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Fixed point theorems for improved α-Geraghty contractions in partial metric spaces

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Abstract

Rosa and Vetro [V. La Rosa, P. Vetro, J. Nonlinear Sci. Appl., **7** (2014), 1–10] established new fixed point results in complete partial metric spaces.

In this paper, we improve the notion of α-Geraghty contraction type mappings and establish some common fixed point theorems for a pair of α-admissible mappings under an improved notion of α-Geraghty contraction type mappings in complete partial metric spaces. We give an example to illustrate these results. An application of main result to the existence of solution of system of integral equations is also presented.

Keywords: Fixed point, α-Geraghty contraction, partial metric space.
2010 MSC: 47H09, 54H25, 47H10.

1. Introduction and preliminaries

In 1973, Geraghty [9] studied a generalization of Banach contraction principle. Hussain et al. [11] generalized the concept of α-admissible mappings and proved fixed point theorems. Subsequently, Abdeljawad [2] introduced a pair of $\alpha-$admissible mappings satisfying new sufficient contractive conditions different from those in [12] and obtained fixed point and common fixed point theorems. Thereafter, many papers have been published on Geraghty's contraction in both metric spaces and partial metric spaces [6, 7, 16].

Matthews [18] introduced the concept of partial metric spaces and proved an analogue of Banach's fixed point theorem in partial metric spaces. In fact, a partial metric space is a generalization of usual metric

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space in which the self distances $d(x,x)$ of elements of space are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see [1–11, 13, 15, 17]).

In this paper, the notion of improved α-Geraghty contraction type mappings is used to show the existence and uniqueness of a common fixed point of two mappings in the settings of complete partial metric spaces. The results proved in this paper will generalize many existing results in the literature (see [7, 9, 11, 16, 19]). We explain hypotheses of our result through an example. An application to the existence of solution of system of integral equations is also discussed.

Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$. Following concepts and results will be required for the proofs of main results.

Definition 1.1 ([18])**.** Let X be a nonempty set and let $p : X \times X \to \mathbb{R}_0^+$ satisfies the following properties: for all $x, y, z \in X$,

(P_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(P_2) $p(x,x) \leq p(x,y)$;

(P_3) $p(x,y) = p(y,x)$;

(P_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then p is called a partial metric on X and the pair (X,p) is known as partial metric space.

In [18], Matthews proved that every partial metric p on X induces a metric $d_p : X \times X \to \mathcal{R}_0^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

for all $x, y \in X$.

Notice that a metric on a set X is a partial metric d such that $d(x,x) = 0$ for all $x \in X$.

In [18], Matthews established that each partial metric p on X generates a T_0 topology $\tau(p)$ on X. The base of the topology $\tau(p)$ is the family of open p-balls $\{B_p(x,\epsilon) : x \in X,\ \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

Definition 1.2 ([18])**.** Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(2) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 1.3 ([21])**.** Let S be a self map defined on X and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. S is said to be an α-admissible mapping if

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(S(x),S(y)) \geq 1, \text{ for all } x,y \in X.$$

Definition 1.4 ([14])**.** Let S be a self map defined on X and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. S is said to be a triangular α-admissible mapping if following conditions:

(1) $\alpha(x,y) \geq 1$ implies $\alpha(S(x),S(y)) \geq 1$, $x, y \in X$,

(2) $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$,

hold for all $x, y, z \in X$.

Definition 1.5 ([2])**.** Let S and T be two self maps defined on X and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. The pair (S,T) of mappings is known as a triangular α-admissible pair of mappings if it satisfies following conditions:

(1) $\alpha(x,y) \geq 1$, implies $\alpha(S(x),T(y)) \geq 1$ and $\alpha(T(x),S(y)) \geq 1$,

(2) $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$,

for all $x,y,z \in X$.

Definition 1.6 ([20])**.** Let S be a self map defined on X and $\alpha,\eta : X \times X \to \mathbb{R}_0^+$ be two functions. S is said to be an α-admissible mapping with respect to η if following

$$\alpha(x,y) \geq \eta(x,y) \ \text{ implies } \alpha(S(x),S(y)) \geq \eta(S(x),S(y)),$$

holds for all $x,y \in X$.

Note that if we take $\eta(x,y) = 1$, then this definition reduces to Definition 1.3. Also if we take $\alpha(x,y) = 1$, then we say that S is a η-subadmissible mapping.

