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Global attractivity of a rational difference equation of order ten

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Abstract

In this paper, we study qualitative properties and periodic nature of the solutions of the difference equation

$$x_{n+1} = ax_{n-4} + \frac{bx_{n-4}^2}{cx_{n-4} + dx_{n-9}}, \quad n = 0, 1, ..,$$

where the initial conditions x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers and a, b, c, d are constants. Also we obtain the form of solutions of some special cases of this equation.

Keywords: Periodicity, stability, rational difference equations.
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1. Introduction

This paper deals with the solution behaviour of the difference equation

$$x_{n+1} = ax_{n-4} + \frac{bx_{n-4}^2}{cx_{n-4} + dx_{n-9}}, \quad n = 0, 1, ..., \tag{1.1}$$

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where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are constants. Also we obtain the form of solution of some special cases.

The study and solution of nonlinear rational recursive sequence of high order is quite challenging and rewarding because some prototypes for the development of the solution and global behavior of nonlinear difference equation come from the results of non-linear difference equations and there is increasingly a lot of interest in studying these equations. Furthermore, diverse nonlinear trend occurring in science and engineering can be modeled by such equations and the solution about such equations offer prototypes towards the development of the theory, see for example [10–32, 43].

A. El-Moneam and Alamoudy [8] examined the positive solutions of the equation in terms of its periodicity, boundedness and the global stability. The considered difference equation is given by

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + fx_{n-3} + rx_{n-4}}{dx_{n-1} + ex_{n-2} + gx_{n-3} + sx_{n-4}}.$$

In [9] Elsayed investigated the solution of the following non-linear difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}}.$$

Keratas et al. [29] gave the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Saleh et al. [42] studied the behavior of the solution of the following difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}.$$

Yalçınkaya [49] has studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{ax_{n-k}}{b + cx_n^p}.$$

Yalçınkaya [50] has explored the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

See also [1–7]. Other related work on rational difference equations see in Refs. [33–42, 44–54].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1 (Equilibrium point)**.** A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2 (Periodicity)**.** A Sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.3 (Fibonacci sequence)**.** The sequence $\{F_m\}_{m=1}^{\infty} = \{1, 2, 3, 5, 8, 13, ...\}$, that is, $F_m = F_{m-1} + F_{m-2} \geq 0$, $F_{-2} = 0$, $F_{-1} = 1$ *is called Fibonacci Sequence.*

Definition 1.4 (Stability)**.**

(i) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq. (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq. (1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq. (1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq. (1.2).

(v) The equilibrium point $\overline{x}$ of Eq. (1.2) is unstable if $\overline{x}$ is not locally stable.

(vi) The linearized equation of Eq. (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A ([35])**.** *Assume that* $p_i \in R$, $i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \quad n = 0, 1, ... \,.$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B ([35])**.** *Let* $[\alpha, \beta]$ *be an interval of real numbers and assume that* $g : [\alpha, \beta]^2 \to [\alpha, \beta]$, *is a continuous function and consider the following equation*

$$x_{n+1} = g(x_n, x_{n-1}), \qquad n = 0, 1, ..., \tag{1.3}$$

satisfying the following conditions:

(a) *$g(x,y)$ is non-decreasing in $x \in [\alpha,\beta]$ for each fixed $y \in [\alpha,\beta]$ and $g(x,y)$ is non-increasing in $y \in [\alpha,\beta]$ for each fixed $x \in [\alpha,\beta]$.*

(b) *If $(m,M) \in [\alpha,\beta] \times [\alpha,\beta]$ is a solution of the system*

$$M = g(M,m) \qquad \text{and} \qquad m = g(m,M),$$

then

$$m = M,$$

then Eq. (1.3) has a unique equilibrium $\overline{x} \in [\alpha,\beta]$ and every solution of Eq. (1.3) converges to $\overline{x}$.

