Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5790–5805
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed points of some set-valued F-contractions

Zeqing Liu[a], Xue Na[a], Young Chel Kwun[b,*], Shin Min Kang[c,d]

[a] *Department of Mathematics, Liaoning Normal University, Dalian, Liaoning 116029, P. R. China.*
[b] *Department of Mathematics, Dong-A University, Busan 49315, Korea.*
[c] *Department of Mathematics and the RINS, Gyeongsang National University, Jinju 52828, Korea.*
[d] *Center for General Education, China Medical University, Taiwan, Taichung 40402, Taiwan.*

Communicated by Z. Kadelburg

Abstract

Fixed point theorems of several set-valued F-contractions without using the Hausdorff metric are provided. Our results extend substantially the results due to Nadler [S. B. Nadler, Jr., Pacific J. Math., **30** (1969), 475–488] and Mizoguchi and Takahashi [N. Mizoguchi, W. Takahashi, J. Math. Anal. Appl., **141** (1989), 177–188]. Five nontrivial examples are given.

Keywords: Set-valued F-contraction, fixed point, complete metric space.
2010 MSC: 54H25.

1. Introduction

In 1969, Nadler [13] initiated the study of fixed points for set-valued contraction mappings and proved the following result, which extends the Banach contraction principle.

Theorem 1.1 ([13]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfies*

$$H(Tx, Ty) \le \alpha d(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where

$$\alpha \in [0, 1) \text{ is a constant.} \tag{1.2}$$

Then T has a fixed point.

Since then many researchers [2, 3, 5–7, 9, 10, 12, 15] have continued the study of Nadler and extended Theorem 1.1 in various directions. Using a function k to replace the constant α in (1.1), Mizoguchi and Takahashi [12] generalized Theorem 1.1 and gave the following result.

*Corresponding author
Email addresses: zeqingliu@163.com (Zeqing Liu), xuena08@163.com (Xue Na), yckwun@dau.ac.kr (Young Chel Kwun), smkang@gnu.ac.kr (Shin Min Kang)

Received 2016-09-29

Theorem 1.2 ([14])**.** *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfies that*

$$H(Tx, Ty) \le k(d(x, y))d(x, y), \quad \forall x, y \in X \text{ with } x \ne y, \tag{1.3}$$

where

$$k : (0, +\infty) \to [0, 1) \text{ with } \limsup_{s \to t^+} k(s) < 1, \quad \forall t \in \mathbb{R}^+. \tag{1.4}$$

Then T has a fixed point.

In 1995, Daffer [5] provided an alternative and somewhat more straightforward proof of Theorem 1.2. In 2006, Feng and Liu [7] obtained an interesting generalization of Theorem 1.1.

In 2012, Wardowski [16] introduced the concept of F-contraction for single-valued mappings and proved a fixed point theorem for the F-contraction, which extends the Banach contraction principle. Afterwards, a few researchers [1, 4, 8, 11, 14] introduced new F-contractions for single-valued and set-valued mappings and proved the existence of fixed points for these F-contractions. In 2014, Acar et al. [1] gave a fixed point result for the generalized multi-valued F-contraction mappings and Cosentino and Vetro [4] got fixed point theorems for the Hardy-Rogers-type F-contractions in complete metric spaces and complete ordered metric spaces. In 2014, Minak et al. [11] showed the existence and uniqueness of fixed points for the Ciric type generalized F-contraction and almost F-contraction in complete metric spaces.

In this paper we establish the existence of fixed points for a few set-valued F-contractions without using the Hausdorff metric in complete metric spaces. The results obtained in the paper extend substantially Theorems 1.1 and 1.2. Five examples are included.

2. Preliminaries

Now we present some notions, notations and results used in this paper. Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Let (X, d) be a metric space, and $CL(X)$ and $CB(X)$ denote the classes of all nonempty closed and all nonempty bounded closed subsets of X, respectively. For every $A, B \in CL(X)$, $x \in X$ and $T : X \to CL(X)$, put

$$d(x, B) = \inf\{d(x, y), y \in B\}, \quad f(x) = d(x, Tx),$$

$$H(A, B) = \begin{cases} \max\left\{ \sup_{x\in A} d(x, B), \sup_{y \in B} d(y, A)\right\}, & \text{if the maximum exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Such a mapping H is called a generalized Hausdorff metric induced by d in $CL(X)$. A point $p \in X$ is said to be a fixed point of T if $p \in Tp$. A sequence $\{x_n\}_{n\in\mathbb{N}_0} \subseteq X$ is said to be an orbit of T if $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$. A function $h : X \to \mathbb{R}^+$ is said to be T-orbitally lower semicontinuous at $z \in X$ if $h(z) \le \liminf_{n\to\infty} h(x_n)$ for each orbit $\{x_n\}_{n\in\mathbb{N}_0} \subseteq X$ of T with $\lim_{n\to\infty} x_n = z$. A function $g : \mathbb{R}^+ - \{0\} \to \mathbb{R}$ is said to be upper semicontinuous from right in $\mathbb{R}^+ - \{0\}$ if $\limsup_{s\to t^+} g(s) \le g(t)$ for all $t \in \mathbb{R}^+ - \{0\}$.

Definition 2.1 ([16])**.** Let $F : (0, +\infty) \to \mathbb{R}$ be a mapping satisfying

(F1) F is strictly increasing;
(F2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Denote by $\mathcal{F}$ the family of all functions F that satisfy (F1)-(F3).

