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Stability of Pexiderized quadratic functional equation on a set of measure zero

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Abstract

Let $\mathbb{R}$ be the set of real numbers and Y a Banach space. We prove the Hyers-Ulam stability theorem when $f, h : \mathbb{R} \to Y$ satisfy the following Pexider quadratic inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2h(y)\| \leq \epsilon,$$

in a set $\Omega \subset \mathbb{R}^2$ of Lebesgue measure $m(\Omega) = 0$.

Keywords: Pexider quadratic functional equation, Hyers-Ulam stability, first category Lebesgue measure, Baire category theorem.
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1. Introduction and preliminaries

In 1940, Ulam proposed the general Ulam stability problem (see [29]):

Let $(G, \cdot)$ be a group and let $(G', \cdot, d)$ be a metric group with the metric d. Given $\delta > 0$, does there exist $\epsilon > 0$ such that if a mapping $h : G \to G'$ satisfies the inequality

$$d(h(xy), h(x)h(y)) \leq \delta$$

for all $x, y \in G$, then there is a homomorphism $H : G \to G'$ with

$$d(h(x), H(x)) \leq \epsilon$$

for all $x \in G$?

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In 1941, Hyers [13] considered the case of approximately additive mappings $f : E \to F$, where E and F are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$ and $\epsilon > 0$. He proved that there exists a unique additive mapping $T : E \to F$ satisfying

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in E$. Aoki [1] and Bourgin [3] considered the stability problem with unbounded Cauchy differences. In 1978, Th. M. Rassias [23] provided a generalization of Hyers theorem which allows the Cauchy difference to be unbounded.

Theorem 1.1. *Let $f : E \to F$ be a mapping from a real normed vector space E into a Banach space F satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E\backslash\{0\}$, where θ and p are constants with $\theta > 0$ and $p \neq 1$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p$$

for all $x \in E\backslash\{0\}$.

Theorem 1.1 is due to Aoki [1] for $0 < p < 1$ (see also [23]); Gajda [12] for $p > 1$; Hyers [13] for $p = 0$ and Th. M. Rassias [24] for $p < 0$ (see [27, page 326], and [3]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [28], for mappings $f : E \to F$, where E is a normed space and F is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain E is replaced by an Abelian group. Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 10, 11, 14, 15, 17–19, 21, 25, 26]).

We say that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Drygas equation if

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in \mathbb{R}. \tag{1.2}$$

The above equation was introduced in [8] in order to obtain a characterization of the quasi-inner-product spaces. Ebanks, Kannappan and Sahoo [9] have obtained the general solution of the Eq. (1.2) as

$$f(x) = A(x) + Q(x), \quad x \in \mathbb{R},$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive mapping and $Q : \mathbb{R} \to \mathbb{R}$ is a quadratic mapping.

The stability in the Hyers-Ulam sense of the Drygas equation has been investigated by Jung and Sahoo in [16].

Theorem 1.2 ([16])**.** *Let $f, g : E \to F$ be a mapping from a real normed vector space E into a Banach space F satisfying the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - g(2y)\| \leq \epsilon, \quad x, y \in E$$

for same $\epsilon > 0$, then there exist a unique additive function $A : E \to F$ and a unique quadratic function $Q : E \to F$ such that

$$\|f(x) - A(x) - 2Q(x) - f(0)\| \leq \frac{37\epsilon}{6},$$

and

$$\|g(x) - Q(x)\| \leq \frac{13\epsilon}{3}$$

for allx $\in E$. If, in particular, f satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \epsilon, \quad x, y \in E,$$

then there exist a unique additive function $A : E \to F$ and a unique quadratic function $Q : E \to F$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \frac{25\epsilon}{3}$$

for allx $\in E$.

Piszczek and Szczawińska [22] obtained the stability of the equation (1.2) on a restricted domain. In 2013, Chung [5] investigated the stability of a conditional Cauchy equation on a set of measure zero.

In 2014, Chung and J. M. Rassias [6] proved the stability of the quadratic functional equation (1.1) in a set of measure zero.

Throughout this paper, let X be a normed space and Y a Banach space.

