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Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mappings in Hilbert spaces

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Abstract

The purpose of this paper is to introduce the implicit midpoint rule of asymptotically nonexpansive mappings in Hilbert spaces. The strong convergence of this viscosity method is proved under certain assumptions imposed on the sequence of parameters. Moreover, it is shown that the limit solves an additional variational inequality. Applications to nonlinear variational inclusion problem, nonlinear Volterra integral equations, variational inequality problem and hierarchical minimization problems are included. The results presented in the paper extend and improve some recent results announced in the current literature.

Keywords: Viscosity, implicit midpoint rule, asymptotically nonexpansive mapping, projection, variational inequality.
2010 MSC: 47J25, 47H09.

1. Introduction

Let H be a Hilbert space, $T : H \to H$ be a nonexpansive mapping and $f : H \to H$ be a contraction. The viscosity approximation method for nonexpansive mapping in Hilbert spaces was introduced by Moudafi [9], following the ideas of Attouch [2]. Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [15].

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The explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where I is the identity of H. It is well known [9, 15] that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point q of T which also solves the variational inequality

$$\langle (I-f)q, x-q\rangle \geq 0, \quad x \in F(T), \tag{1.2}$$

where $F(T)$ is the set of fixed points of T.

The implicit midpoint rule is one of the powerful methods for solving ordinary differential equations; see [3, 4, 7, 10, 11, 13] and the references cited therein. For instance, consider the initial value problem for the differential equation $y'(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$. The implicit midpoint rule is that which generates a sequence $\{y_n\}$ via the relation

$$\frac{1}{h}(y_{n+1} - y_n) = f\left(\frac{y_{n+1}+y_n}{2}\right).$$

The implicit midpoint rule has been extended [1] to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the implicit procedure:

$$x_{n+1} = (1-t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0. \tag{1.3}$$

Recently, Xu et al [16] in a Hilbert spaces introduced the following process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.4}$$

where T is a nonexpansive mapping. They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T, which, in addition, also solves the variational inequality (1.2).

Motivated and inspired by the research going on in this direction. The purpose of this paper is to introduce the *viscosity implicit midpoint rule* for asymptotically nonexpansive mapping in Hilbert space. More precisely, we consider the following implicit iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T^n\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 1. \tag{1.5}$$

Under suitable conditions, some strong converge theorems to a fixed point of the asymptotically nonexpansive mapping are proved. Also, it is shown that the limit solves an additional variational inequality. Applications to nonlinear variational inclusion problem, nonlinear Volterra integral equations, variational inequality problem and hierarchical minimization problems are included. The results presented in the paper extend and improve some recent results announced in the current literature.

2. preliminaries

In the sequel, we always assume that H is a real Hilbert space and C is a nonempty, closed, and convex subset of H. The nearest point projection from H onto C, P_C, is defined by

$$P_C(x) := arg \min_{z\in C} \|x-z\|^2, x \in H. \tag{2.1}$$

Namely, $P_C(x)$ is the only point in C that minimizes the objective $\|x-z\|$ over $z \in C$.

Note that $P_C(x)$ is characterized as follows:

$$P_C(x) \in C \quad \text{and} \quad \langle x - P_C(x), z - P_C(x)\rangle \leq 0 \qquad \text{for all } z \in C. \tag{2.2}$$

Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C.$$

Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad x, y \in C, \quad n \geq 1.$$

The set of fixed points of T is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

Note that if $T : C \to C$ is an asymptotically nonexpansive mapping, then $\mathrm{Fix}(T)$ is always closed and convex. Further if, in addition, C is bounded, then $\mathrm{Fix}(T)$ is nonempty.

The demiclosedness principle of asymptotically nonexpansive mappings is quite helpful in verifying the weak convergence of an algorithm to a fixed point of a asymptotically nonexpansive mapping.

Lemma 2.1 ([6])**.** (Demiclosedness principle). *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H, and $T : C \to C$ be a asymptotically nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that (i) $\{x_n\}$ weakly converges to x and (ii) $(I - T)x_n$ converges strongly to 0, then $x = Tx$.*

The following lemmas play an important role in our paper.