For more details and examples of α-admissible mappings, see [2, 14–16, 21].

The following lemma will be helpful in the sequel.

Lemma 1.7 ([18])**.**

(1) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.*

(2) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.*

(3) *If $\lim_{n\to\infty} x_n = \upsilon$ such that $p(\upsilon,\upsilon) = 0$, then $\lim_{n\to\infty} p(x_n,y) = p(\upsilon,y)$ for every $y \in X$.*

Lemma 1.8 ([7])**.** *Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,S(x_0)) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = S(x_n)$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}\cup\{0\}$ with $n < m$.*

Lemma 1.9 ([2])**.** *Let $S,T : X \to X$ be triangular α-admissible mappings. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,S(x_0)) \geq 1$. Define sequence $x_{2i+1} = S(x_{2i})$, and $x_{2i+2} = T(x_{2i+1})$, where $i = 0,1,2,\cdots$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}\cup\{0\}$ with $n < m$.*

We denote by Ω the family of all functions $\beta : [0,\infty) \to [0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

2. Main results

In this section, we prove some fixed point theorems for α-Geraghty contraction type mappings in complete partial metric space. We begin with the following definition.

Definition 2.1. Let (X,p) be a partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. The mappings $S,T : X \to X$ form a pair of improved α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x,y \in X$,

$$\alpha(x,y)p(S(x),T(y)) \leq \beta\left(M(x,y)\right)M(x,y), \tag{2.1}$$

where

$$M(x,y) = \max\left\{p(x,y), p(x,S(x)), p(y,T(y)), \frac{p(y,S(x)) + p(x,T(y))}{2}\right\}.$$

If $S = T$, then T is called a generalized α-Geraghty contraction type mapping if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y)p(S(x),T(y)) \leq \beta\left(N(x,y)\right)N(x,y),$$

where

$$N(x,y) = \max\left\{p(x,y), p(x,T(x)), p(y,T(y)), \frac{p(x,T(y)) + p(y,T(x))}{2}\right\}.$$

The following theorem is one of our main results.

Theorem 2.2. *Let (X,p) be a complete partial metric space, $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S, T : X \to X$ are two continuous mappings satisfying following conditions:*

(1) *(S,T) is a pair of improved α-Geraghty contraction type mappings;*

(2) *(S,T) is triangular α-admissible;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$.*

Then (S,T) has a common fixed point $\upsilon \in X$.

Proof. We begin with the following observation:

$$M(x,y) = 0 \text{ if and only if } x = y \text{ is a common fixed point of } (S,T).$$

Indeed, if $x = y$ is a common fixed point of (S,T), then $T(y) = T(x) = x = y = S(y) = S(x)$ and

$$M(x,y) = \max\left\{p(x,x), p(x,x), p(x,x), \frac{p(x,x) + p(x,x)}{2}\right\} = p(x,x).$$

From the contractive condition (2.1) we get

$$p(x,x) = p(S(x),T(y)) \leq \alpha(x,y)p(S(x),T(y)) \leq \beta\left(M(x,y)\right)M(x,y).$$

This is only possible if $p(x,x) = 0$, which implies $M(x,y) = 0$. Conversely, if $M(x,y) = 0$, then using (P_1) and (P_2), it is easy to check that $x = y$ is a fixed point of S and T.

On the other hand, if $M(x,y) > 0$, we construct an iterative sequence $\{x_n\}$ of points in such a way that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$, where $i = 0, 1, 2, \cdots$. We observe that if $x_n = x_{n+1}$, then x_n is a common fixed point of S and T. Thus suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Due to assumptions (2), (3) and using Lemma 1.9, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Then

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &= p(S(x_{2i}), T(x_{2i+1})) \leq \alpha(x_{2i}, x_{2i+1})p(S(x_{2i}), T(x_{2i+1})) \\ &\leq \beta\left(M(x_{2i}, x_{2i+1})\right)M(x_{2i}, x_{2i+1}) \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{p(x_{2i}, x_{2i+1}), p(x_{2i}, S(x_{2i})), p(x_{2i+1}, T(x_{2i+1})), \frac{p(x_{2i+1}, S(x_{2i})) + p(x_{2i}, T(x_{2i+1}))}{2}\right\} \\ &= \max\left\{p(x_{2i}, x_{2i+1}), p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2}), \frac{p(x_{2i+1}, x_{2i+1}) + p(x_{2i}, x_{2i+2})}{2}\right\} \\ &\leq \max\left\{p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2})\right\}. \end{aligned}$$