2. Local stability of the equilibrium point of Eq. (1.1)

In this section we study the local stability character of the equilibrium point of Eq. (1.1). The equilibrium points of Eq. (1.1) are given by the relation

$$\overline{x} = a\overline{x} + \frac{\overline{x}^2}{c\overline{x} + d\overline{x}}$$

or

$$\overline{x}^2(1-a)(c+d) = b\overline{x}^2.$$

If $(1-a)(c+d) \neq b$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0,\infty)^2 \longrightarrow (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v) = au + \frac{bu^2}{cu + dv}.$$

Therefore, at $\overline{x} = 0$, we get

$$\left(\frac{\partial f}{\partial u}\right)_{\overline{x}} = a + \frac{bc + 2bd}{(c+d)^2}, \qquad \left(\frac{\partial f}{\partial v}\right)_{\overline{x}} = \frac{-bd}{(c+d)^2}.$$

Then the linearized equation of Eq. (1.1) about $\overline{x}$ is

$$y_{n+1} - \left(a + \frac{bc + 2bd}{(c+d)^2}\right) y_{n-4} + \left(\frac{bd}{(c+d)^2}\right) y_{n-9} = 0. \tag{2.1}$$

Theorem 2.1. *Assume that*

$$b(c+3d) < (c+d)^2(1-a), \qquad a < 1.$$

Then the equilibrium point $\overline{x} = 0$ of Eq. (1.1) is locally asymptotically stable.

Proof. It is followed by Theorem A that, Eq. (2.1) is asymptotically stable if

$$\left| a + \frac{bc + 2bd}{(c+d)^2} \right| + \left| \frac{bd}{(c+d)^2} \right| < 1$$

or

$$a + \frac{bc + 3bd}{(c+d)^2} < 1,$$

and so

$$b(c+3d) < (c+d)^2(1-a),$$

which completes the proof. □

3. Global attractivity of the equilibrium point of Eq. (1.1)

In this section we investigate the global attractivity character of solutions of Eq. (1.1).

Theorem 3.1. *The equilibrium point $\overline{x}$ of Eq.* (1.1) *is global attractor if $c(1-a) \neq b$.*

Proof. Let α, β are real numbers and assume that $g : [\alpha, \beta]^2 \to [\alpha, \beta]$, be a function defined by

$$g(u, v) = au + \frac{bu^2}{cu + dv}.$$

Suppose that (m, M) is a solution of the system

$$M = g(M, m) \quad and \quad m = g(m, M).$$

Then from Eq. (1.1), we see that

$$M = aM + \frac{bM^2}{cM + dm}, \quad m = am + \frac{bm^2}{cm + dM}.$$

Therefore,

$$M(1-a) = \frac{bM^2}{cM + dm}, \quad m(1-a) = \frac{bm^2}{cm + dM},$$

or,

$$c(1-a)(M^2 - m^2) = b(M^2 - m^2), \qquad c(1-a) \neq b.$$

Thus

$$M = m.$$

It follows by the Theorem B that $\overline{x}$ is a global attractor of Eq. (1.1) and then the proof is complete. □

4. Boundedness of solutions of Eq. (1.1)

In this section we study the boundedness of solution of Eq. (1.1).

Theorem 4.1. *Every solution of Eq.* (1.1) *is bounded if*

$$\left(\alpha + \frac{\beta}{\gamma}\right) < 1.$$

Proof: Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$x_{n+1} = ax_{n-4} + \frac{bx_{n-4}^2}{cx_{n-4} + dx_{n-9}} \leq ax_{n-4} + \frac{bx_{n-4}^2}{cx_{n-4}} = \left(a + \frac{b}{c}\right) x_{n-4}.$$

Then,

$$x_{n+1} \leq x_{n-4} \text{ for all } n \geq 0.$$

Then the subsequences $\{x_{5n-4}\}_{n=0}^{\infty}$, $\{x_{5n-3}\}_{n=0}^{\infty}$, $\{x_{5n-2}\}_{n=0}^{\infty}$, $\{x_{5n-1}\}_{n=0}^{\infty}$, and $\{x_{5n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0\}.$$

In order to confirm the result of this section we consider some numerical examples for $x_{-9} = 6$, $x_{-8} = 11$, $x_{-7} = 10$, $x_{-6} = 5$, $x_{-5} = 8$, $x_{-4} = 2$, $x_{-3} = 9$, $x_{-2} = 5$, $x_{-1} = 9$, $x_0 = 6$, and $a = 0.5$, $b = 6$, $c = 9$, $d = 10$. (See Figure 1) and if we put $x_{-9} = 10$, $x_{-8} = 6$, $x_{-7} = 5$, $x_{-6} = 11$, $x_{-5} = 10$, $x_{-4} = 2$, $x_{-3} = 8$, $x_{-2} = 2$, $x_{-1} = 9$, $x_0 = 7$, and $a = 0.8$, $b = 6$, $c = 9$, $d = 10$. (See Figure 2.)

