{F} = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathscr{F}\}.$$

It is easy to check that $\kappa\mathscr{F}$ is a Furstenberg family, and is proper if $\mathscr{F}$ is so. Given two Furstenberg families $\mathscr{F}_1$ and $\mathscr{F}_2$, define $\mathscr{F}_1 \cdot \mathscr{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathscr{F}_1, F_2 \in \mathscr{F}_2\}$. A Furstenberg family $\mathscr{F}$ is a *filter* if it is proper and satisfies $\mathscr{F} \cdot \mathscr{F} \subset \mathscr{F}$; and it has the *Ramsey property*, if $F_1 \cup F_2 \in \mathscr{F}$ implies $F_1 \in \mathscr{F}$ or $F_2 \in \mathscr{F}$. It can be verified that Furstenberg $\mathscr{F}$ has the Ramsey property if and only if $\kappa\mathscr{F}$ is a filter.

For $A \subset \mathbb{Z}^+$, define

$$\overline{\text{Dens}}(A) = \limsup_{n\to+\infty} \frac{1}{n} |A \cap [0, n-1]| \text{ and } \underline{\text{Dens}}(A) = \liminf_{n\to+\infty} \frac{1}{n} |A \cap [0, n-1]|.$$

Then, $\overline{\text{Dens}}(A)$ and $\underline{\text{Dens}}(A)$ are the *upper density* and the *lower density* of A, respectively.

Let $\mathscr{F}_{inf}$ be the collection of all infinite subsets of $\mathbb{Z}_+$ and by $\mathscr{F}_{cf}$ the family of cofinite subset, that is, the collection of subsets of $\mathbb{Z}_+$ with finite complements. It is easy to see that $\mathscr{F}_{inf}$ is the largest proper translation invariant family and its dual $\mathscr{F}_{cf} = \kappa\mathscr{F}_{inf}$, clearly as a filter, is the smallest one.

A subset $F = \{a_1 < a_2 < \dots\} \subset \mathbb{Z}_+$ is

(1) *syndetic* if there exists $N \in \mathbb{Z}_+$ such that $a_{i+1} - a_i \le N$ for all $i \in \mathbb{N}$;

(2) *thick* if it contains arbitrarily large blocks of consecutive numbers;

(3) an *IP set* if there is a subset $\{p_i : i \in \mathbb{N}\}$ such that $F \supset \{p_{i_1} + \dots + p_{i_k} : k \in \mathbb{N},\ i_1 < \dots < i_k\}$.

Denote the collection of all syndetic (resp., thick, IP, positive upper density) subsets of $\mathbb{Z}_+$ by $\mathscr{F}_s$ (resp., $\mathscr{F}_t$, $\mathscr{F}_{ip}$, $\mathscr{F}_{pud}$). The Hindman Theorem [10] claims that $\mathscr{F}_{ip}$ has the Ramsey property.

In the same way we can define a Furstenberg family consisting of some subsets of the set $\mathbb{R}^+$ of all nonnegative real numbers and the above concepts related to Furstenberg families.

2.2. Topological dynamics

Let (X, T) be a t.d.s. and $U, V \subset X$. The *return time set* from U to V is defined as

$$N_T(U, V) = \{n \in \mathbb{Z}_+ : T^n(U) \cap V \neq \emptyset\}.$$

In particular, $N_T(x, V) = \{n \in \mathbb{Z}_+ : T^n(x) \in V\}$ for $x \in X$.