Thus

$$\begin{aligned}p(x_{2i+1}, x_{2i+2}) &\leq \beta\left(M(x_{2i}, x_{2i+1})\right) M(x_{2i}, x_{2i+1}) \\ &\leq \beta\left(p(x_{2i}, x_{2i+1})\right) p((x_{2i}, x_{2i+1}) < p(x_{2i}, x_{2i+1}),\end{aligned}$$

and so

$$p(x_{2i+1}, x_{2i+2}) < p(x_{2i}, x_{2i+1}). \tag{2.2}$$

This implies that

$$p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Hence we deduce that the sequence $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is nonnegative and nonincreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$. We assert that $r = 0$. Suppose to the contrary that $r > 0$. Then from (2.2), we have

$$\frac{p(x_{n+1}, x_{n+2})}{M(x_n, x_{n+1})} \leq \beta(M(x_n, x_{n+1})) \leq 1.$$

Applying limit $n \to \infty$, we have

$$1 \leq \beta(M(x_n, x_{n+1})) \leq 1.$$

It follows that $\lim_{n\to\infty} \beta(M(x_n, x_{n+1})) = 1$. Owing to the fact that $\beta \in \Omega$, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}) = 0, \tag{2.3}$$

which yields that $r = \lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Now, we claim that $\{x_n\}$ is a Cauchy sequence in (X, p). Suppose, on contrary, that $\{x_n\}$ is not a Cauchy sequence, that is, $\lim_{n,m\to\infty} p(x_n, x_m) \neq 0$. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$,

$$p(x_{m_k}, x_{n_k}) \geq \epsilon.$$

This means that

$$p(x_{m_k}, x_{n_{k-1}}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned}\epsilon &\leq p(x_{m_k}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) - p(x_{n_{k-1}}, x_{n_{k-1}}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + p(x_{n_{k-1}}, x_{n_k}).\end{aligned}$$

That is,

$$\epsilon < \epsilon + p(x_{n_{k-1}}, x_{n_k}) \tag{2.4}$$

for all $k \in \mathbb{N}$. In the view of (2.4) and (2.3), we have

$$\lim_{k\to\infty} p(x_{m_k}, x_{n_k}) = \epsilon. \tag{2.5}$$

Again using the triangle inequality, we have

$$\begin{aligned}p(x_{m_k}, x_{n_k}) &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) - p(x_{m_{k+1}}, x_{m_{k+1}}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}) \\ &\quad - p(x_{n_{k+1}}, x_{n_{k+1}})\end{aligned}$$

$$\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}),$$

and

$$\begin{aligned} p(x_{m_{k+1}}, x_{n_{k+1}}) &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) - p(x_{m_k}, x_{m_k}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}) \\ &\quad - p(x_{n_k}, x_{n_k}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}). \end{aligned}$$

Taking limit as $k \to +\infty$ and using (2.3) and (2.5), we obtain

$$\lim_{k\to+\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

By Lemma 1.9, since $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$, we have

$$\begin{aligned} p(x_{n_{k+1}}, x_{m_{k+2}}) = p(S(x_{n_k}), T(x_{m_{k+1}})) &\leq \alpha(x_{n_k}, x_{m_{k+1}}) p(S(x_{n_k}), T(x_{m_{k+1}})) \\ &\leq \beta(M(x_{n_k}, x_{m_{k+1}})) M(x_{n_k}, x_{m_{k+1}}). \end{aligned}$$

Finally, we conclude that

$$\frac{p(x_{n_{k+1}}, x_{m_{k+2}})}{M(x_{n_k}, x_{m_{k+1}})} \leq \beta(M(x_{n_k}, x_{m_{k+1}})).$$

By using (2.3), taking limit as $k \to +\infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \beta(p(x_{n_k}, x_{m_{k+1}})) = 1.$$

So $\lim_{k\to\infty} p(x_{n_k}, x_{m_{k+1}}) = 0 < \epsilon$, which is a contradiction. Hence