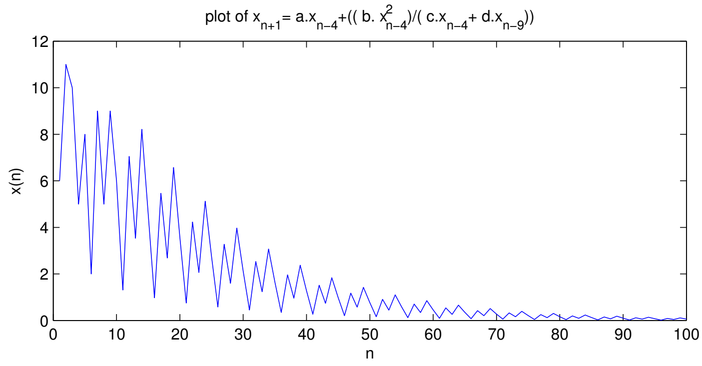

Figure 1

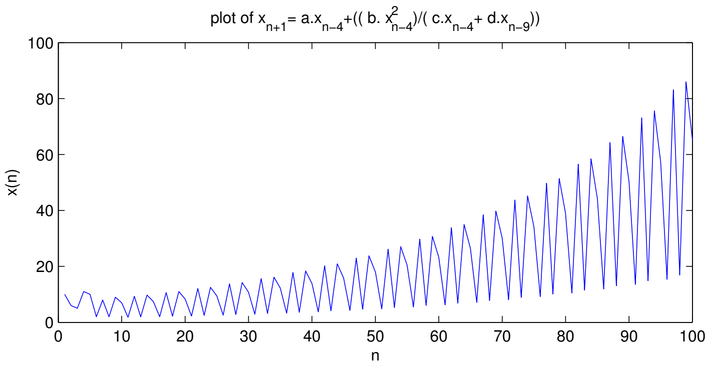

Figure 2

5. Special cases of Eq. (1.1)

5.1. First equation

In this section we study the following special case of Eq. (1.1)

$$x_{n+1} = x_{n-4} + \frac{x_{n-4}^2}{x_{n-4} + x_{n-9}}, \tag{5.1}$$

where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary real numbers.

Theorem 5.1. *Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq. (5.1). Then, for $n = 0, 1, 2, ...$*

$$x_{5n-4} = e\prod_{i=1}^{n}\left(\frac{f_{2i+1}e + f_{2i}j}{f_{2i}e + f_{2i-1}j}\right), \qquad x_{5n-3} = d\prod_{i=1}^{n}\left(\frac{f_{2i+1}d + f_{2i}i}{f_{2i}d + f_{2i-1}i}\right),$$

$$x_{5n-2} = c\prod_{i=1}^{n}\left(\frac{f_{2i+1}c + f_{2i}h}{f_{2i}c + f_{2i-1}h}\right), \qquad x_{5n-1} = b\prod_{i=1}^{n}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right),$$

$$x_{5n} = a\prod_{i=1}^{n}\left(\frac{f_{2i+1}a + f_{2i}f}{f_{2i}a + f_{2i-1}f}\right),$$

where $x_{-9} = j,\ x_{-8} = i,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ \{f_m\}_{m=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof. For $n = 0$ result holds. Now suppose that $n > 0$ and that our assumption hold for $n-1,\ n-2$. That is,

$$x_{5n-9} = e\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}e + f_{2i}j}{f_{2i}e + f_{2i-1}j}\right), \qquad x_{5n-14} = e\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}e + f_{2i}j}{f_{2i}e + f_{2i-1}j}\right),$$

$$x_{5n-8} = d\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}d + f_{2i}i}{f_{2i}d + f_{2i-1}i}\right), \qquad x_{5n-13} = d\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}d + f_{2i}i}{f_{2i}d + f_{2i-1}i}\right),$$

$$x_{5n-7} = c\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}c + f_{2i}h}{f_{2i}c + f_{2i-1}h}\right), \qquad x_{5n-12} = c\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}c + f_{2i}h}{f_{2i}c + f_{2i-1}h}\right),$$

$$x_{5n-6} = b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right), \qquad x_{5n-11} = b\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right),$$

$$x_{5n-5} = a\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}a + f_{2i}f}{f_{2i}a + f_{2i-1}f}\right), \qquad x_{5n-10} = a\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}a + f_{2i}f}{f_{2i}a + f_{2i-1}f}\right).$$