Our aim is to prove the Hyers-Ulam stability on a set Lebesgue measure 0 of the Pexider quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2h(y), \tag{1.3}$$

where $f, h : X \to Y$ are functions. Using the result, we obtain an asymptotic behavior of the equation.

2. Stability of the Eq. (1.3) in set of measure zero

Throughout this section, we assume that $\Omega \subset X^2$ satisfies the following conditions: for given $x, y \in X$ there exist $t, t' \in X$ such that

(C1) $\{(x+y,t), (x-y,t), (x,y+t), (x,y-t), (0,y+t), (0,y-t), (y,t), (-y,t)\} \subset \Omega$;
(C2) $\{(t', x+y), (t', x-y), (t'+x, y), (t'-x, y), (t', x)\} \subset \Omega$.

We prove the Ulam-Hyers stability of (1.3) in Ω.

Theorem 2.1. *Suppose that $f, h : X \to Y$ satisfy the following Pexider quadratic functional inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2h(y)\| \leq \epsilon \tag{2.1}$$

for all $(x, y) \in \Omega$ and some constant $\epsilon \geq 0$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x) - f(0)\| \leq \frac{100}{3}\epsilon,$$

and

$$\|h(x) - Q(x)\| \leq \frac{3}{2}\epsilon$$

for all $x \in X$.

Proof. Let $D(x, y) = f(x+y) + f(x-y) - 2f(x) - 2h(y)$, $D_1(x, y) = f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) + 2f(0)$ and $f, h : X \to Y$ be functions satisfying (2.1) for all $(x, y) \in \Omega$. Since Ω satisfies $(C1)$, for given $x, y \in X$, there exists $t \in X$ such that

$$\begin{aligned}2D_1(x,y) =& 2f(x+y)+2h(t)-f(x+y+t)-f(x+y-t)\\&+2f(x-y)+2h(t)-f(x-y+t)-f(x-y-t)\\&+f(x+y+t)+f(x-y-t)-2f(x)-2h(y+t)\\&+f(x+y-t)+f(x-y+t)-2f(x)-2h(y-t)\\&+2h(y+t)+2f(0)-f(y+t)-f(-y-t)\\&+2h(y-t)+2f(0)-f(y-t)-f(-y+t)\\&+f(y+t)+f(y-t)-2f(y)-2h(t)\\&+f(-y+t)+f(-y-t)-2f(-y)-2h(t)\\=&-D(x+y,t)-D_1(x-y,t)+D(x,y+t)+D_1(x,y-t)\\&-D(0,y+t)-D_1(0,y-t)+D(y,t)+D(-y,t),\end{aligned}$$

and

$$\begin{aligned}&\|D(x+y,t)\|\le\epsilon,\ \ \|D(x-y,t)\|\le\epsilon,\ \ \|D(x,y+t)\|\le\epsilon,\ \ \|D(x,y-t)\|\le\epsilon,\\&\|D(0,y+t)\|\le\epsilon,\ \ \|D(0,y-t)\|\le\epsilon,\ \ \|D(y,t)\|\le\epsilon,\ \ \|D(-y,t)\|\le\epsilon.\end{aligned}$$

Thus, using the triangle inequality, we have

$$\|D_1(x,y)\|\le 4\epsilon$$

for all $x, y \in X$. Now, by Theorem 1.2, there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q_1 : X \to Y$ such that

$$\|f(x)-f(0)-A(x)-Q_1(x)\|\le\frac{100}{3}\epsilon \tag{2.2}$$

for all $x \in X$.

Let $D_2(x,y) = h(x+y)+h(x-y)-2h(x)-2h(y)$. Since Ω satisfies $(C2)$, for given $x, y \in X$, there exists $t' \in X$ such that

$$2D_2(x,y) = -D(t',x+y)-D(t',x-y)+D(t'+x,y)+D(t'-x,y)+2D(t',x),$$

and

$$\begin{aligned}&\|D(t',x+y)\|\le\epsilon,\ \ \|D(t',x-y)\|\le\epsilon,\ \ \|D(t'+x,y)\|\le\epsilon,\ \ \|D(t'-x,y)\|\le\epsilon,\\&\|2D(t',x)\|\le 2\epsilon.\end{aligned}$$