Lemma 2.2 ([8])**.** *Let H be a real Hilbert space. $x, y \in H$ and $t \in [0, 1]$. Then*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

It is easy to prove that the following lemma holds:

Lemma 2.3. *Let H be a Hilbert space. Then for all $u, x, y \in H$, the following inequality holds*

$$\|x - u\|^2 \leq \|y - u\|^2 + 2\langle x - y, x - u\rangle.$$

Lemma 2.4 ([14])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \ \forall n \geq 0, \tag{2.3}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$;

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed and convex subset of a Hilbert space H, and $T : C \to C$ be a asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $\alpha \in [0, 1)$. For an arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 1. \tag{3.1}$$

where $\{\alpha_n\} \in (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{k_n^2-1}{\alpha_n} = 0$;

If the following condition is satisfied

(iv) $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$;

then the sequence $\{x_n\}$ converges strongly to $\tilde{x} = P_{F(T)} f(\tilde{x})$, which solves the following variational inequality:

$$\langle (I-f)q, x-q\rangle \geq 0, \quad \forall\, x \in F(T).$$

Proof. We divided the proof into six steps.

Step 1. We prove that $\{x_n\}$ is bounded.

In fact, for any $p \in F(T)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n) T^n\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\| \\
&\leq (1-\alpha_n)\left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\| + \alpha_n \|f(x_n) - p\| \\
&\leq (1-\alpha_n) k_n \left\|\frac{x_n + x_{n+1}}{2} - p\right\| + \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) \\
&\leq \frac{(1-\alpha_n)k_n}{2}\left(\|x_n - p\| + \|x_{n+1} - p\|\right) + \alpha_n\left(\alpha\|x_n - p\| + \|f(p) - p\|\right).
\end{aligned}$$

After simplifying, it follows that

$$(1 - \frac{(1-\alpha_n)k_n}{2})\|x_{n+1} - p\| \leq \frac{(1-\alpha_n)k_n + 2\alpha\alpha_n}{2}\|x_n - p\| + \alpha_n \|f(p) - p\|,$$

that is,

$$\begin{aligned}
\frac{1-(k_n-1)+k_n\alpha_n}{2}\|x_{n+1} - p\| &\leq \frac{1+(k_n-1)-\alpha_n k_n + 2\alpha\alpha_n}{2}\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \frac{1+\varepsilon\alpha_n - \alpha_n k_n + 2\alpha\alpha_n}{2}\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= \frac{1-(k_n - 2\alpha - \varepsilon)\alpha_n}{2}\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \frac{1-(1 - 2\alpha - \varepsilon)\alpha_n}{2}\|x_n - p\| + \alpha_n\|f(p) - p\|.
\end{aligned} \tag{3.2}$$

Also by condition (iii), for any given positive number ε, $0 < \varepsilon < 1-\alpha$, there exists a sufficient large positive integer n_0, such that for any $n \geq n_0$, we have

$$k_n^2 - 1 \leq 2\varepsilon\alpha_n, \quad \textit{and} \quad k_n - 1 \leq \frac{k_n+1}{2}(k_n - 1) \leq \frac{k_n^2-1}{2} \leq \varepsilon\alpha_n. \tag{3.3}$$

Since $\{k_n\} \in [1, +\infty)$ and $k_n - 1 \leq \varepsilon\alpha_n$ for all $n \geq n_0$, then we have

$$1-(k_n-1)+k_n\alpha_n \geq 1 - \varepsilon\alpha_n + k_n\alpha_n = 1 + (k_n - \varepsilon)\alpha_n \geq 1 + (1-\varepsilon)\alpha_n. \tag{3.4}$$

Substituting (3.4) into (3.2), after simplifying we have

$$\|x_{n+1} - p\| \leq \frac{1-(1-2\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}\|x_n - p\| + \frac{2\alpha_n}{1+(1-\varepsilon)\alpha_n}\|f(p) - p\|$$

$$
\begin{aligned}
&\leq \left(1-\frac{2(1-\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}\right)\|x_n-p\|+\frac{2\alpha_n}{1+(1-\varepsilon)\alpha_n}\|f(p)-p\| \\
&\leq \left(1-\frac{2(1-\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}\right)\|x_n-p\|+\frac{2(1-\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}\left(\frac{1}{1-\alpha-\varepsilon}\|f(p)-p\|\right) \\
&\leq \max\left\{\|x_n-p\|,\frac{1}{1-\alpha-\varepsilon}\|f(p)-p\|\right\},\ \forall n\geq n_0.
\end{aligned}
$$

By induction we readily obtain

$$
\|x_n-p\|\leq \max\left\{\|x_0-p\|,\frac{1}{1-\alpha-\varepsilon}\|f(p)-p\|\right\},\ \forall n\geq n_0.
$$

Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{T^n x_n\}$ and $\left\{T^n(\frac{x_n+x_{n+1}}{2})\right\}$.