A t.d.s. (X, T) is *transitive* if for each pair non-empty open subsets $U, V \subset X$, $N_T(U, V) \neq \emptyset$; and it is *weakly mixing* if $(X \times X, T \times T)$ is transitive. We say that $x \in X$ is a *transitive point* if its orbit $\text{Orb}^+(x, T) := \{x, T(x), T^2(x), \dots\}$ is dense in X. The set of all transitive points of T is denoted by Tran_T. When $\text{Tran}_T = X$ we say (X, T) is *minimal.* A point $x \in X$ is called a *minimal point* if $(\overline{\text{Orb}^+(x, T)}, T)$ is a minimal subsystem of (X, T); and is called a *periodic point* if $(\overline{\text{Orb}^+(x, T)}, T)$ is a minimal subsystem with finite cardinality, that is, there is $n \in \mathbb{N}$ such that $T^n(x) = x$. A t.d.s. (X, T) is called a P-*system* if it is transitive and the set of periodic points is dense; and it is an M-*system* if it is transitive and the set of minimal points is dense.

Recall that for $\delta > 0$, we denote

$$N_T(U, \delta) = \{n \in \mathbb{N} : \text{diam}(T^n(U)) > \delta\}.$$

for any non-empty open subset $U \subset X$. Let $\mathscr{F}$, F_1, F_2 be proper Furstenberg families. According to Moothathu [19], Tan and Zhang [20], a dynamical system (X, T) is said to be

(1) *$\mathscr{F}$-sensitive* if there exists $\varepsilon > 0$ –$\mathscr{F}$-sensitive constant–such that for any non-empty open subset $U \subset X$, $N_T(U, \varepsilon) \in \mathscr{F}$;

(2) *ergodically sensitive* if there exists $\varepsilon > 0$ –$\mathscr{F}$-ergodically sensitive constant–such that for any non-empty open subset $U \subset X$, $N_T(U, \varepsilon) \in \mathscr{F}_{pud}$;

(3) *multi-sensitive* if there is $\varepsilon > 0$ –multi-sensitive constant–such that for any $k \in \mathbb{N}$ and any non-empty open subsets $U_1, \ldots, U_k \subset X$, $\bigcap_{i=1}^{k} N_T(U_i, \varepsilon) \neq \emptyset$;

(4) *multi-$\mathscr{F}$-sensitive* if there is $\varepsilon > 0$ –multi-$\mathscr{F}$-sensitive constant–such that for any $k \in \mathbb{N}$ and any non-empty open subsets $U_1, \ldots, U_k \subset X$, $\bigcap_{i=1}^{k} N_T(U_i, \varepsilon) \in \mathscr{F}$, that is, $\{n \in \mathbb{Z}_+ : \min_{1 \leq i \leq k} \operatorname{diam}(T^n(U_i)) \geq \varepsilon\} \in \mathscr{F}$;

(5) *$(\mathscr{F}_1, \mathscr{F}_2)$-sensitive* if there is $\varepsilon > 0$ such that for any $x \in X$ and any $\delta > 0$, there exists $y \in B(x, \delta)$ such that for any $\lambda > 0$, $\{n \in \mathbb{Z}_+ : d(T^n(x), T^n(y)) < \lambda\} \in \mathscr{F}_1$ and $\{n \in \mathbb{Z}_+ : d(T^n(x), T^n(y)) \geq \varepsilon\} \in \mathscr{F}_2$.

Clearly, every $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive dynamical system is $\mathscr{F}_2$-sensitive. Moreover, in [3, Corollary 3.9], [12, Lemma 2.5] and [26, Theorem 2.1], the following results are obtained:

(1) every nontrivial weakly mixing system is Li-Yorke sensitive;

(2) a dynamical system is sensitive if and only if it is $\mathscr{F}_{inf}$-sensitive;

(3) a dynamical system is Li-Yorke sensitive if and only if it is $(\mathscr{F}_{inf}, \mathscr{F}_{inf})$-sensitive;

(4) a dynamical system is multi-sensitive if and only if it is multi-$\mathscr{F}_{ip}$-sensitive if and only if it is multi-$\mathscr{F}_{inf}$-sensitive.

Thus, $\mathscr{F}$-sensitivity, $(\mathscr{F}_1, F_2)$-sensitivity, and multi-$\mathscr{F}$-sensitivity are natural generalizations of sensitivity, Li-Yorke sensitivity, and multi-sensitivity, respectively. In [12, Example 4.2], Huang et al. constructed a sensitive dynamical system which is not multi-sensitive, showing that the multi-sensitivity is strictly stronger than the sensitivity.