Now, it follows from Eq. (5.1) that,

$$\begin{aligned}
x_{5n-1} =& x_{5n-6} + \frac{x_{5n-6}^2}{x_{5n-6} + x_{5n-11}}\\
=& b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right) + \frac{b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right) b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right)}{b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right) + b\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right)}\\
=& b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right) + \frac{\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right)\left(\frac{f_{2n-1}b + f_{2n-2}g}{f_{2n-2}b + f_{2n-3}g}\right)}{\left(\frac{f_{2n-1}b + f_{2n-2}g}{f_{2n-2}b + f_{2n-3}g}\right) + 1}\\
=& b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right)\left(1 + \frac{f_{2n-1}b + f_{2n-2}g}{f_{2n-1}b + f_{2n-2}g + f_{2n-2}b + f_{2n-3}g}\right)\\
=& b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right)\left(1 + \frac{f_{2n-1}b + f_{2n-2}g}{f_{2n}b + f_{2n-1}g}\right)\\
=& b\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right)\left(\frac{f_{2n+1}b + f_{2n}g}{f_{2n}b + f_{2n-1}g}\right).
\end{aligned}$$

Therefore,

$$x_{5n-1} = b\prod_{i=1}^{n}\left(\frac{f_{2i+1}b + f_{2i}g}{f_{2i}b + f_{2i-1}g}\right).$$

Also, we see from Eq. (5.1) that,

$$\begin{aligned}
x_{5n} =& x_{5n-5} + \frac{x_{5n-5}}{x_{5n-5} + x_{5n-10}}\\
=& a\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}a + f_{2i}f}{f_{2i}a + f_{2i-1}f}\right) + \frac{a\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}a + f_{2i}f}{f_{2i}a + f_{2i-1}f}\right) a\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}a + f_{2i}f}{f_{2i}a + f_{2i-1}f}\right)}{\left(\frac{f_{2n-1}a + f_{2n-2}f}{f_{2n-2}a + f_{2n-3}f}\right) + 1}
\end{aligned}$$

$$
\begin{aligned}
&=a\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}a+f_{2i}f}{f_{2i}a+f_{2i-1}f}\right)\left(1+\frac{f_{2n-1}a+f_{2n-2}f}{f_{2n-1}a+f_{2n-2}f+f_{2n-2}a+f_{2n-3}f}\right)\\
&=a\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}a+f_{2i}f}{f_{2i}a+f_{2i-1}f}\right)\left(\frac{f_{2n+1}a+f_{2n}f}{f_{2n}a+f_{2n-1}f}\right).
\end{aligned}
$$

Therefore,

$$x_{5n}=a\prod_{i=1}^{n}\left(\frac{f_{2i+1}a+f_{2i}f}{f_{2i}a+f_{2i-1}f}\right).$$

Similarly, one can prove other relations. Hence, the proof is completed. □

5.2. Second equation

In this section we solve the specific form of the Eq. (1.1).

$$x_{n+1}=x_{n-4}+\frac{x_{n-4}^{2}}{x_{n-4}-x_{n-9}}, \tag{5.2}$$

where the initial conditions x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary real numbers.

Theorem 5.2. *Suppose that* $\{x_n\}_{n=-9}^{\infty}$ *be a solution of Eq.* (5.2)*. Then for* $n=0,1,2,\ldots$ *we see that*

$$x_{5n-4}=e\prod_{i=1}^{n}\left(\frac{f_{2i+1}e-f_{2i}j}{f_{2i}e-f_{2i-1}j}\right),\qquad x_{5n-3}=d\prod_{i=1}^{n}\left(\frac{f_{2i+1}d-f_{2i}i}{f_{2i}d-f_{2i-1}i}\right),$$

$$x_{5n-2}=c\prod_{i=1}^{n}\left(\frac{f_{2i+1}c-f_{2i}h}{f_{2i}c-f_{2i-1}h}\right),\qquad x_{5n-1}=b\prod_{i=1}^{n}\left(\frac{f_{2i+1}b-f_{2i}g}{f_{2i}b-f_{2i-1}g}\right),$$

$$x_{5n}=a\prod_{i=1}^{n}\left(\frac{f_{2i+1}a-f_{2i}f}{f_{2i}a-f_{2i-1}f}\right).$$