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in (X, p). From (1.1), we infer that $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore, $\lim_{n,m\to\infty} d_p(x_n, x_m) = 0$ and thus by Lemma 1.7 $\{x_n\}$ is a Cauchy sequence in both (X, p) and (X, d_p). Since (X, p) is a complete partial metric space so by Lemma 1.7, (X, d_p) is also a complete metric space. Completeness of (X, d_p) implies that there exists $\upsilon \in X$ such that $x_n \to \upsilon$, that is, $\lim_{n\to\infty} d_p(x_n, \upsilon) = 0$. By Lemma 1.7, we get

$$\lim_{n\to\infty} p(\upsilon, x_n) = p(\upsilon, \upsilon) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.6}$$

Due to $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, we infer from (2.6) that $p(\upsilon, \upsilon) = 0$ and $\{x_n\}$ converges to υ with respect to $\tau(p)$, moreover, $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$. Now the continuity of T implies

$$\upsilon = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T(x_{2n+1}) = T(\lim_{n\to\infty} x_{2n+1}) = T(\upsilon).$$

Analogously, $\upsilon = S(\upsilon)$. Thus we have $S(\upsilon) = T(\upsilon) = \upsilon$. Hence (S, T) has a common fixed point. Now we show that υ is the unique common fixed point of S and T. Assume the contrary, that is, there exists $\omega \in X$ such that $\upsilon \neq \omega$ and $\omega = T(\omega)$. From the contractive condition (2.1), we have

$$p(\upsilon, \omega) \leq \beta(M(\upsilon, \omega)) M(\upsilon, \omega) < M(\upsilon, \omega),$$

and

$$M(\upsilon, \omega) = \max\left\{p(\upsilon, \omega), p(\upsilon, S(\upsilon)), p(\omega, T(\omega)), \frac{p(\omega, S(\upsilon)) + p(\upsilon, T(\omega))}{2}\right\},$$

which gives

$$M(\upsilon, \omega) = p(\upsilon, \omega).$$

This means that $p(\upsilon, \omega) < p(\upsilon, \omega)$, which is a contradiction, so $p(\upsilon, \omega) = 0$. Consequently, υ is a unique common fixed point of the pair (S, T). □

It is possible to remove the continuity of mappings S and T by replacing a weaker condition:
(C): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to \upsilon \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, \upsilon) \geq 1$ for all k.

Theorem 2.3. *Let (X,p) be a complete partial metric space, $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S,T : X \to X$ are two mappings such that*

(1) *(S,T) is a pair of improved α-Geraghty contraction type mappings;*

(2) *(S,T) is triangular α-admissible;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$;*

(4) *(C) holds.*

Then (S,T) has a common fixed point $\upsilon \in X$.

Proof. Following the proof of Theorem 2.2, we know that $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$ as $n \to +\infty$. We only have to show that υ is a common fixed point of S and T. Due to hypotheses (4), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, \upsilon) \geq 1$ for all k. Now by using (2.1) for all k, we have

$$\begin{aligned} p(x_{2n_k+1}, T(\upsilon)) = p(S(x_{2n_k}), T(\upsilon)) &\leq \alpha(x_{2n_k}, \upsilon) p(S(x_{2n_k}), T(\upsilon)) \\ &\leq \beta\left(M(x_{2n_k}, \upsilon)\right) M(x_{2n_k}\upsilon), \end{aligned}$$

and so

$$p(x_{2n_k+1}, T(\upsilon)) \leq \beta\left(M(x_{2n_k}, \upsilon)\right) M(x_{2n_k}, \upsilon).$$

On the other hand, we obtain

$$M(x_{2n_k}, \upsilon) = \max\left\{ p(x_{2n_k}, \upsilon), p(x_{2n_k}, S(x_{2n_k})), p(\upsilon, T(\upsilon)), \frac{p(\upsilon, S(x_{2n_k})) + p(x_{2n_k}, T(\upsilon))}{2} \right\}.$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = \max\left\{ p(\upsilon, S(\upsilon)), p(\upsilon, T(\upsilon)) \right\}. \tag{2.7}$$

Case I: Assume that $\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = p(\upsilon, T(\upsilon))$.
Suppose that $p(\upsilon, T(\upsilon)) > 0$. From (2.7), for a large k, we have $M(x_{2n_k}, \upsilon) > 0$, which implies that

$$\beta(M(x_{2n_k}, \upsilon)) < M(x_{2n_k}, \upsilon).$$

Then we have

$$p(x_{2n_k}, T(\upsilon)) < M(x_{2n_k}, \upsilon). \tag{2.8}$$

Letting $k \to \infty$ in (2.8), we obtain that $p(\upsilon, T(\upsilon)) < p(\upsilon, T(\upsilon))$, which is a contradiction. Thus we obtain that $p(\upsilon, T(\upsilon)) = 0$. Due to (P_1) and (P_2), we have $\upsilon = T(\upsilon)$.