Lemma 2.2. *Let (X, d) be a metric space, $B \in CL(X)$ and $F : \mathbb{R}^+ - \{0\} \to \mathbb{R}$ satisfies that*

$$F(d(x, A)) \ge \inf\{F(d(x, a)) : a \in A\}, \quad \forall (A, x) \in CL(X) \times (X - A). \tag{a1}$$

Then, for each $(\varepsilon, x) \in (\mathbb{R}^+ - \{0\}) \times (X - B)$ there exists $b \in B$ such that

$$F(d(x, b)) < F(d(x, B)) + \varepsilon.$$

Proof. Suppose that there exists $(\varepsilon, x) \in (\mathbb{R}^+ - \{0\}) \times (X - B)$ such that

$$F(d(x_0, b)) \geq F(d(x_0, B)) + \varepsilon_0, \quad \forall b \in B. \tag{2.1}$$

It follows from (a1) and (2.1) that

$$\begin{aligned} F(d(x_0, B)) &\geq \inf\{F(d(x_0, b)) : b \in B\} \\ &\geq F(d(x_0, B)) + \varepsilon_0 \\ &> F(d(x_0, B)), \end{aligned}$$

which is a contradiction. This completes the proof. □

Lemma 2.3. *Let (X, d) be a metric space and $F : \mathbb{R}^+ - \{0\} \to \mathbb{R}$ be upper semi-continuous from the right. Then* (a1) *holds.*

Proof. Let $(A, x) \in CL(X) \times (X - A)$ and put $r = d(x, A)$. Now we prove that there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ satisfying

$$d(x, a_n) \downarrow r \quad \text{as } n \to \infty. \tag{2.2}$$

Suppose that there exits $a \in A$ satisfying $r = d(x, a)$. Let $a_n := a$ for all $n \in \mathbb{N}$. Then (2.2) holds. Suppose that $r \neq d(x, a)$ for all $a \in A$. It is clear that for $\varepsilon_1 = 1$, there exists $a_1 \in A$ satisfying

$$r < d(x, a_1) < r + 1$$

for $\varepsilon_n = \min\left\{\frac{1}{n}, d(x, a_{n-1}) - r\right\} > 0$, there exists $a_n \in A$ satisfying

$$r < d(x, a_n) < r + \varepsilon_n = \min\left\{r + \frac{1}{n}, d(x, a_{n-1})\right\}, \quad \forall n \geq 2,$$

which implies that (2.2) holds. Note that

$$F(d(x, a_n)) \geq \inf\{F(d(x, a)) : a \in A\}, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

Combining (2.2) and (2.3) and using the right upper semi-continuity of F, we conclude that

$$F(r) \geq \limsup_{s\to r^+} F(s) \geq \limsup_{n\to\infty} F(d(x, a_n)) \geq \inf\{F(d(x, a)) : a \in A\}.$$

This completes the proof. □

Lemma 2.4. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfies* (1.3) *and* (1.4)*. Then $f(x) = d(x, Tx)$ is continuous in X.*

Proof. Let x be an arbitrary point in X. For any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $\lim_{n\to\infty} x_n = x$, by (1.3) and (1.4) we get that

$$\begin{aligned} |f(x_n) - f(x)| &= |d(x_n, Tx_n) - d(x, Tx)| \leq d(x_n, x) + H(Tx_n, Tx) \\ &\leq d(x_n, x) + k(d(x_n, x))d(x_n, x) \leq 2d(x_n, x) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which yields that

$$\lim_{n\to\infty} f(x_n) = f(x),$$

that is, f is continuous in X. This completes the proof. □

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfies that*

$$H(Tx, Ty) \le k_1(d(x, y))d(x, y), \quad \forall x, y \in X \text{ with } x \ne y, \tag{2.4}$$

where

$$k_1 : (0, +\infty) \to \Big(\frac{1}{2}, 1\Big) \text{ with } \limsup_{s\to t^+} k_1(s) < 1, \quad \forall t \in \mathbb{R}^+. \tag{2.5}$$

Then T has a fixed point.

Remark 2.6. Theorems 1.2 and 2.5 are equivalent. In fact, if (2.4) and (2.5) hold, by putting $k(t) := k_1(t), \forall t \in (0, +\infty)$ we obtain that (1.3) and (1.4) are satisfied; conversely, if (1.3) and (1.4) hold, by selecting $k_1(t) := \frac{1+k(t)}{2}$ $\forall t \in (0, +\infty)$ we get that

$$\frac{1}{2} \le k_1(t) = \frac{1 + k(t)}{2} < 1, \quad \forall t \in \mathbb{R}^+ - \{0\},$$
$$H(Tx, Ty) \le k(d(x, y))d(x, y) \le k_1(d(x, y))d(x, y), \quad \forall x, y \in X \text{ with } x \ne y$$

and

$$\limsup_{s\to t^+} k_1(s) = \limsup_{s\to t^+} \frac{1 + k(s)}{2} \le \frac{1}{2}\Big(1 + \limsup_{s\to t^+} k(s)\Big) < 1, \quad \forall t \in \mathbb{R}^+,$$

that is, (2.4) and (2.5) hold.

From Lemma 2.4, Theorem 2.5 and Remark 2.6, we get the following:

Lemma 2.7. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfies* (1.1) *and* (1.2)*. Then $f(x) = d(x, Tx)$ is continuous in X.*

Theorem 2.8. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfies that*

$$H(Tx, Ty) \le \alpha_1 d(x, y), \quad \forall x, y \in X, \tag{2.6}$$

where

$$\alpha_1 \in \Big(\frac{1}{2}, 1\Big) \text{ is a constant.} \tag{2.7}$$

Then T has a fixed point.

Remark 2.9. Theorems 1.1 and 2.8 are equivalent.

3. Main results

Now we prove a few fixed point theorems for the set-valued F-contractions (a1), (a4), (a9), and (a10) below without using the Hausdorff metric in complete metric spaces.

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ satisfies that*

$$\varphi(d(x, y)) + F(d(y, Ty)) \le F(d(x, y)), \ \forall (x, y) \in (X - Tx) \times (Tx - (Ty \cup \{x\})), \tag{a2}$$

where $F \in \mathcal{F}$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy (a1) *and*

$$\liminf_{s\to t^+} \varphi(s) > 0, \ \forall t \in \mathbb{R}^+. \tag{a3}$$

Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.