Step 2. We show that $\lim_{n\to\infty}||x_{n+1}-x_n||=0$. Observe that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq \|x_{n+1}-T^nx_n\|+\|T^nx_n-x_n\| \\
&= \left\|\alpha_nf(x_n)+(1-\alpha_n)T^n\left(\frac{x_n+x_{n+1}}{2}\right)-T^nx_n\right\|+\|T^nx_n-x_n\| \\
&\leq (1-\alpha_n)\left\|T^n\left(\frac{x_n+x_{n+1}}{2}\right)-T^nx_n\right\|+\alpha_n\|f(x_n)-T^nx_n\|+\|T^nx_n-x_n\| \\
&\leq (1-\alpha_n)k_n\left\|\frac{x_n+x_{n+1}}{2}-x_n\right\|+\alpha_n\|f(x_n)-T^nx_n\|+\|T^nx_n-x_n\| \\
&\leq \frac{(1-\alpha_n)k_n}{2}\|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-T^nx_n\|+\|T^nx_n-x_n\| \\
&\leq \frac{(1-\alpha_n)k_n}{2}\|x_{n+1}-x_n\|+\alpha_nM+\|T^nx_n-x_n\|.
\end{aligned}
$$

Here $M>0$ is a constant such that

$$
M\geq \sup\{\|f(x_n)-T^nx_n\|,\ n\geq 1\}.
$$

It turns out that

$$
(1-\frac{(1-\alpha_n)k_n}{2})\|x_{n+1}-x_n\|\leq \alpha_nM+\|T^nx_n-x_n\|.
$$

By (3.3) $1-\frac{(1-\alpha_n)k_n}{2}=\frac{1-(k_n-1)+k_n\alpha_n}{2}\geq\frac{1+(1-\varepsilon)\alpha_n}{2}$ for all $n\geq n_0$. Consequently, we arrive at

$$
\|x_{n+1}-x_n\|\leq \frac{2\alpha_n}{1+(1-\varepsilon)\alpha_n}M+\frac{1}{1+(1-\varepsilon)\alpha_n}\|T^nx_n-x_n\|.
$$

By virtue of the conditions (i) and (iv), we have

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.5}
$$

Step 3. We show that $\lim_{n\to\infty}||x_n-Tx_n||=0$. In fact, since

$$
\begin{aligned}
\|x_n-T^{n-1}x_n| &= \left\|(\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})T^{n-1}\left(\frac{x_{n-1}+x_n}{2}\right)-T^{n-1}x_n\right\| \\
&\leq \alpha_{n-1}\left\|f(x_{n-1})-T^{n-1}x_n\right\|+(1-\alpha_{n-1})\left\|T^{n-1}\left(\frac{x_{n-1}+x_n}{2}\right)-T^{n-1}x_n\right\| \\
&\leq \alpha_{n-1}\left\|f(x_{n-1})-T^{n-1}x_n\right\|+(1-\alpha_{n-1})k_{n-1}\left\|\frac{x_{n-1}+x_n}{2}-x_n\right\| \\
&\leq \alpha_{n-1}\left\|f(x_{n-1})-T^{n-1}x_n\right\|+\frac{(1-\alpha_{n-1})k_{n-1}}{2}\left\|x_{n-1}-x_n\right\|,
\end{aligned}
$$

by condition (i) and (3.5), we have

$$\lim_{n\to\infty} \|x_n - T^{n-1}x_n\| = 0. \tag{3.6}$$

Therefore

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ &\le \|x_n - T^n x_n\| + k_1\|T^{n-1}x_n - x_n\| \to 0.\end{aligned}$$

Step 4. From Step 3 and Lemma 2.1, it is a straightforward consequence that the following weak $\omega-$limit set of $\{x_n\}$:

$$\omega_w(x_n) = \{x \in H : \text{ there exists a subsequence of } \{x_n\} \text{ weakly converging to } x\}$$

is contained in $Fix(T)$.