Case II: Assume that $\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = p(\upsilon, S(\upsilon))$. Then arguing like above, we get $\upsilon = S(\upsilon)$. Thus $\upsilon = T(\upsilon) = S(\upsilon)$. □

If we set $S = T$ and $M(x,y) = \max\left\{ p(x,y), p(x,T(x)), p(y,T(y)), \frac{p(x,T(y)) + p(y,T(x))}{2} \right\}$ in Theorem 2.2 and Theorem 2.3, then we obtain results presented by Rosa and Vetro [16].

Corollary 2.4 ([16]). *Let (X,p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ a function. Suppose that $S : X \to X$ is a continuous mapping such that*

(1) *S is a generalized α-Geraghty contraction type mapping;*

(2) *S is triangular α-admissible;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$.*

Then S has a fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

Corollary 2.5 ([16])**.** *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ a function. Suppose that S satisfies the following conditions:*

(1) *S is a generalized α-Geraghty contraction type mapping;*

(2) *S is triangular α-admissible;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$;*

(4) *(C) holds.*

Then S has a fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

If we set $M(x, y) = \max\{p(x, y), p(x, S(x)), p(y, S(y))\}$ and $p(x, x) = 0$ ($\forall\ x \in X$) in Theorem 2.2 and Theorem 2.3, then the results presented by Cho et al. [7] can be viewed as particular cases of Theorem 2.2 and Theorem 2.3.

Corollary 2.6 ([7])**.** *Let (X, p) be a complete metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S : X \to X$ is a mapping which satisfies the following conditions:*

(1) *S is a generalized α-Geraghty contraction type mapping;*

(2) *S is triangular α-admissible;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$;*

(4) *T is continuous.*

Then S has a fixed point $p \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

Corollary 2.7 ([7])**.** *Let (X, p) be a complete metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Let $S : X \to X$ be a mapping such that*

(1) *S is a generalized α-Geraghty contraction type mapping;*

(2) *S is triangular α-admissible;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$;*

(4) *(C) holds.*

Then S has a fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

Definition 2.8. Let (X, p) be a partial metric space and $\alpha, \eta : X \times X \to \mathbb{R}_0^+$ be two functions. A pair of mappings $S, T : X \to X$ is called a pair of improved (α, η)-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \text{ implies } p(S(x), T(y)) \leq \beta(M(x, y))M(x, y),$$

where

$$M(x, y) = \max\left\{p(x, y), p(x, S(x)), p(y, T(y)), \frac{p(y, S(x)) + p(x, T(y))}{2}\right\}.$$

Theorem 2.9. *Let (X, p) be a complete partial metric space and $\alpha, \eta : X \times X \to \mathbb{R}_0^+$ be two functions. Suppose that $S, T : X \to X$ are mappings such that*

(1) *(S, T) is a pair of improved (α, η)-Geraghty contraction type mappings;*

(2) *(S, T) is triangular α-admissible with respect to η;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq \eta(x_0, S(x_0))$;*

(4) *S and T are continuous.*

Then (S, T) has a common fixed point $\upsilon \in X$.

Proof. Let x_1 in X be such that $x_1 = S(x_0)$ and $x_2 = T(x_1)$. Continuing this process, we construct a sequence x_n of points in X such that

$$x_{2i+1} = S(x_{2i}), \text{ and } x_{2i+2} = T(x_{2i+1}), \text{ where } i = 0, 1, 2, \ldots.$$

The assumptions (2) and (3) lead to $\alpha(S(x_0), T(x_1)) \geq \eta(S(x_0), T(x_1))$, we deduce that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ which implies that $\alpha(T(x_1), S(x_2)) \geq \eta(T(x_1), S(x_2))$. Continuing in this way, we obtain $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &= p(S(x_{2i}), T(x_{2i+1})) \leq \alpha(x_{2i}, x_{2i+1}) p(S(x_{2i}), T(x_{2i+1})) \\ &\leq \beta(M(x_{2i}, x_{2i+1})) M(x_{2i}, x_{2i+1}). \end{aligned}$$