Example 2.1. Let $T : [0, 1] \longrightarrow [0, 1]$ be sensitive. Applying [19, Theorem 2] yields that T is $\mathscr{F}_{cf}$-sensitive, thus multi-$\mathscr{F}_{cf}$-sensitive.

Example 2.2. Let $\beta \notin \mathbb{Q}$ and (X, T) be given by $X = \mathbb{R}^2/\mathbb{Z}^2$ and $T(x, y) = (x+\beta, x+y)$ for any $(x, y) \in X$. It follows from [12, Example 4.7] that (X, T) is an invertible minimal distal system (and hence containing no Li-Yorke pairs) and multi-$\mathscr{F}_{inf}$-sensitive. Meanwhile, it follows from [12, Example 5.2] that there exists a nonminimal E-system [1] such that (1) it contains a fixed point as its unique minimal set, and hence the system is Li-Yorke sensitive; (2) it is not thickly sensitive, and hence not multi-sensitive by [12, Theorem 4.6]. These mean that the multi-$\mathscr{F}$-sensitivity and the $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity don't imply each other.

Denote

$$\mathrm{LY}_{\mathscr{F}_1, \mathscr{F}_2}(\varepsilon, T) = \left\{ (x, y) \in X \times X : \begin{array}{l} \{n \in \mathbb{Z}_+ : d(T^n(x), T^n(y)) < \lambda\} \in \mathscr{F}_1 \text{ for any} \\ \lambda > 0 \text{ and } \{n \in \mathbb{Z}_+ : d(T^n(x), T^n(y)) \geq \varepsilon\} \in \mathscr{F}_2 \end{array} \right\}.$$

In the same way one can define the above notions and give the above notations for semi-flows, where for the notion of a semi-flow and the notations related to semi-flows we refer to [18].

2.3. Product dynamical systems

Given two maps $T : X \longrightarrow X$ and $S : Y \longrightarrow Y$ on compact metric spaces X and Y with metrics d_1 and d_2 respectively, their product $T \times S : X \times Y \longrightarrow X \times Y$ is defined by $T \times S(x, y) = (T(x), S(y))$ for all $(x, y) \in X \times Y$, the product metric d on $X \times Y$ is defined by $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. More results on product dynamical systems can be found in [7, 18, 25].

[1] A dynamical system (X, T) is an *E-system* if (X, T) is a transitive system admitting an invariant probability Borel measure with full support.

3. Sensitivity for product systems

In this section we discuss some forms of sensitivity for product systems by using family theory. More precisely, we study $\mathscr{F}$-sensitivity, multi-$\mathscr{F}$-sensitivity, $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity and Li-Yorke sensitivity of product systems. Consequently, our main results extend and improve the existing ones.

Theorem 3.1. *Let (X, T) and (Y, S) be two dynamical systems and $\mathscr{F}$ be a Furstenberg family with the Ramsey property. Then, $(X \times Y, T \times S)$ is multi-$\mathscr{F}$-sensitive if and only if (X, T) or (Y, S) is multi-$\mathscr{F}$-sensitive.*

Proof. By the definition of multi-$\mathscr{F}$-sensitivity, it suffices to prove the necessity.

Let $\varepsilon > 0$ be a multi-$\mathscr{F}$-sensitive constant of $T \times S$ and assume that both (X, T) and (Y, S) are not multi-$\mathscr{F}$-sensitive. Then, there are nonempty open subsets $U_1, \ldots, U_{k_1} \subset X$ and $V_1, \ldots, V_{k_2} \subset Y$ such that

$$\left\{ n \in \mathbb{Z}_+ : \min_{1 \le i \le k_1} \operatorname{diam}(T^n(U_i)) < \frac{\varepsilon}{2} \right\} \in \kappa\mathscr{F},$$

and

$$\left\{ n \in \mathbb{Z}_+ : \min_{1 \le i \le k_2} \operatorname{diam}(S^n(V_i)) < \frac{\varepsilon}{2} \right\} \in \kappa\mathscr{F}.$$