Step 5. Now we prove

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle \le 0, \tag{3.7}$$

where $q \in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $q = P_{Fix(T)}f(q)$.

As a matter of fact, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a point p and moreover

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \lim_{j\to\infty}\langle f(q) - q, x_{n_j} - q\rangle. \tag{3.8}$$

Since $p \in Fix(T)$, by using (2.2), (3.7), (3.8) and by virtue of Step 4, we can conclude that

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \langle f(q) - q, p - q\rangle \le 0. \tag{3.9}$$

Step 6. Finally, we prove that $x_n \to q \in F(T)$ as $n \to \infty$.

Indeed, for any $n \ge 1$, we set $z_n = \alpha_n q + (1-\alpha_n)T^n\left(\frac{x_n+x_{n+1}}{2}\right)$. It follows from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \|z_n - q\|^2 + 2\langle x_{n+1} - z_n, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)^2\left\|T^n(\frac{x_n + x_{n+1}}{2}) - q\right\|^2 + 2\langle x_{n+1} - z_n, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - q\right\|^2 + 2\langle \alpha_n(f(x_n) - q), x_{n+1} - q\rangle \\ &= (1-\alpha_n)^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - q\right\|^2 + 2\alpha_n\langle f(x_n) - f(q), x_{n+1} - q\rangle \\ &\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - q\right\|^2 + 2\alpha_n\|f(x_n) - f(q)\| \cdot \|x_{n+1} - q\| \\ &\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - q\right\|^2 + 2\alpha_n\alpha\|x_n - q\| \cdot \|x_{n+1} - q\| \\ &\quad + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)^2 k_n^2\left(\frac{1}{2}\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 - \frac{1}{4}\|x_{n+1} - x_n\|^2\right) \\ &\quad + \alpha_n\alpha\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{(1-\alpha_n)^2k_n^2}{2}+\alpha_n\alpha\right)\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right)\\
&\quad + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle\\
&\leq \frac{(1-2\alpha_n)k_n^2+2\alpha_n\alpha}{2}\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right)+\alpha_n^2M_1\\
&\quad + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle.
\end{aligned}
$$

Here $M_1>0$ is a constant such that

$$M_1 \geq \sup\left\{k_n^2\|x_n-q\|^2,\, n\geq 1\right\}.$$

It follows from (3.3) that for all $n \geq n_0$

$$
\begin{aligned}
\left(1-\frac{(1-2\alpha_n)k_n^2+2\alpha_n\alpha}{2}\right)\|x_{n+1}-q\|^2 &\leq \frac{(1-2\alpha_n)k_n^2+2\alpha_n\alpha}{2}\|x_n-q\|^2+\alpha_n^2M_1\\
&\quad + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle\\
&= \frac{1+(k_n^2-1)-2(k_n^2-\alpha)\alpha_n}{2}\|x_n-q\|^2+\alpha_n^2M_1\\
&\quad + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle\\
&\leq \frac{1+2\varepsilon\alpha_n-2(1-\alpha)\alpha_n}{2}\|x_n-q\|^2+\alpha_n^2M_1\\
&\quad + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle\\
&= \frac{1-2(1-\varepsilon-\alpha)\alpha_n}{2}\|x_n-q\|^2+\alpha_n^2M_1\\
&\quad + 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle.
\end{aligned}
$$

Since

$$
\begin{aligned}
1-\frac{(1-2\alpha_n)k_n^2+2\alpha\alpha_n}{2} &= \frac{1-(k_n^2-1)+2(k_n^2-\alpha)\alpha_n}{2}\\
&\geq \frac{1-2\varepsilon\alpha_n+2(1-\alpha)\alpha_n}{2}\\
&= \frac{1+2(1-\varepsilon-\alpha)\alpha_n}{2},\ \forall n\geq n_0,
\end{aligned}
$$

consequently, we arrive at

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \frac{1-2(1-\varepsilon-\alpha)\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}\|x_n-q\|^2+\frac{2\alpha_n^2}{1+2(1-\varepsilon-\alpha)\alpha_n}M_1\\
&\quad + \frac{4\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}\langle f(q)-q, x_{n+1}-q\rangle\\
&= \left(1-\frac{4(1-\varepsilon-\alpha)\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}\right)\|x_n-q\|^2+\frac{2\alpha_n^2}{1+2(1-\varepsilon-\alpha)\alpha_n}M_1\\
&\quad + \frac{4\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}\langle f(q)-q, x_{n+1}-q\rangle.
\end{aligned}
$$