Proof. Same as the proof of Theorem 5.1 and will be omitted. □

We will confirm our result by considering some numerical examples assume for Eq. (5.1) that $x_{-9}=5$, $x_{-8}=1$, $x_{-7}=4$, $x_{-6}=3$, $x_{-5}=7$, $x_{-4}=8$, $x_{-3}=9$, $x_{-2}=2$, $x_{-1}=9$, $x_0=4$ (See Figure 3) and when we take $x_{-9}=11$, $x_{-8}=9$, $x_{-7}=5$, $x_{-6}=2$, $x_{-5}=6$, $x_{-4}=2$, $x_{-3}=10$, $x_{-2}=8$, $x_{-1}=5$, $x_0=12$ for Eq. (5.2) (see Figure 4).

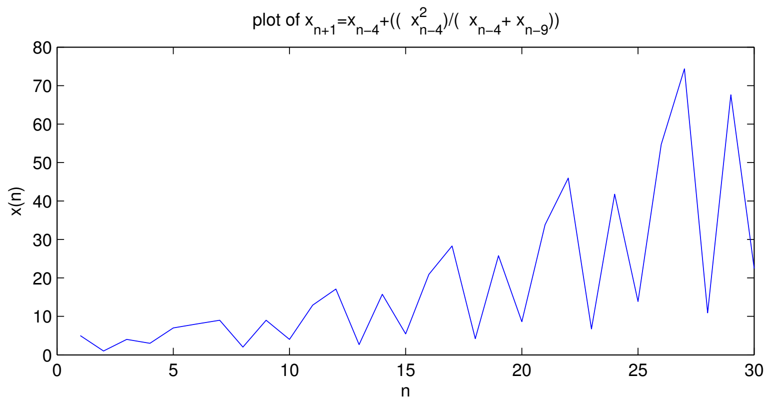

Figure 3

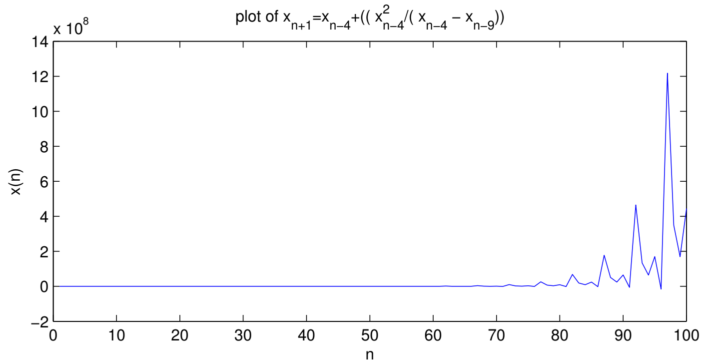

Figure 4

5.3. Third equation

In this section we deal with the form of the solutions of Eq. (1.1).

$$x_{n+1} = x_{n-4} - \frac{x_{n-4}^2}{x_{n-4} + x_{n-9}}, \tag{5.3}$$

where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary real numbers.

Theorem 5.3. *The solution of Eq.* (5.3) *will takes the following formulas for* $n = 0, 1, 2, ...$

$$\begin{aligned} x_{5n-1} &= \frac{bg}{f_n b + f_{n+1} g}, & x_{5n-2} &= \frac{ch}{f_n c + f_{n+1} h}, \\ x_{5n-3} &= \frac{di}{f_n d + f_{n+1} i}, & x_{5n-4} &= \frac{ej}{f_n e + f_{n+1} j}, \\ x_{5n} &= \frac{af}{f_n a + f_{n+1} f}. \end{aligned}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1,\ n-2$. That is,

$$\begin{aligned} x_{5n-9} &= \frac{ej}{f_{n-1} e + f_n j}, & x_{5n-14} &= \frac{ej}{f_{n-2} e + f_{n-1} j}, \\ x_{5n-8} &= \frac{di}{f_{n-1} d + f_n i}, & x_{5n-13} &= \frac{di}{f_{n-2} d + f_{n-1} i}, \\ x_{5n-7} &= \frac{ch}{f_{n-1} c + f_n h}, & x_{5n-12} &= \frac{ch}{f_{n-2} c + f_{n-1} h}, \\ x_{5n-6} &= \frac{bg}{f_{n-1} b + f_n g}, & x_{5n-11} &= \frac{bg}{f_{n-2} b + f_{n-1} g}, \\ x_{5n-5} &= \frac{af}{f_{n-1} a + f_n f}, & x_{5n-10} &= \frac{af}{f_{n-2} a + f_{n-1} f}. \end{aligned}$$