Proof. Let x_0 be an arbitrary point of X. If $x_0 \in Tx_0$, then x_0 is a fixed point of T and $\lim_{n\to\infty} x_n = x_0$, where $x_n := x_0$ for all $n \geq 1$, the proof is finished. Suppose that $x_0 \in X - Tx_0$. Choose $x_1 \in Tx_0 - \{x_0\}$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and $\lim_{n\to\infty} x_n = x_1$, where $x_n := x_1$ for all $n \geq 2$, the proof is finished. Suppose that $x_1 \in Tx_0 - (Tx_1 \cup \{x_0\})$. It follows from (a2) that

$$F(d(x_1, Tx_1)) \leq F(d(x_0, x_1)) - \varphi(d(x_0, x_1)). \tag{3.1}$$

Put $\varepsilon_1 = \frac{1}{2}\varphi(d(x_0, x_1))$. Lemma 2.2 guarantees that there exists $x_2 \in Tx_1 - \{x_1\}$ such that

$$F(d(x_1, x_2)) < F(d(x_1, Tx_1)) + \frac{1}{2}\varphi(d(x_0, x_1)). \tag{3.2}$$

Making use of (3.1) and (3.2), we deduce that

$$\begin{aligned} F(d(x_1, x_2)) &< F(d(x_0, x_1)) - \varphi(d(x_0, x_1)) + \frac{1}{2}\varphi(d(x_0, x_1)) \\ &= F(d(x_0, x_1)) - \frac{1}{2}\varphi(d(x_0, x_1)). \end{aligned}$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T and $\lim_{n\to\infty} x_n = x_2$, where $x_n := x_2$ for all $n \geq 3$, the proof is finished. Suppose that $x_2 \in Tx_1 - (Tx_2 \cup \{x_1\})$. Clearly, (a2) ensures that

$$F(d(x_2, Tx_2)) \leq F(d(x_1, x_2)) - \varphi(d(x_1, x_2)). \tag{3.3}$$

Put $\varepsilon_2 = \frac{1}{2}\varphi(d(x_1, x_2))$. It follows from Lemma 2.2 that there exists $x_3 \in Tx_2 - \{x_2\}$ such that

$$F(d(x_2, x_3)) < F(d(x_2, Tx_2)) + \frac{1}{2}\varphi(d(x_1, x_2)). \tag{3.4}$$

By means of (3.3) and (3.4), we conclude that

$$\begin{aligned} F(d(x_2, x_3)) &< F(d(x_1, x_2)) - \varphi(d(x_1, x_2)) + \frac{1}{2}\varphi(d(x_1, x_2)) \\ &= F(d(x_1, x_2)) - \frac{1}{2}\varphi(d(x_1, x_2)). \end{aligned}$$

Repeating this process, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}}$ of T such that either there exists $k \in \mathbb{N}_0$ with $x_k \in Tx_k$, $x_i \in Tx_{i-1} - (Tx_i \cup \{x_{i-1}\})$, $i \in \{1, 2, \cdots, k-1\}$ and $\lim_{n\to\infty} x_n = x_k$, where $x_n = x_k$ for all $n \geq k+1$, the proof is finished, or

$$x_n \in Tx_{n-1} - (Tx_n \cup \{x_{n-1}\}), \quad \forall n \in \mathbb{N}, \tag{3.5}$$

$$F(d(x_n, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \tag{3.6}$$

$$F(d(x_n, x_{n+1})) < F(d(x_n, Tx_n)) + \frac{1}{2}\varphi(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \tag{3.7}$$

By virtue of (3.6) and (3.7), we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &< F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) + \frac{1}{2}\varphi(d(x_{n-1}, x_n)) \\ &= F(d(x_{n-1}, x_n)) - \frac{1}{2}\varphi(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.8}$$

Using (3.5), (3.8), (F1), and $\varphi(\mathbb{R}^+ - \{0\}) \subseteq \mathbb{R}^+ - \{0\}$, we obtain that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{3.9}$$

It follows from (3.9) that the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is positive and decreasing. Consequently, there exists some $a \in \mathbb{R}^+$ satisfying

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = a. \tag{3.10}$$

By virtue of (a3) there exists a constant $b > 0$ satisfying

$$\liminf_{s\to a^+} \varphi(s) = 2b,$$

which means that for $\varepsilon = b$, there exists $\delta > 0$ satisfying

$$\varphi(s) - 2b > -\varepsilon, \quad \forall s \in (a, a+\delta),$$

that is,

$$\varphi(s) > b, \quad \forall s \in (a, a+\delta). \tag{3.11}$$

It is easy to see that (3.9) and (3.10) yield that there exists $n_0 \in \mathbb{N}$ satisfying

$$a < d(x_n, x_{n+1}) < a + \delta, \quad \forall n \geq n_0,$$

which together with (3.11) means that

$$\varphi(d(x_n, x_{n+1})) > b, \quad \forall n \geq n_0. \tag{3.12}$$

In view of (3.8) and (3.12), we get that

$$\begin{aligned}
F(d(x_n, x_{n+1})) &< F(d(x_{n-1}, x_n)) - \frac{1}{2}\varphi(d(x_{n-1}, x_n)) \\
&< F(d(x_{n-2}, x_{n-1})) - \frac{1}{2}\varphi(d(x_{n-2}, x_{n-1})) - \frac{1}{2}\varphi(d(x_{n-1}, x_n)) \\
&\vdots \\
&< F(d(x_{n_0}, x_{n_0+1})) - \frac{1}{2}\varphi(d(x_{n_0}, x_{n_0+1})) \\
&\quad - \frac{1}{2}\varphi(d(x_{n_0+1}, x_{n_0+2})) - \cdots - \frac{1}{2}\varphi(d(x_{n-1}, x_n)) \\
&< F(d(x_{n_0}, x_{n_0+1})) - \frac{1}{2}(n - n_0)b, \quad \forall n \geq n_0.
\end{aligned} \tag{3.13}$$

Taking limit in (3.13), we acquire that

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty,$$

which together with (F2) and (3.10) gives that

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.14}$$

Clearly, (F3) and (3.14) ensure that there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} [d^k(x_n, x_{n+1}) F(d(x_n, x_{n+1}))] = 0. \tag{3.15}$$