Now, take $\gamma_n = \frac{4(1-\varepsilon-\alpha)\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}$, $\delta_n = \frac{2\alpha_n^2}{1+2(1-\varepsilon-\alpha)\alpha_n}M_1 + \frac{4\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}\langle f(q)-q, x_{n+1}-q\rangle$. It follows from conditions (i), (ii) and (3.7) that $\{\gamma_n\}\subset(0,1)$, $\sum_{n=1}^{\infty}\gamma_n=\infty$ and

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty}\frac{1}{2(1-\varepsilon-\alpha)}\Big(\alpha_nM_1+2\langle f(q)-q, x_{n+1}-q\rangle\Big)\leq 0.$$

From Lemma 2.4 we have that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Remark 3.2. Since every nonexpansive mapping is an asymptotically nonexpansive mapping, Theorem 3.1 is an improvement and generalization of the main results in Alghamdi et al. [1] and Xu et al. [16].

The following result can be obtained from Theorem 3.1 immediately.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let f be a contraction on C with coefficient $k \in [0, 1)$, and for the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{3.10}$$

where $\{\alpha_n\} \in (0, 1)$ satisfies the conditions: (i), (ii) *and* (iii) *in Theorem* 3.1. *Then the sequence $\{x_n\}$ defined by* (3.10) *converges strongly to q such that $q = P_{Fix(T)}f(q)$ which is also a solution of the following variational inequality:*

$$\langle q - f(q),\ x - q\rangle \geq 0, \quad \forall\, x \in Fix(T).$$

Proof. It suffices to prove that the following condition is satisfied:

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.11}$$

In fact, by the same method as given in [16] we can prove that $\|x_n - x_{n+1}\| \to 0$ (as $n \to \infty$). Therefore we have

$$\begin{aligned}
\|x_n - Tx_n\| &= \left\|\alpha_{n-1}f(x_{n-1}) + (1 - \alpha_{n-1})T\left(\frac{x_{n-1} + x_n}{2}\right) - Tx_n\right\| \\
&\leq \alpha_{n-1}\|f(x_{n-1}) - Tx_n\| + \frac{(1 - \alpha_{n-1})}{2}\|x_{n-1} - x_n\| \\
&\leq \alpha_{n-1}M + \frac{(1 - \alpha_{n-1})}{2}\|x_{n-1} - x_n\| \to 0 \ \ (as\ n \to \infty),
\end{aligned}$$

where $M = \sup_{n\geq 2}\|f(x_{n-1}) - Tx_n\|$. This completes the proof of Theorem 3.3. □

4. Applications

4.1. Application to nonlinear variational inclusion problem

Let H be a real Hilbert space, $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then, the *resolvent mapping $J^M_\lambda : H \to H$ associated with M*, is defined by

$$J^M_\lambda(x) := (I + \lambda M)^{-1}(x),\ \forall x \in H \tag{4.1}$$

for some $\lambda > 0$, where I stands identity operator on H. We note that for all $\lambda > 0$ the resolvent operator J^M_λ is a single-valued nonexpansive mapping.