Therefore,

$$p(x_{2i+1}, x_{2i+2}) \leq \alpha(x_{2i}, x_{2i+1}) p(S(x_{2i}), T(x_{2i+1}))$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{ p(x_{2i}, x_{2i+1}), p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2}), \frac{p(x_{2i+1}, x_{2i+1}) + p(x_{2i}, x_{2i+2})}{2} \right\} \\ &\leq \max\left\{ p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2}) \right\}. \end{aligned}$$

From the definition of β, the case $M(x_{2i}, x_{2i+1}) = p(x_{2i+1}, x_{2i+2})$ is not possible. Indeed, if $x_{2i+1} \neq x_{2i+2}$, then

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &\leq \beta(M(x_{2i}, x_{2i+1})) M(x_{2i}, x_{2i+1}) \\ &\leq \beta(p(x_{2i+1}, x_{2i+2})) p(x_{2i+1}, x_{2i+2}) < p(x_{2i+1}, x_{2i+2}), \end{aligned}$$

which is a contradiction. Therefore, $M(x_{2i}, x_{2i+1}) = p(x_{2i}, x_{2i+1})$ and so

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &\leq \beta(M(x_{2i}, x_{2i+1})) M(x_{2i}, x_{2i+1}) \\ &\leq \beta(p(x_{2i}, x_{2i+1})) p((x_{2i}, x_{2i+1}) < p(x_{2i}, x_{2i+1}). \end{aligned}$$

This implies that

$$p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

The rest of the proof follows from the proof of Theorem 2.2. □

It is possible to remove the continuity of mappings S and T by replacing a weaker condition:
(C1) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to \upsilon \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, \upsilon) \geq \eta(x_{n_k}, \upsilon)$ for all k.

Theorem 2.10. *Let (X, p) be a complete partial metric space and $\alpha, \eta : X \times X \to \mathbb{R}_0^+$ be two functions. Suppose that $S, T : X \to X$ are mappings such that*

(1) *(S,T) is a pair of improved (α,η)-Geraghty contraction type mappings;*

(2) *(S,T) is triangular α-admissible with respect to η;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq \eta(x_0, S(x_0))$;*

(4) *(C1) holds.*

Then S and T have a common fixed point $\upsilon \in X$.

Proof. Following the proofs of Theorem 2.3 and Theorem 2.9, it is easy to prove the existence and uniqueness of a common fixed point of S and T. □

If we set $M(x,y) = \max\left\{p(x,y), p(x,S(x)), p(y,S(y)), \dfrac{p(y,S(x)) + p(x,S(y))}{2}\right\}$ and $S = T$ in Theorem 2.9 and Theorem 2.10, then we get the following results.

Corollary 2.11. *Let (X,p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S : X \to X$ is a continuous mapping such that*

(1) *S is a generalized (α,η)-Geraghty contraction type mapping;*

(2) *S is triangular α-admissible with respect to η;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq \eta(x_0, S(x_0))$.*

Then S has a fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

Corollary 2.12. *Let (X,p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ a function. Suppose that $S : X \to X$ is a mapping such that*

(1) *S is a generalized (α,η)-Geraghty contraction mapping;*

(2) *S is triangular α-admissible with respect to η;*

(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq \eta(x_0, S(x_0))$;*

(4) *(C1) holds.*

Then S has a fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to υ for every $x \in X$.

To illustrate the results proved in this paper and to show the superiority of a pair of improved α-Geraghty contraction type mappings over contractions used in [7, 16], we present the following example.