In terms of (3.13)-(3.15), we arrive at

$$\begin{aligned}
0 &\leq \frac{1}{2}(n - n_0) b d^k(x_n, x_{n+1}) \\
&< [F(d(x_{n_0}, x_{n_0+1})) - F(d(x_n, x_{n+1}))] d^k(x_n, x_{n+1}) \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which connotes that

$$\lim_{n\to\infty}\left[(n-n_0)bd^k(x_n,x_{n+1})\right]=0,$$

that is,

$$\lim_{n\to\infty}\left[nd^k(x_n,x_{n+1})\right]=0. \tag{3.16}$$

From (3.16) we deduce that there exists $n_1>n_0$ such that

$$nd^k(x_n,x_{n+1})\le 1,\quad \forall n\ge n_1,$$

that is,

$$d(x_n,x_{n+1})\le \frac{1}{n^{\frac{1}{k}}},\quad \forall n\ge n_1,$$

which gives that

$$\begin{aligned}d(x_n,x_m)&\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)\\&\le\sum_{i=n}^{m-1}d(x_i,x_{i+1})\\&\le\sum_{i=n}^{\infty}d(x_i,x_{i+1})\\&\le\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}},\quad \forall m>n\ge n_1,\end{aligned}$$

which together with the convergence of the series $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ implies that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since (X,d) is complete, it follows that the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ converges to some point $z\in X$, that is,

$$\lim_{n\to\infty}x_n=z. \tag{3.17}$$

Suppose that f is T-orbitally lower semicontinuous at z. It follows from (3.14) and (3.17) that

$$0\le d(z,Tz)=f(z)\le\liminf_{n\to\infty}f(x_n)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}d(x_n,Tx_n)\le\lim_{n\to\infty}d(x_n,x_{n+1})=0,$$

that is, $z\in X$ is a fixed point of T.

Conversely, suppose that $z\in X$ is a fixed point of T. For each orbit $\{y_n\}_{n\in\mathbb{N}_0}$ of T with $\lim_{n\to\infty}y_n=z$, we derive that

$$f(z)=d(z,Tz)=0\le\liminf_{n\to\infty}f(y_n),$$

which implies that f is T-orbitally lower semicontinuous at z. This completes the proof. □

Theorem 3.2. *Let (X,d) be a complete metric space and $T:X\to CL(X)$ satisfies that*

$$\varphi(d(x,Tx))+F(d(y,Ty))\le F(d(x,y)),\ \forall(x,y)\in(X-Tx)\times(Tx-(Ty\cup\{x\})), \tag{a4}$$

where $F\in\mathcal{F}$ and $\varphi:\mathbb{R}^+-\{0\}\to\mathbb{R}^+-\{0\}$ satisfy (a1), (a3), *and*

$$\varphi \textit{ is locally bounded at } 0. \tag{a5}$$

Then, for each $x_0\in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z\in X$ such that $\lim_{n\to\infty}x_n=z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x)=d(x,Tx)$ is T-orbitally lower semicontinuous at z.

Proof. Let x_0 be an arbitrary point of X. If $x_0 \in Tx_0$, then x_0 is a fixed point of T and $\lim_{n\to\infty} x_n = x_0$, where $x_n := x_0$ for all $n \geq 1$, the proof is finished. Suppose that $x_0 \in X - Tx_0$. For $\varepsilon_1 = \frac{1}{2}\varphi(d(x_0, Tx_0)) > 0$, it follows from Lemma 2.2 that there exists $x_1 \in Tx_0 - \{x_0\}$ satisfying

$$F(d(x_0, x_1)) < F(d(x_0, Tx_0)) + \frac{1}{2}\varphi(d(x_0, Tx_0)). \tag{3.18}$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T and $\lim_{n\to\infty} x_n = x_1$, where $x_n := x_1$ for all $n \geq 2$, the proof is finished. Suppose that $x_1 \in Tx_0 - (Tx_1 \cup \{x_0\})$. It follows from (a4) that

$$F(d(x_1, Tx_1)) \leq F(d(x_0, x_1)) - \varphi(d(x_0, Tx_0)). \tag{3.19}$$

In light of (3.18) and (3.19), we get that

$$\begin{aligned} F(d(x_1, Tx_1)) &< F(d(x_0, Tx_0)) + \frac{1}{2}\varphi(d(x_0, Tx_0)) - \varphi(d(x_0, Tx_0)) \\ &= F(d(x_0, Tx_0)) - \frac{1}{2}\varphi(d(x_0, Tx_0)). \end{aligned}$$

For $\varepsilon_2 = \frac{1}{2}\varphi(d(x_1, Tx_1)) > 0$, it follows from Lemma 2.2 that there exists $x_2 \in Tx_1 - \{x_1\}$ satisfying

$$F(d(x_1, x_2)) < F(d(x_1, Tx_1)) + \frac{1}{2}\varphi(d(x_1, Tx_1)). \tag{3.20}$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T and $\lim_{n\to\infty} x_n = x_2$, where $x_n := x_2$ for all $n \geq 3$, the proof is finished. Suppose that $x_2 \in Tx_1 - (Tx_2 \cup \{x_1\})$. Equation (a4) ensures that

$$F(d(x_2, Tx_2)) \leq F(d(x_1, x_2)) - \varphi(d(x_1, Tx_1)). \tag{3.21}$$

By means of (3.20) and (3.21), we obtain that

$$\begin{aligned} F(d(x_2, Tx_2)) &< F(d(x_1, Tx_1)) + \frac{1}{2}\varphi(d(x_1, Tx_1)) - \varphi(d(x_1, Tx_1)) \\ &= F(d(x_1, Tx_1)) - \frac{1}{2}\varphi(d(x_1, Tx_1)). \end{aligned}$$