This, together with the Ramsey property of $\mathscr{F}$, implies that $F_1 := \{n \in \mathbb{Z}_+ : \min_{1 \le i \le k_1} \operatorname{diam}(T^n(U_i)) < \varepsilon/2\} \cap \{n \in \mathbb{Z}_+ : \min_{1 \le i \le k_2} \operatorname{diam}(S^n(V_i)) < \varepsilon/2\} \in \kappa\mathscr{F}$. Since each $U_i \times V_j$ $(1 \le i \le k_1,\ 1 \le j \le k_2)$ is a nonempty open subset of $X \times Y$, then the multi-$\mathscr{F}$-sensitivity of $T \times S$ implies that

$$F_2 := \left\{ n \in \mathbb{Z}_+ : \min_{1 \le i \le k_1, 1 \le j \le k_2} \operatorname{diam}((T \times S)^n(U_i \times V_j)) \ge \varepsilon \right\} \in \mathscr{F}.$$

Fix any $n \in F_1 \cap F_2 \neq \emptyset$. It can be verified that there are $1 \le i \le k_1$ and $1 \le j \le k_2$ such that $\operatorname{diam}(T^n(U_i)) < \varepsilon/2$ and $\operatorname{diam}(S^n(V_j)) < \varepsilon/2$. Then,

$$\varepsilon \le \operatorname{diam}(T \times S)^n(U_i \times V_j) = \operatorname{diam}(T^n(U_i) \times S^n(V_j)) \le \frac{\varepsilon}{\sqrt{2}},$$

which is a contradiction, since $\varepsilon > 0$. □

Lemma 3.2 ([12])**.** *A dynamical system (X, T) is multi-sensitive if and only if (X, T) is multi-$\mathscr{F}_{inf}$-sensitive.*

Corollary 3.3. *Let (X, T) and (Y, S) be two dynamical systems and $\mathscr{F}$ be a Furstenberg family with the Ramsey property. Then, $(X \times Y, T \times S)$ is $\mathscr{F}$-sensitive if and only if (X, T) or (Y, S) is $\mathscr{F}$-sensitive.*

Since both $\mathscr{F}_{inf}$ and $\mathscr{F}_{pud}$ have the Ramsey property, using Theorem 3.1 and Lemma 3.2, one immediately has:

Corollary 3.4 ([24, Theorem 10])**.** *Let (X, T) and (Y, S) be two dynamical systems. Then, $(X \times Y, T \times S)$ is multi-sensitive if and only if (X, T) or (Y, S) is multi-sensitive.*

Corollary 3.5. *Let (X, T) and (Y, S) be two dynamical systems. Then, $(X \times Y, T \times S)$ is multi-$\mathscr{F}_{pud}$-sensitive if and only if (X, T) or (Y, S) is multi-$\mathscr{F}_{pud}$-sensitive.*

Corollary 3.6 ([18, Lemma 3.3-4])**.** *Let (X, T) and (Y, S) be two dynamical systems. Then, $(X \times Y, T \times S)$ is ergodically sensitive if and only if (X, T) or (Y, S) is ergodically sensitive.*

Theorem 3.7. *Let (X, T) and (Y, S) be two dynamical systems and $\mathscr{F}_1$ and $\mathscr{F}_2$ be Furstenberg families such that $\mathscr{F}_2$ has the Ramsey property. Then, $(X \times Y, T \times S)$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive if and only if (X, T) or (Y, S) is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive.*

Proof. The sufficiency follows immediately from the definition of $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity and the hereditary upwards properties of $\mathscr{F}_1$ and $\mathscr{F}_2$.