The "so-called" *monotone variational inclusion problem (in short, MVIP)* is to find $x^* \in H$ such that

$$0 \in M(x^*). \tag{4.2}$$

From the definition of resolvent mapping J^M_λ, it is easy to know that (MVIP) (4.2) is equivalent to find $x^* \in H$ such that

$$x^* \in Fix(J^M_\lambda) \ \ for\ some\ \lambda > 0. \tag{4.3}$$

For any given function $x_0 \in H$, define a sequence by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J^M_\lambda\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geq 0. \tag{4.4}$$

From Theorem 3.3 we have the following,

Theorem 4.1. *Let M, J_λ^M be the same as above. Let $f : H \to H$ be a contraction. Let $\{x_n\}$ be the sequence defined by* (4.4)*. If the sequence $\{\alpha_n\} \in (0,1)$ satisfies the conditions:* (i), (ii) *and* (iii) *in Theorem* 3.1 *and $Fix(J_\lambda^M) \neq \emptyset$, then $\{x_n\}$ converges strongly to the solution of monotone variational inclusion* (4.2)*, which is also a solution of the following variational inequality:*

$$\langle \tilde{x} - f(\tilde{x}), x - \tilde{x} \rangle \geq 0, \quad \forall\, x \in Fix(J_\lambda^M).$$

4.2. Application to nonlinear Volterra integral equations

Let us consider the following nonlinear Volterra integral equation

$$x(t) = g(t) + \int_0^t F\left(t, s, x(s)\right) ds, \quad t \in [0,1], \tag{4.5}$$

where g is a continuous function on $[0,1]$ and $F : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition

$$|F(t,s,x) - F(t,s,y)| \leq |x-y|, \quad t,s \in [0,1] \quad x,y \in \mathbb{R}.$$

Define a mapping $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tx)(t) = g(t) + \int_0^t F(t,s,x(s))ds, \quad t \in [0,1]. \tag{4.6}$$

It is easy to see that T is a nonexpansive mapping. This means that to find the solution of integral equation (4.5) is reduced to find a fixed point of the nonexpansive mapping T in $L^2[0,1]$.

For any given function $x_0 \in L^2[0,1]$, define a sequence of functions $\{x_n\}$ in $L^2[0,1]$ by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geq 0. \tag{4.7}$$

From Theorem 3.3 we have the following,

Theorem 4.2. *Let $F,\ g,\ T, L^2[0,1]$ be the same as above. Let f be a contraction on $L^2[0,1]$ with coefficient $k \in [0,1)$. Let $\{x_n\}$ be the sequence defined by* (4.7)*. If the sequence $\{\alpha_n\} \in (0,1)$ satisfies the conditions:* (i), (ii) *and* (iii) *in Theorem* 3.1 *and $Fix(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly in $L^2[0,1]$ to the solution of integral equation* (4.5) *which is also a solution of the following variational inequality:*

$$\langle \tilde{x} - f(\tilde{x}), x - \tilde{x} \rangle \geq 0, \quad \forall\, x \in Fix(T).$$

4.3. Application to variational inequalities

Consider the variational inequality (VI)

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad x \in C, \tag{4.8}$$

where A is a (single-valued) monotone operator in Hilbert space H and C is a closed convex subset of H with $C \subset \mathrm{dom}(A)$.

An example of (4.8) is the constrained minimization problem

$$\min_{x \in C} \varphi(x), \tag{4.9}$$

where $\varphi : H \to \mathbb{R}$ is a proper convex and lower-semicontinuous function. If φ is (Fréchet) differentiable, then the minimization problem (4.9) is equivalently reformulated as (4.8) with $A = \nabla\varphi$.

Notice that the VI (4.8) is equivalent to the fixed point problem, for any $\lambda > 0$,

$$Tx^* = x^*, \qquad Tx := P_C(I - \lambda A)x. \tag{4.10}$$

If A is Lipschitzian and strongly monotone, then, for $\lambda > 0$ small enough, T is a contraction and its unique fixed point is also the unique solution of the VI (4.8). However, if A is not strongly monotone, T is no longer a contraction, in general. In this case we must deal with nonexpansive mappings for solving the VI (4.8). More precisely, we assume

(A1) A is $L-$Lipschitzian for some $L > 0$, that is,

$$\|Ax - Ay\| \leq L\|x - y\|, \quad x, y \in H.$$

(A2) A is $\mu-$inverse strongly monotone ($\mu-$ism) for some $\mu > 0$, namely,

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax - Ay\|^2, \quad x, y \in H.$$

Note that if $\nabla\varphi$ is $L-$Lipschtzian, then $\nabla\varphi$ is $\frac{1}{L}-$ism.