Now, it follows from Eq. (5.3) that,

$$\begin{aligned} x_{5n-1} =& x_{5n-6} + \frac{x_{5n-6}^2}{x_{5n-6} + x_{5n-11}} \\ =& \frac{bg}{f_{n-1} b + f_n g} - \left(\frac{\frac{bg}{f_{n-1} b + f_n g} \frac{bg}{f_{n-1} b + f_n g}}{\frac{bg}{f_{n-1} b + f_n g} + \frac{bg}{f_{n-2} b + f_{n-1} g}} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{bg}{f_{n-1}b+f_ng} - \left(\frac{\frac{bg}{f_{n-2}b+f_{n-1}g}\left(f_{n-2}b+f_{n-1}g\right)}{f_{n-2}b+f_{n-1}g+f_{n-1}b+f_ng} \right) \\
&= \frac{bg}{f_{n-1}b+f_ng}\left(1-\frac{f_{n-2}b+f_{n-1}g}{f_{n-2}b+f_{n-1}g+f_{n-1}b+f_ng}\right) \\
&= \frac{bg}{f_{n-1}b+f_ng}\left(\frac{f_{n-2}b+f_{n-1}g+f_{n-1}b+f_ng+f_{n-2}b+f_{n-1}g}{f_{n-2}b+f_{n-1}g+f_{n-1}b+f_ng}\right) \\
&= \frac{bg}{f_{n-1}b+f_ng}\left(\frac{f_{n-1}b+f_ng}{f_nb+f_{n+1}g}\right).
\end{aligned}
$$

Thus

$$x_{5n-1} = \frac{bg}{f_nb+f_{n+1}g}.$$

Also, from Eq. (5.3), we see that

$$
\begin{aligned}
x_{5n} &= x_{5n-5} - \frac{x_{5n-10}^2}{x_{5n-10}+x_{5n-12}} \\
&= \frac{af}{f_{n-1}a+f_nf} - \left(\frac{\frac{af}{f_{n-1}a+f_nf}\frac{af}{f_{n-1}a+f_nf}}{\frac{af}{f_{n-1}a+f_nf}+\frac{af}{f_{n-2}a+f_{n-1}f}} \right) \\
&= \frac{af}{f_{n-1}a+f_nf} - \left(\frac{\frac{af}{f_{n-1}a+f_nf}\left(f_{n-2}a+f_{n-1}f\right)}{f_{n-2}a+f_{n-1}f+f_{n-1}a+f_nf} \right) \\
&= \frac{af}{f_{n-1}a+f_nf}\left(\frac{f_{n-1}a+f_nf}{f_na+f_{n+1}f}\right).
\end{aligned}
$$

Then,

$$x_{5n} = \frac{af}{f_na+f_{n+1}f}.$$

Hence, the proof is completed. □

We consider a numerical example of this special case assume $x_{-9} = 5$, $x_{-8} = 8$, $x_{-7} = 2$, $x_{-6} = 7$, $x_{-5} = 9$, $x_{-4} = 12$, $x_{-3} = 9$, $x_{-2} = 11$, $x_{-1} = 6$, $x_0 = 12$ (See Figure 5).

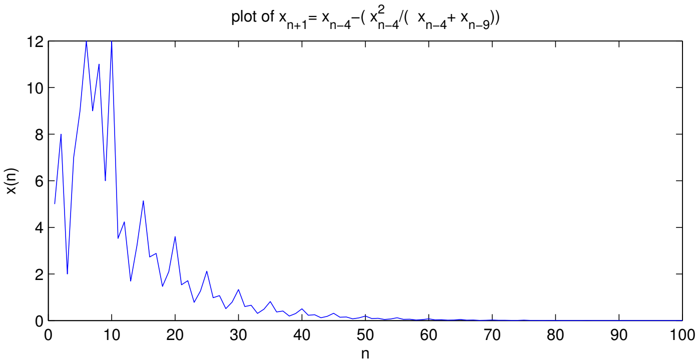

Figure 5

5.4. *Fourth equation*

In this section we obtain the expressions of the solutions of Eq. (1.1).

$$x_{n+1} = x_{n-4} - \frac{x_{n-4}^2}{x_{n-4} - x_{n-9}}, \tag{5.4}$$

where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary real numbers.