Continuing this process, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}}$ of T such that either there exists $k \in \mathbb{N}_0$ with $x_k \in Tx_k$, $x_i \in Tx_{i-1} - (Tx_i \cup \{x_{i-1}\})$, $i \in \{1, 2, \cdots, k-1\}$ and $\lim_{n\to\infty} x_n = x_k$, where $x_n = x_k$ for all $n \geq k+1$, the proof is finished, or (3.5) and the following conditions (3.22) and (3.23) hold:

$$F(d(x_{n-1}, x_n)) < F(d(x_{n-1}, Tx_{n-1})) + \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})), \quad \forall n \in \mathbb{N} \tag{3.22}$$

and

$$F(d(x_n, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, Tx_{n-1})), \quad \forall n \in \mathbb{N}. \tag{3.23}$$

On account of (3.22) and (3.23), we derive that

$$\begin{aligned} F(d(x_n, Tx_n)) &< F(d(x_{n-1}, Tx_{n-1})) + \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) - \varphi(d(x_{n-1}, Tx_{n-1})) \\ &= F(d(x_{n-1}, Tx_{n-1})) - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.24}$$

In terms of (3.24), (F1), and $\varphi(\mathbb{R}^+ - \{0\}) \subseteq \mathbb{R}^+ - \{0\}$, we have

$$0 < d(x_n, Tx_n) < d(x_{n-1}, Tx_{n-1}), \quad \forall n \in \mathbb{N}. \tag{3.25}$$

It follows from (3.25) that the sequence $\{d(x_n, Tx_n)\}_{n\in\mathbb{N}}$ converges to a constant $a \in \mathbb{R}^+$, that is,

$$\lim_{n\to\infty} d(x_n, Tx_n) = a. \tag{3.26}$$

As in the proof of Theorem 3.1, we conclude that (3.11) holds. It follows from (3.11) and (3.26) that there exists $n_0 \in \mathbb{N}$ satisfying

$$a < d(x_n, Tx_n) < a + \delta, \quad \forall n \geq n_0,$$

which together with (3.11) means that

$$\varphi(d(x_n, Tx_n)) > b, \quad \forall n \geq n_0. \tag{3.27}$$

In light of (3.24) and (3.27), we obtain that

$$\begin{aligned} F(d(x_n, Tx_n)) &< F(d(x_{n-1}, Tx_{n-1})) - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) \\ &< F(d(x_{n-2}, Tx_{n-2})) - \frac{1}{2}\varphi(d(x_{n-2}, Tx_{n-2})) - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) \\ &\vdots \\ &< F(d(x_{n_0}, Tx_{n_0})) - \frac{1}{2}\varphi(d(x_{n_0}, Tx_{n_0})) - \frac{1}{2}\varphi(d(x_{n_0+1}, Tx_{n_0+1})) \\ &\quad - \cdots - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) \\ &< F(d(x_{n_0}, Tx_{n_0})) - \frac{1}{2}(n - n_0)b, \quad \forall n \geq n_0. \end{aligned} \tag{3.28}$$

Taking limits in (3.28), we get that

$$\lim_{n\to\infty} F(d(x_n, Tx_n)) = -\infty,$$

which together with (F2) and (3.26) gives that

$$a = \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.29}$$

By virtue of (a5) and $\varphi(\mathbb{R}^+ - \{0\}) \subseteq \mathbb{R}^+ - \{0\}$, we know that there exist positive constants c and σ satisfying

$$0 < \varphi(t) \leq c, \quad \forall t \in (0, \sigma). \tag{3.30}$$

It follows from (3.25) and (3.29) that there exists $n_1 > n_0$ satisfying

$$0 < d(x_n, Tx_n) < \sigma, \quad \forall n \geq n_1,$$

which together with (3.30) means that

$$0 < \varphi(d(x_n, Tx_n)) \leq c, \quad \forall n \geq n_1. \tag{3.31}$$

By means of (3.22), (3.23), (3.27), and (3.31), we deduce that

$$\begin{aligned} F(d(x_n, x_{n+1})) &< F(d(x_n, Tx_n)) + \frac{1}{2}\varphi(d(x_n, Tx_n)) \\ &\leq F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, Tx_{n-1})) + \frac{1}{2}\varphi(d(x_n, Tx_n)) \\ &< F(d(x_{n-1}, Tx_{n-1})) - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) + \frac{1}{2}\varphi(d(x_n, Tx_n)) \\ &\leq F(d(x_{n-2}, x_{n-1})) - \varphi(d(x_{n-2}, Tx_{n-2})) - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) + \frac{1}{2}\varphi(d(x_n, Tx_n)) \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&< F(d(x_{n_1}, x_{n_1+1})) - \frac{1}{2}\varphi(d(x_{n_1}, Tx_{n_1})) - \cdots - \frac{1}{2}\varphi(d(x_{n-2}, Tx_{n-2}))\\
&\quad - \frac{1}{2}\varphi(d(x_{n-1}, Tx_{n-1})) + \frac{1}{2}\varphi(d(x_n, Tx_n))\\
&< F(d(x_{n_1}, x_{n_1+1})) - \frac{1}{2}(n-n_1)b + \frac{1}{2}c\\
&\to -\infty \quad \text{as } n \to \infty.
\end{aligned}
\tag{3.32}
$$

That is,

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty,$$

which together with (F2) gives that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.33}$$

It is easy to see that (F3) and (3.33) yield (3.15). Using (3.15), (3.32), and (3.33), we get that

$$
\begin{aligned}
0 &\le \frac{1}{2}(n-n_1)bd^k(x_n, x_{n+1})\\
&< \left[F(d(x_{n_1}, x_{n_1+1})) + \frac{1}{2}c - F(d(x_n, x_{n+1}))\right] d^k(x_n, x_{n+1})\\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

which means (3.16). The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.3. *Let (X, d) be a complete metric space, $F \in \mathcal{F}$, $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a1)*,* (a3)*,* (a4)*, and*

$$\limsup_{t\to 0^+} \varphi(t) < +\infty. \tag{a6}$$

Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.