Thus, using the triangle inequality, we have

$$\|D_2(x,y)\|\le 3\epsilon \tag{2.3}$$

for all $x, y \in X$. Now, by [6, Theorem 1.1], there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\|h(x)-Q_2(x)\|\le\frac{3}{2}\epsilon \tag{2.4}$$

for all $x \in X$. It remains to prove that $Q_1 = Q_2$. From condition (C1), for given $y \in X$, there exists $t \in X$ such that

$$\begin{aligned}2f(y)+2f(-y)-4f(0)-4f(y) =& f(y+t)+f(-y-t)-2f(0)-2h(y+t)\\&+f(y-t)+f(-y+t)-2f(0)-2h(y-t)\\&+2h(y+t)+2h(y-t)-4h(y)-4h(t)\\&-f(y+t)-f(y-t)+2f(y)+2h(t)\\&-f(-y+t)-f(-y-t)+2f(y)+2h(t).\end{aligned}$$

It flows from (2.1) and (2.3) that

$$\|f(y)+f(-y)-2f(0)-2h(y)\| \le 5\epsilon \tag{2.5}$$

for all $y \in X$. Using inequalities (2.2), (2.4) and (2.5), we have

$$\begin{aligned}\|2Q_1(x)-2Q_2(x)\| \le & \|Q_1(x)+A(x)+f(0)-f(x)\| + \|Q_1(x)+A(-x)+f(0)-f(-x)\| \\ & + \|-2Q_2(x)+2h(x)\| + \|f(x)+f(-x)-2f(0)-2h(x)\| \\ \le & \frac{224}{3}\epsilon,\end{aligned}$$

and using the bi-additivity of Q_1 and Q_2, we have $Q_1 = Q_2$. This completes the proof. □

Corollary 2.2. *Suppose that $f, h : X \to Y$ satisfy*

$$f(x+y)+f(x-y)-2f(x)-2h(y) = 0 \tag{2.6}$$

for all $(x, y) \in \Omega$. Then Eq. (2.6) holds for all $x, y \in X$.

3. Applications

In this section, we construct some sets Ω of measure zero satisfying the conditions $(C1)$ and $(C2)$ when $X = \mathbb{R}$. The following lemma is a crucial key of the construction given in [20, Theorem 1.6].

Lemma 3.1. *The set $\mathbb{R}$ of real numbers can be partitioned as*

$$\mathbb{R} = F \cup L,$$

where F is of first Baire category, that is, F is a countable union of nowhere dense subsets of $\mathbb{R}$, and L is of Lebesgue measure 0.

Lemma 3.2 ([6])**.** *Let L be a subset of $\mathbb{R}$ of measure 0 such that $L^c := \mathbb{R}\backslash L$ is of first Baire category. Then, for any countable subsets $S \subset \mathbb{R}$, $T \subset \mathbb{R}\backslash\{0\}$ and $d > 0$, there exists $\lambda \ge d$ such that*

$$S + \lambda T = \{s + \lambda\tau : s \in S, \tau \in T\} \subset L. \tag{3.1}$$

Theorem 3.3. *Let $\Omega = e^{-i\frac{\pi}{3}}(L \times L)$ be the rotation of $L \times L$ by $\frac{-\pi}{3}$. Then Ω satisfies the conditions $(C1)$ and $(C2)$ which has two-dimensional Lebesgue measure 0.*

Proof. Let $\Omega = e^{-i\frac{\pi}{3}}(L \times L)$, that is,

$$\Omega = \{(p,q) \in \mathbb{R}^2 : \frac{1}{2}p - \frac{\sqrt{3}}{2}q \in L,\ \frac{\sqrt{3}}{2}p + \frac{1}{2}q \in L\}.$$

Then Ω satisfies all the conditions $(C1)$ and $(C2)$. Let $x, y, t, t' \in \mathbb{R}$ and let

$$P_{x,y,t} = \{(x+y,t),(x-y,t),(x,y+t),(x,y-t),(0,y+t),(0,y-t),(y,t),(-y,t)\},$$

and

$$Q_{x,y,t'} = \{(t',x+y),(t',x-y),(t'+x,y),(t'-x,y),(t',x)\}.$$

Then by the construction of Ω, $(C1)$ is equivalent to the condition that for every $x, y \in \mathbb{R}$ there exists $t \in \mathbb{R}$ such that

$$e^{i\frac{\pi}{3}}P_{x,y,t} \subset L \times L. \tag{3.2}$$

Equation (3.2) is equivalent to

$$B_1 := \{\frac{1}{2}p - \frac{\sqrt{3}}{2}q,\ \frac{\sqrt{3}}{2}p + \frac{1}{2}q : (p,q) \in P_{x,y,t}\} \subset L.$$