Example 2.13. Let $X = \{1,2,3\}$. Define $p : X \times X \to \mathbb{R}_0^+$ by

$$p(1,3) = p(3,1) = \frac{5}{7}, p(1,1) = \frac{1}{10}, p(2,2) = \frac{2}{10}, p(3,3) = \frac{3}{10},$$
$$p(1,2) = p(2,1) = \frac{3}{7}, p(2,3) = p(3,2) = \frac{4}{7}.$$

It is easy to check that p is a partial metric and define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in X; \\ 0 & \text{if otherwise.} \end{cases}$$

Define mappings $S, T : X \to X$ as follow:

$$S(x) = 1 \text{ for each } x \in X,$$

$$T(1) = T(3) = 1,\ T(2) = 3,$$

and define $\beta : \mathbb{R}_0^+ \to [0,1)$ by $\beta\left(M(x,y)\right) = \dfrac{9}{10}$ for all $x, y \in X$. Note that $S(x)$, $T(x)$ belong to X and are continuous. The pair (S,T) is α-admissible. Indeed, $\alpha(x,y) = 1$ implies $\alpha(S(x), T(y)) = 1$. We shall show that the condition (2.1) in Theorem 2.2 is satisfied. If $x = 2$, $y = 3$, then $\alpha(2,3) = 1$ and

$$\begin{aligned} M(2,3) &= \max\left\{p(2,3), p(2,S(2)), p(3,T(3)), \frac{p(3,S(2)) + p(2,T(3))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{3}{7}, \frac{5}{7}, \frac{4}{7}\right\} = \frac{5}{7}, \end{aligned}$$

$p\left(S(2), T(3)\right) = p\left(1,1\right) = \dfrac{1}{10}$. Now

$$\frac{1}{10} = \alpha(2,3)p\left(S(2), T(3)\right) \leq \beta(M(2,3))M(2,3) = \frac{9}{14},$$

holds. Similarly, for other cases ($x = 1$, $y = 3$ and $x = 2$, $y = 1$ etc.), it is easy to check that the contractive condition (2.1) in Theorem 2.2 is satisfied. All the conditions (1)–(3) of Theorem 2.2 are satisfied. Hence (S,T) has a unique common fixed point ($x = 1$). However, the contractive condition (3) in [7] does not hold for this particular case. Indeed, for $x = 2$, $y = 3$,

$$\begin{aligned} M(2,3) &= \max\left\{d(2,3), d(2,T(2)), d(3,T(3))\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}\right\} = \frac{5}{7}, \end{aligned}$$

$$\alpha(2,3)d(T(2),T(3)) = \frac{5}{7} \nleq \frac{9}{14} = \beta(M(2,3))M(2,3).$$

Here we have assumed that $p(x,y) = d(x,y)$ for all $x, y \in X$ such that $x \neq y$. Similarly, the contractive condition (3.5) in [16] does not hold for this particular case. Indeed, for $x = 2$, $y = 3$,

$$\begin{aligned} M(2,3) &= \max\left\{p(2,3), p(2,T(2)), p(3,T(3)), \frac{p(3,T(2)) + p(2,T(3))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{51}{140}\right\} = \frac{5}{7}, \end{aligned}$$

$$\alpha(2,3)\left(p\left(T(2),T(3)\right)\right) = \frac{5}{7} \nleq \frac{9}{14} = \beta((M(2,3)))\left(M(2,3)\right).$$

3. Application to system of integral equations

In this section, we shall apply Theorem 2.2 to show the existence of solution of a pair of simultaneous Volterra-Hammerstein integral equations

$$x(t) = f(t) + \lambda \int_0^1 K(t,s)F_n(s,x(s))\,ds, \tag{3.1}$$

$$y(t) = f(t) + \lambda \int_0^1 K(t,s)G_n(s,y(s))\,ds, \tag{3.2}$$

for all $t \in [0,1]$, where $f(t)$ is known, $K(t,s)$, $F_n(s,x(s))$ and $G_n(s,y(s))$ are real valued functions that are measurable both in t and s on $[0,1]$, λ is real number.

Let $X = L^1([0,1],\mathbb{R})$ and $p(x,y) = d(x,y) + c_n$ for all $x, y \in X$, where

$$d(x,y) = \|x(s) - y(s)\|_X = \int_0^1 |x(s) - y(s)|\, ds,$$

and $\{c_n\}$ is a sequence of positive real numbers satisfying, $c_n \to 0$ as $n \to \infty$. It is easy to verify that (X, p) is a complete partial metric space.

Let Φ represent the class of functions $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with properties:

(1) ϕ is increasing.

(2) For each $t > 0$, $\phi(t) < t$.

(3) $\int_0^1 \phi(t)\, dt \leq \phi\left(\int_0^1 t\, dt\right)$.

(4) $\beta(t) = \frac{\phi(t)}{t} \in S$.