Proof. It follows from (a6) that there is $c \in \mathbb{R}^+$ with

$$\limsup_{t\to 0^+} \varphi(t) = c,$$

which means that for $\varepsilon = 1$, there exists $\delta > 0$ satisfying

$$\varphi(t) - c < 1, \quad \forall t \in (0, \delta),$$

which together with $\varphi(\mathbb{R}^+ - \{0\}) \subseteq \mathbb{R}^+ - \{0\}$ gives that

$$0 < \varphi(t) < 1 + c, \quad \forall t \in (0, \delta),$$

which yields (a5). Thus Theorem (3.3) follows from Theorem 3.2. This completes the proof. □

Theorem 3.4. *Let (X, d) be a complete metric space, $F \in \mathcal{F}$, $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a1)*,* (a4)*, and*

$$\varphi \textit{ is nondecreasing and } \lim_{t\to 0^+} \varphi(t) > 0. \tag{a7}$$

Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.

Proof. Since φ is nondecreasing and $\varphi(\mathbb{R}^+ - \{0\}) \subseteq \mathbb{R}^+ - \{0\}$, it follows that

$$\liminf_{s\to t^+} \varphi(s) = \lim_{s\to t^+} \varphi(s) \geq \varphi(t) > 0, \quad \forall t \in \mathbb{R}^+ - \{0\},$$

which together with (a7) yields (a3) and (a5). Thus Theorem 3.4 follows from Theorem 3.2. This completes the proof. □

Theorem 3.5. *Let (X, d) be a complete metric space, $F \in \mathcal{F}$, and $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a1), (a4), *and*

$$\varphi \text{ is nonincreasing and } \lim_{t\to 0^+} \varphi(t) < +\infty. \tag{a8}$$

Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.

Proof. Using (a8) and $\varphi(\mathbb{R}^+ - \{0\}) \subseteq \mathbb{R}^+ - \{0\}$, we have

$$+\infty > \lim_{s\to 0^+} \varphi(s) = \limsup_{s\to 0^+} \varphi(s) = \liminf_{s\to 0^+} \varphi(s) \geq \varphi(1) > 0,$$
$$\liminf_{s\to t^+} \varphi(s) = \lim_{s\to t^+} \varphi(s) \geq \varphi(2t) > 0, \quad \forall t \in \mathbb{R}^+ - \{0\},$$

that is, (a3) and (a6) hold. Consequently, Theorem 3.5 follows from Theorem 3.3. This completes the proof. □

Letting $F(t) = \ln t$ for all $t \in \mathbb{R}^+ - \{0\}$ and making use of Lemmas 2.2 and 2.3, and Theorems 3.1–3.5 and their proofs, we get the following results.

Corollary 3.6. *Let (X, d) be a complete metric space, and $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a3) *and*

$$d(y, Ty) \leq e^{-\varphi(d(x,y))} d(x, y), \ \forall (x, y) \in X \times (Tx - \{x\}). \tag{a9}$$

Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.

Corollary 3.7. *Let (X, d) be a complete metric space, and $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a3), (a5), *and*

$$d(y, Ty) \leq e^{-\varphi(d(x,Tx))} d(x, y), \ \forall (x, y) \in (X - Tx) \times Tx. \tag{a10}$$

Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.

Corollary 3.8. *Let (X, d) be a complete metric space, and $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a3), (a6), *and* (a10). *Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.*

Corollary 3.9. *Let (X, d) be a complete metric space, and $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a10) *and* (a7). *Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.*

Corollary 3.10. *Let (X, d) be a complete metric space, and $T : X \to CL(X)$ and $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ satisfy* (a10) *and* (a8). *Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, z is a fixed point of T in X if and only if the function $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at z.*

4. Remarks and examples

Now we give some remarks and examples to illustrate the results in Section 3.

Remark 4.1. Corollary 3.6 extends Theorem 2.5. Assume that the conditions of Theorem 2.5 are satisfied. Put

$$\varphi(t) = -\ln k_1(t), \quad \forall t \in \mathbb{R}^+ - \{0\}. \tag{4.1}$$

It follows from (2.4) and (4.1) that

$$\begin{aligned} d(y,Ty) &\le H(Tx,Ty) \le k_1(d(x,y))d(x,y) \\ &= e^{-\ln \frac{1}{k_1(d(x,y))}} d(x,y) \\ &= e^{-\varphi(d(x,y))} d(x,y), \quad \forall (x,y) \in X \times (Tx - \{x\}), \end{aligned}$$

which gives (a9). Combining (2.5) and (4.1), we deduce that

$$\liminf_{s\to t^+} \varphi(s) = -\limsup_{s\to t^+} \big(\ln k_1(s)\big) = -\ln\left(\limsup_{s\to t^+} k_1(s)\right) > 0, \quad \forall t \in \mathbb{R}^+,$$

which yields (a3). That is, the conditions of Corollary 3.6 are fulfilled. It follows from Corollary 3.6 and Lemma 2.4 that the set-valued mapping T has a fixed point in X.

Remark 4.2. It follows from Remark 2.6 that Theorems 2.5 and 1.2 are equivalent. On account of Remark 4.1, we know that Corollary 3.6 extends Theorem 1.2, which, in turn, generalizes Theorem 1.1. The following example shows that Corollary 3.6 extends substantially Theorems 1.1 and 1.2.

Example 4.3. Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$Tx = \begin{cases} [2x-2, x], & \forall x \in (-\infty, 0], \\ [0, \frac{x}{3}], & \forall x \in \mathbb{R}^+ - \{0\}. \end{cases}$$

Now we assert that Theorems 1.1 and 1.2 cannot be used to prove the existence of fixed points for the set-valued mapping T. In fact,

$$\begin{aligned} H(T(-1), T(-10)) &= H([-4,-1],[-22,-10]) = 18 \\ &\not\le 9\alpha = \alpha d(-1,-10), \quad \forall \alpha \in [0,1) \end{aligned}$$

and

$$\begin{aligned} H(T(-1), T(-10)) &= H([-4,-1],[-22,-10]) = 18 \\ &\not\le 9k(9) = k(d(-1,-10))d(-1,-10) \end{aligned}$$

for any $k : (0,+\infty) \to [0,1)$ with $\limsup_{s\to t^+} k(s) < 1$ for all $t \in \mathbb{R}^+$.