Under the conditions (A1) and (A2), it is well known [5] that the operator $T = P_C(I-\lambda A)$ is nonexpansive provided $0 < \lambda < 2\mu$. Applying Theorem 3.3 we can get the following result:

Theorem 4.3. *Assume the VI* (4.8) *is solvable. Assume also A satisfies (A1) and (A2), and $0 < \lambda < 2\mu$. Let $f : C \to C$ be a contraction. Define a sequence $\{x_n\}$ by the viscosity implicit midpoint rule:*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)P_C(I - \lambda A)\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0.$$

In addition, assume $\{\alpha_n\}$ satisfies the conditions (i)–(iii) *in Theorem* 3.1. *Then $\{x_n\}$ converges in norm to a solution x^* of the VI* (4.8) *which is also a solution to the VI*

$$\langle(I - f)(x^*), x - x^*\rangle \geq 0, \quad x \in A^{-1}(0). \tag{4.11}$$

4.4. Application to hierarchical minimization

We next consider a *hierarchical minimization problem* (see [12] and references cited therein). Let $\varphi_0, \varphi_1 : H \to R$ be a lower semicontinuous convex function. Consider the following hierarchical minimization problem:

$$\min_{x\in S_0} \varphi_1(x), \qquad S_0 := \arg\min_{x\in H} \varphi_0(x). \tag{4.12}$$

Here we always assume that S_0 is nonempty. Let $S := \arg\min_{x\in S_0} \varphi_1(x)$ and assume $S \neq \emptyset$.

Assume φ_0 and φ_1 are differentiable and their gradients satisfy the Lipschitz continuity conditions:

$$\|\nabla\varphi_0(x) - \nabla\varphi_0(y)\| \leq L_0\|x - y\|, \qquad \|\nabla\varphi_1(x) - \nabla\varphi_1(y)\| \leq L_1\|x - y\|. \tag{4.13}$$

Note that the condition (4.13) implies that $\nabla\varphi_i$ is $\frac{1}{L_i} - ism(i = 0, 1)$. Now let

$$T_0 = I - \gamma_0\nabla\varphi_0, \quad T_1 = I - \gamma_1\nabla\varphi_1,$$

where $\gamma_0 > 0$ and $\gamma_1 > 0$. Note that T_i is nonexpansive [5] if $0 < \gamma_i < \frac{2}{L_i}(i = 0, 1)$. Also, it is easily seen that $S_0 = Fix(T_0)$.

The optimality condition for $x^* \in S_0$ to be a solution of the hierarchical minimization (4.12) is the VI:

$$x^* \in S_0, \ \langle\nabla\varphi_1(x^*), x - x^*\rangle \geq 0, \quad x \in S_0. \tag{4.14}$$

This is the VI (4.8) with $C = S_0$ and $A = \nabla\varphi_1$. From Theorem 3.3 we have the following result.

Theorem 4.4. *Assume the hierarchical minimization problem* (4.12) *is solvable. Let $f : C \to C$ be a contraction. Define a sequence $\{x_n\}$ by the viscosity implicit midpoint rule:*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)P_{S_0}(I - \lambda\nabla\varphi_1)\left(\frac{x_n + x_{n+1}}{2}\right).$$

In addition, assume $\{\alpha_n\}$ satisfies the conditions (i)–(iii) *in Theorem* 3.1. *If the condition* (4.13) *is satisfied and $0 < \gamma_i < \frac{2}{L_i}(i = 0, 1)$, then $\{x_n\}$ converges in norm to a solution x^* of the VI* (4.14) *that is, a solution of hierarchical minimization problem* (4.12) *which also solves the VI*

$$\langle(I - f)x^*, x - x^*\rangle \geq 0, \quad x \in S.$$

Remark 4.5. As we have observed that Theorem 3.1 can be viewed as an extension of the main result in [1, 16]. It remains an open question whether Theorem 3.1 holds without the condition (iv), that is, we have the following:

Open Question Let C be a nonempty closed and convex subset of a Hilbert space H, and $T : C \to C$ be a asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $\alpha \in [0, 1)$. For an arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be the sequence generated by (3.1). If the sequence $\{\alpha_n\} \in (0, 1)$ satisfies the conditions (i)-(iii) in Theorem 3.1, does the conclusion of Theorem 3.1 hold?

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