For example, $\phi(t) = \frac{1}{5}t$, $\phi(t) = \frac{t}{1+t}$ are elements of Φ.

Now we present the main result regarding application of Theorem 2.2.

Theorem 3.1. *Assume that the following hypotheses are satisfied:*

(C_1) $\int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\, dt = R_1 < +\infty$.

(C_2) $F, G \in L^1[0,1]$ *are such that, as* $n \to \infty$ $|F_n(s, x(s)) - G_n(s, y(s))| \leq \phi(x(s) - y(s))$, *for all* $s \in [0,1]$ *and* $x, y \in L^1[0,1]$.

Then the system of integral equations (3.1) *and* (3.2) *has a solution for each* λ *with* $\lambda R_1 < 1$.

Proof. We define the operators, for all $x, y \in X$

$$Sx(t) = f(t) + \lambda \int_0^1 K(t,s) F_n(s, x(s))\, ds.$$

$$Ty(t) = f(t) + \lambda \int_0^1 K(t,s) G_n(s, y(s))\, ds.$$

Then S and T are operators from X into itself. Indeed, we have

$$\begin{aligned} |Sx| &\leq |f(t)| + |\lambda| \int_0^1 |K(t,s) F_n(s, x(s))|\, ds, \\ &\leq |f(t)| + |\lambda| \sup_{0\leq s\leq 1} |K(t,s)| \int_0^1 |F_n(s, x(s))|\, ds. \end{aligned}$$

By assumptions (C_1) and (C_2), we obtain

$$\int_0^1 |Sx|\, dt \leq |\lambda| \int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\, dt \int_0^1 |F_n(s, x(s))|\, ds + \int_0^1 |f(t)|\, dt < +\infty.$$

This implies that $Sx \in X$, similarly $Ty \in X$. Now consider for all $x, y \in X$ that

$$\begin{aligned} p(Sx, Ty) &= d(Sx, Ty) + c_n \\ &= \|Sx - Ty\| + c_n \\ &= \int_0^1 |Sx(t) - Ty(t)|\, dt + c_n \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left| \lambda \int_0^1 K(t,s)F_n(s,x(s))\,ds - \lambda \int_0^1 K(t,s)G_n(s,y(s))\,ds \right| dt + c_n \\
&= \int_0^1 \left| \lambda \int_0^1 K(t,s)\left[F_n(s,x(s)) - G_n(s,y(s))\right]\,ds \right| dt + c_n \\
&\leq |\lambda| \int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\,dt \int_0^1 |F_n(s,x(s)) - G_n(s,y(s))|\,ds + c_n.
\end{aligned}
$$

Letting $n \to \infty$, we get

$$
\begin{aligned}
p(Sx,Ty) &\leq |\lambda| R_1 \int_0^1 \phi\left(|x(s)-y(s)|\right)\,ds \\
&\leq |\lambda| R_1 \phi\left(d(x,y)\right) < \phi\left(d(x,y)\right) \leq \phi\left(p(x,y)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&p(Sx,Ty) \leq \phi\left(p(x,y)\right) \leq \phi\left(M(x,y)\right) = \frac{\phi\left(M(x,y)\right)}{M(x,y)} M(x,y), \\
&p(Sx,Ty) \leq \beta\left(M(x,y)\right) M(x,y), \quad \text{for all } \; x,y \in X.
\end{aligned}
$$

Lastly, we define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$
\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in X; \\ 0 & \text{otherwise.} \end{cases}
$$

Then for all $x,y \in X$, we have

$$
\alpha(x,y)p(Sx,Ty) \leq \beta\left(M(x,y)\right) M(x,y).
$$

Apparently, $\alpha(x,y) = 1$ and $\alpha(y,z) = 1$ implies $\alpha(x,z) = 1$ for all $x,y,z \in X$, moreover, $\alpha(x,y) = 1$ implies $\alpha(Sx,Ty) = 1$ and $\alpha(Tx,Sy) = 1$, so (S,T) is triangular α-admissible pair of mappings. Hence hypotheses of Theorem 2.2 are satisfied. Consequently, the mappings S and T have common fixed point which is the solution of system of integral equations (3.1) and (3.2). □

Remark 3.2. For more details, applications and examples see [16] and references therein. Our results are more general than those in [5–7, 16] and improve several results existing in the literature.

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