Next we verify that the conditions of Corollary 3.6 hold. Define $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ by

$$\varphi(t) = \ln\left(3 - \frac{1}{1+t}\right), \quad \forall t \in \mathbb{R}^+ - \{0\}.$$

It is easy to see that

$$f(x) = d(x,Tx) = \begin{cases} 0, & \forall x \in (-\infty, 0], \\ \frac{2x}{3}, & \forall x \in \mathbb{R}^+ - \{0\} \end{cases}$$

is continuous in X and

$$\varphi(\mathbb{R}^+ - \{0\}) \subseteq (\ln 2, \ln 3), \quad \liminf_{s\to t^+} \varphi(s) = \ln\left(3 - \frac{1}{1+t}\right) > 0, \quad \forall t \in \mathbb{R}^+.$$

In order to verify (a9), we consider the following two possible cases:

Case 1. Let $x \in (-\infty, 0]$ and $y \in Tx - \{x\} = [2x-2, x)$. It follows that

$$d(y, Ty) = 0 \le e^{-\varphi(d(x,y))} d(x, y);$$

Case 2. Let $x \in (0, +\infty)$ and $y \in Tx - \{x\} = [0, \frac{x}{3}]$. It follows that

$$0 \le y \le \frac{x}{3}, \quad x \ge x - y \ge \frac{2}{3}x,$$
$$d(y, Ty) = \frac{2y}{3} \le \frac{2x}{9} = e^{-\ln 3}\frac{2}{3}x \le e^{-\varphi(d(x,y))} d(x, y).$$

That is, (a9) is satisfied. It follows from Corollary 3.6 that the set-valued mapping T has a fixed point in X.

Remark 4.4. We claim that each of Corollaries 3.7–3.10 generalizes Theorem 2.8. Assume that the conditions of Theorem 2.8 hold. Put

$$\varphi(t) = -\ln \alpha_1, \quad \forall t \in \mathbb{R}^+ - \{0\}. \tag{4.2}$$

It is clear that (2.7) and (4.2) ensure that (a3), (a5), (a6), (a7), and (a8) hold.

It follows from (2.6) and (4.2) that

$$\begin{aligned} d(y, Ty) &\le H(Tx, Ty) \le \alpha_1 d(x, y) \\ &= e^{-\ln \frac{1}{\alpha_1}} d(x, y) \\ &= e^{-\varphi(d(x,Tx))} d(x, y), \quad \forall (x, y) \in (X - Tx) \times Tx, \end{aligned}$$

which gives (a10). That is, the conditions of Corollaries 3.7–3.10 are fulfilled. It follows from each of Corollaries 3.7–3.10 and Lemma 2.7 that the set-valued mapping T has a fixed point in X.

Remark 4.5. It follows from Remark 2.9 that Theorem 2.8 is equivalent to Theorem 1.1. By means of Remark 4.4, we get that each of Corollaries 3.7–3.10 extends Theorem 1.1. The following examples prove that each of Corollaries 3.7–3.10 extend indeed Theorem 1.1 and differs from Theorem 1.2.

Example 4.6. Let $X = (-\infty, 1]$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$Tx = \begin{cases} (-\infty, 0], & \forall x \in (-\infty, 0], \\ [\frac{x}{3}, \frac{x}{2}], & \forall x \in (0, 1]. \end{cases}$$

Since $T0 = (-\infty, 0] \notin CB(X)$, it is clear that Theorems 1.1 and 1.2 are useless in proving the existence of fixed points for the set-valued mapping T. Define $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ by

$$\varphi(t) = \begin{cases} \ln(2-t), & \forall t \in (0, \frac{1}{2}), \\ 6t - \sin t, & \forall t \in [\frac{1}{2}, +\infty). \end{cases}$$

It is easy to see that

$$f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (-\infty, 0], \\ \frac{x}{2}, & \forall x \in (0, 1] \end{cases}$$

is continuous in X and

$$\liminf_{s \to t^+} \varphi(s) = \begin{cases} \ln(2-t) \ge \ln \frac{3}{2} > 0, & \forall t \in [0, \frac{1}{2}), \\ 6t - \sin t > 2, & \forall t \in [\frac{1}{2}, +\infty), \end{cases}$$

that is, φ satisfies (a3) and (a5).

Let $(x, y) \in (X - Tx) \times Tx$. It follows that $(x, y) \in (0, 1] \times [\frac{x}{3}, \frac{x}{2}]$ and

$$d(y, Ty) = \frac{y}{2} \le \frac{x}{4} \le e^{-\varphi(\frac{x}{2})}\frac{x}{2} \le e^{-\varphi(d(x,Tx))} d(x, y),$$

that is, (a10) holds. It follows from Corollary 3.7 that the set-valued mapping T has a fixed point in X.

Example 4.7. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$Tx = \begin{cases} [\frac{1}{3}, 1], & \forall x \in [0, 1), \\ \{x\} \cup [x^2, +\infty), & \forall x \in [1, +\infty). \end{cases}$$

Obviously, Theorems 1.1 and 1.2 are useless in proving the existence of fixed points for the set-valued mapping T because $T1 = [1, +\infty) \notin CB(X)$. Define $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ by

$$\varphi(t) = 1 + t, \quad \forall t \in \mathbb{R}^+ - \{0\}.$$

It is easy to see that

$$f(x) = d(x, Tx) = \begin{cases} \frac{1}{3} - x, & \forall x \in [0, \frac{1}{3}), \\ 0, & \forall x \in [\frac{1}{3}, +\infty) \end{cases}$$

is continuous in X and

$$\liminf_{s \to t^+} \varphi(s) = \begin{cases} \ln(2 - t) \geq \ln \frac{3}{2} > 0, & \forall t \in [0, \frac{1}{2}), \\ 6t - \sin t > 2, & \forall t \in [\frac{1}{2}, +\infty), \end{cases}$$

that is, (a3) and (a6) hold.