Necessity. Fix a $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive constant $\varepsilon > 0$ of $T \times S$ and assume that both (X, T) and (Y, S) are not $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive. This implies that there are $x_1 \in X$, $y_1 \in Y$ and $\delta_1 > 0$ such that for any $x \in B(x_1, \delta_1)$ and any $y \in B(y_1, \delta_1)$, $(x_1, x) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon/2, T)$ and $(y_1, y) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon/2, S)$. Take a positive number δ such that $B((x_1, y_1), \delta) \subset B(x_1, \delta_1) \times B(y_1, \delta_1)$. To prove that for any $(x, y) \in B((x_1, y_1), \delta)$, $((x_1, y_1), (x, y)) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon, T \times S)$, consider two cases as follows.

Case 1. If there is $\lambda > 0$ such that $\{n \in \mathbb{Z}_+ : d(T^n(x), T^n(x_1)) \geq \lambda\} \in \kappa\mathscr{F}_1$ or $\{n \in \mathbb{Z}_+ : d(S^n(y), S^n(y_1)) \geq \lambda\} \in \kappa\mathscr{F}_1$, then one has $\{n \in \mathbb{Z}_+ : d((T \times S)^n(x, y), (T \times S)^n(x_1, y_1)) \geq \lambda\} \in \kappa\mathscr{F}_1$, implying that $((x_1, y_1), (x, y)) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon, T \times S)$.

Case 2. If for any $\lambda > 0$, $\{n \in \mathbb{Z}_+ : d(T^n(x), T^n(x_1)) < \lambda\} \in \mathscr{F}_1$ and $\{n \in \mathbb{Z}_+ : d(S^n(y), S^n(y_1)) < \lambda\} \in \mathscr{F}_1$, noting that $(x_1, x) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon/2, T)$ and $(y_1, y) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon/2, S)$, it follows that

$$F_1 := \left\{n \in \mathbb{Z}_+ : d(T^n(x), T^n(x_1)) < \frac{\varepsilon}{2}\right\} \in \kappa\mathscr{F}_2,$$

and

$$F_2 := \left\{n \in \mathbb{Z}_+ : d(S^n(y), S^n(y_1)) < \frac{\varepsilon}{2}\right\} \in \kappa\mathscr{F}_2.$$

Clearly, for any $n \in F_1 \cap F_2$, $d((T \times S)^n(x_1, y_1), (T \times S)^n(x, y)) \leq \varepsilon/\sqrt{2} < \varepsilon$. This, together with the Ramsey property of $\mathscr{F}_2$, implies that

$$\{n \in \mathbb{Z}_+ : d((T \times S)^n(x_1, y_1), (T \times S)^n(x, y)) < \varepsilon\} \in \kappa\mathscr{F}_2,$$

that is, $\{n \in \mathbb{Z}_+ : d((T \times S)^n(x_1, y_1), (T \times S)^n(x, y)) \geq \varepsilon\} \notin \mathscr{F}_2$. So, $((x_1, y_1), (x, y)) \notin \mathrm{LY}_{\mathscr{F}_1,\mathscr{F}_2}(\varepsilon, T \times S)$.

Summing up Case 1 and Case 2 yields that $T \times S$ is not $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive. A contradiction with the $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity of $T \times S$, completing the proof of the necessity. □

Corollary 3.8. *Let (X, T) and (Y, S) be two dynamical systems. Then, $(X \times Y, T \times S)$ is Li-Yorke sensitive if and only if (X, T) or (Y, S) is Li-Yorke sensitive.*

Remark 3.9. By the definitions, the above results and their proofs, one can easily verify that the above results hold for continuous semi-flows.

Acknowledgment

Xinxing Wu was supported by scientific research starting project of Southwest Petroleum University (No. 2015QHZ029) and the Scientific Research Fund of the Sichuan Provincial Education Department (No. 14ZB0007), and Risong Li was supported by the Project of Enhancing School With Innovation of Guangdong Ocean University (Grant NO. GDOU2016050207) and the Key Scientific and Technological Research Project of Science and Technology Department of Zhanjiang City (Grant 2010C3112005).

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