Theorem 5.4. *Assume that $\{x_n\}_{n=-9}^{\infty}$ be a solution of Eq. (5.4). Then every solution of Eq. (5.4) is periodic with period 30. Moreover, $\{x_n\}_{n=-9}^{\infty}$ takes the form*

$$\left\{ \begin{array}{c} j, i, h, g, f, e, d, c, b, a, \frac{-ej}{e-j}, \frac{-di}{d-i}, \frac{-ch}{c-h}, \frac{-bg}{b-g}, \frac{-af}{a-f}, \\ -j, -i, -h, -g, -f, -e, -d, -c, -b, -a, \frac{ej}{e-j}, \frac{di}{d-i}, \\ \frac{ch}{c-h}, \frac{bg}{b-g}, \frac{af}{a-f}, j, i, h, g, f, e, d, c, b, a, ..., \end{array} \right\}$$

or,

$$\begin{array}{lll} x_{30n-9} = j, & x_{30n-8} = i, & x_{30n-7} = h, \\ x_{30n-6} = g, & x_{30n-5} = f, & x_{30n-4} = e, \\ x_{30n-3} = d, & x_{30n-2} = c, & x_{30n-1} = b, \\ x_{30n} = a, & x_{30n+1} = \frac{-ej}{e-j}, & x_{30n+2} = \frac{-di}{d-i}, \\ x_{30n+3} = \frac{-ch}{c-h}, & x_{30n+4} = \frac{-bg}{b-g}, & x_{30n+5} = \frac{-af}{a-f}, \\ x_{30n+6} = -j, & x_{30n+7} = -i, & x_{30n+8} = -h, \\ x_{30n+9} = -g, & x_{30n+10} = -f, & x_{30n+11} = -e, \\ x_{30n+12} = -d, & x_{30n+13} = -c, & x_{30n+14} = -b, \\ x_{30n+15} = -a, & x_{30n+16} = \frac{ej}{e-j}, & x_{30n+17} = \frac{di}{d-i}, \\ x_{30n+18} = \frac{ch}{c-h}, & x_{30n+19} = \frac{bg}{b-g}, & x_{30n+20} = \frac{af}{a-f}, \end{array}$$

where $x_{-9} = j,\ x_{-8} = i,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof. Same as the proof of Theorem 5.3 and will be omitted. □

Figure 6 shows the solution of Eq. (5.4) when $x_{-9} = 8$, $x_{-8} = 4$, $x_{-7} = 2$, $x_{-6} = 3$, $x_{-5} = 10$, $x_{-4} = 14$, $x_{-3} = 19$, $x_{-2} = 5$, $x_{-1} = 9$, $x_0 = 13$.

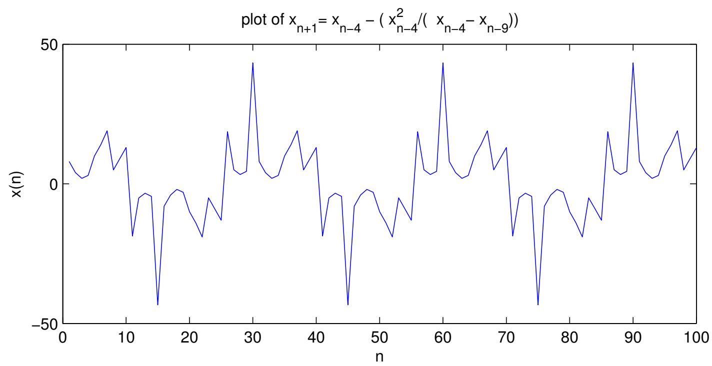

Figure 6

6. Conclusion

In This paper we studied global stability, boundedness and the form of solutions of some special cases of Eq. (1.1). In Section 2, we proved when $b(c+3d) < (c+d)^2(1-a)$, Eq. (1.1) has local stability. In Section 3, we showed that the unique equilibrium of Eq. (1.1) is globally asymptotically stable if $c(1-a) \neq b$. In Section 4, we proved that the solution of Eq. (1.1) is bounded if $\left(a+\frac{b}{c}\right) < 1$. In Section 5, we obtained the form of the solution of four special cases of Eq. (1.1) and gave numerical examples of each of the case with different initial values.

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