Let $(x, y) \in (X - Tx) \times Tx$. It follows that $(x, y) \in [0, \frac{1}{3}) \times [\frac{1}{3}, 1]$ and

$$d(y, Ty) = 0 \leq e^{-\varphi(d(x,Tx))} d(x, y),$$

that is, (a10) holds. It follows from Corollary 3.8 that the set-valued mapping T has a fixed point in X.

Example 4.8. Let $X = [-1, +\infty)$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$Tx = \begin{cases} [\frac{x^2}{8}, \frac{x^2}{4}], & \forall x \in [-1, 0), \\ [x, +\infty), & \forall x \in [0, +\infty). \end{cases}$$

It is clear that Theorems 1.1 and 1.2 are useless in proving the existence of fixed points of the set-valued mapping T because $T1 = [1, +\infty) \notin CB(X)$. Define $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ by

$$\varphi(t) = 1 + \ln(1 + t), \quad \forall t \in \mathbb{R}^+ - \{0\}.$$

It is easy to see that

$$f(x) = d(x, Tx) = \begin{cases} \frac{x^2}{8} - x, & \forall x \in [-1, 0), \\ 0, & \forall x \in [0, +\infty) \end{cases}$$

is continuous in X,

$$\lim_{t \to 0^+} \varphi(t) = \lim_{t \to 0^+} [1 + \ln(1 + t)] = 1$$

and φ is nondecreasing, that is, (a7) holds.

Let $(x, y) \in (X - Tx) \times Tx$. It follows that $(x, y) \in [-1, 0) \times \left[\frac{x^2}{8}, \frac{x^2}{4}\right]$ and

$$d(y, Ty) = 0 \leq e^{-\varphi(d(x,Tx))} d(x, y),$$

that is, (a10) holds. It follows from Corollary 3.9 that the set-valued mapping T has a fixed point in X.

Example 4.9. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$Tx = \begin{cases} [0, +\infty), & x = 0, \\ [\frac{x}{20}, \frac{3x}{10}], & \forall x \in (0, +\infty). \end{cases}$$

Obviously, Theorems 1.1 and 1.2 are useless in proving the existence of fixed points of the set-valued mapping T because $T0 = [0, +\infty) \notin CB(X)$. Define $\varphi : \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$ by

$$\varphi(t) = \ln\left(3 + \frac{1}{3+t}\right), \quad \forall t \in \mathbb{R}^+ - \{0\}.$$

It is easy to see that

$$f(x) = d(x, Tx) = \begin{cases} 0, & x = 0, \\ \frac{7x}{10}, & \forall x \in \mathbb{R}^+ - \{0\} \end{cases}$$

is continuous in X,

$$\lim_{t\to 0^+} \varphi(t) = \lim_{t\to 0^+} \ln\left(3 + \frac{1}{3+t}\right) = \ln\frac{10}{3} < +\infty,$$

and

$$\varphi'(t) = -\frac{1}{3(3+t)^3 + 3 + t} < 0, \quad \forall t \in \mathbb{R}^+ - \{0\},$$

which gives that φ is nonincreasing, that is, (a8) holds.

Let $(x, y) \in (X - Tx) \times Tx$. It follows that $(x, y) \in (0, +\infty) \times [\frac{x}{20}, \frac{3x}{10}]$ and

$$d(y, Ty) = \frac{7}{10}y \leq \frac{21}{100}x \leq e^{-\varphi(\frac{7}{10}x)}\frac{7}{10}x \leq e^{-\varphi(d(x,Tx))}d(x, y),$$

that is, (a10) holds. It follows from Corollary 3.10 that the set-valued mapping T has a fixed point in X.

Acknowledgment

The authors would like to thank the editor and referees for useful comments and suggestions.
This work was supported by the Dong-A University research fund.

References

[1] Ö. Acar, G. Durmaz, G. Minak, *Generalized multivalued F-contractions on complete metric spaces*, Bull. Iranian Math. Soc., **40** (2014), 1469–1478. 1
[2] H. Aydi, M. Abbas, C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl., **159** (2012), 3234–3242. 1
[3] L. Ćirić, *Multi-valued nonlinear contraction mappings*, Nonlinear Anal., **71** (2009), 2716–2723. 1
[4] M. Cosentino, P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-type*, Filomat, **28** (2014), 715–722. 1
[5] P. Z. Daffer, H. Kaneko, *Fixed points of generalized contractive multi-valued mappings*, J. Math. Anal. Appl., **192** (1995), 655–666. 1, 1
[6] A. A. Eldred, J. Anuradha, P. Veeramani, *On equivalence of generalized multi-valued contractions and Nadler's fixed point theorem*, J. Math. Anal. Appl., **336** (2007), 751–757.
[7] Y. Q. Feng, S. Y. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl., **317** (2006), 103–112 1, 1
[8] N. Hussain, P. Salimi, *uzuki-Wardowski type fixed point theorems for α-GF-contractions*, Taiwanese J. Math., **18** (2014), 1879–1895. 1
[9] D. Klim, D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **334** (2007), 132–139. 1
[10] P. S. Macansantos, *A generalized Nadler-type theorem in partial metric spaces*, Int. J. Math. Anal. (Ruse), **7** (2013), 343–348. 1
[11] G. Mınak, A. Helvacı, I. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results*, Filomat, **28** (2014), 1143–1151. 1
[12] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141** (1989), 177–188. 1
[13] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1, 1.1

[14] D. Paesano, C. Vetro, *Multi-valued F-contractions in 0-complete partial metric spaces with application to Volterra type integral equation*, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, **108** (2014), 1005–1020. 1.2, 1

[15] D. Wardowski, *On set-valued contractions of Nadler type in cone metric spaces*, Appl. Math. Lett., **24** (2011), 275–278. 1

[16] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), 7 pages. 1, 2.1