The set B_1 is contained in a set of form $S + tT$, where

$$S = \{\frac{1}{2}(x+y), \frac{\sqrt{3}}{2}(x+y), \frac{1}{2}(x-y), \frac{\sqrt{3}}{2}(x-y), \frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y, \pm\frac{\sqrt{3}}{2}y, \pm\frac{1}{2}y\},$$
$$T = \{\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\}.$$

By Lemma 3.2, for given $x, y \in \mathbb{R}$ and $d > 0$, there exists $t \geq d$ such that

$$B_1 \subset S + tT \subset L. \tag{3.3}$$

Thus Ω satisfies $(C1)$.

Similarly,

$$B_2 := \{\frac{1}{2}p - \frac{\sqrt{3}}{2}q,\ \frac{\sqrt{3}}{2}p + \frac{1}{2}q : (p,q) \in Q_{x,y,t'}\} \subset S + t'T \subset L \tag{3.4}$$

for some $t' \in \mathbb{R}$, where

$$S = \{-\frac{\sqrt{3}}{2}(x+y), \frac{1}{2}(x+y), -\frac{\sqrt{3}}{2}(x-y), \frac{1}{2}(x-y), \frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y, -\frac{1}{2}x - \frac{\sqrt{3}}{2}y,$$
$$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y, -\frac{\sqrt{3}}{2}x, \frac{1}{2}x\},$$
$$T = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}.$$

Then Ω satisfies $(C2)$. This completes the proof. □

Corollary 3.4. *Let $\alpha > 0$ and $\Omega_\alpha := \{(p,q) \in \Omega : |p| + |q| \geq \alpha\}$. Then Ω_α satisfies the conditions $(C1)$ and $(C2)$.*

Proof. In view of the proof of Theorem 3.3, (3.3) and (3.4) imply that for every $x, y \in \mathbb{R}$ and $d > 0$ there exist $t, t' \geq d$ such that

$$P_{x,y,t} \subset \Omega \text{ and } Q_{x,y,t'} \subset \Omega. \tag{3.5}$$

For given $x, y \in \mathbb{R}$, if we take $d = \alpha + |x| + |y|$ and if $t, t' \geq d$, then we have

$$P_{x,y,t} \subset \{(p,q) : |p| + |q| \geq \alpha\} \text{ and } Q_{x,y,t'} \subset \{(p,q) : |p| + |q| \geq \alpha\}. \tag{3.6}$$

It follows from (3.5) and (3.6) that for every $x, y \in \mathbb{R}$ there exist $t, t' \in \mathbb{R}$ such that

$$P_{x,y,t} \subset \Omega_\alpha \text{ and } Q_{x,y,t'} \subset \Omega_\alpha.$$

Thus Ω_α satisfies $(C1)$ and $(C2)$. This completes the proof. □

Remark 3.5. As a consequence of Theorem 2.1 and Corollary 3.4, we obtain the asymptotic behavior of f, h satisfying

$$\|f(x+y) + f(x-y) - 2f(x) - 2h(y)\| \to 0, \tag{3.7}$$

as $(x, y) \in \Omega$, $|x| + |y| \to \infty$.

Corollary 3.6. *Suppose that $f, h : \mathbb{R} \to \mathbb{R}$ satisfy the condition (3.7). Then there exist a unique additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a unique quadratic mapping $Q : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = A(x) + Q(x) + f(0) \ \ \text{and} \ \ h(x) = Q(x)$$

for all $x \in \mathbb{R}$.

Proof. By (3.7), for each $n \in \mathbb{N}^*$, there exists $\alpha_n > 0$

$$|f(x+y) + f(x-y) - 2f(x) - 2h(y)| \le \frac{1}{n}$$

for all $(x, y) \in \Omega$, $|x|+|y| \ge \alpha_n$. Note that $\Omega_{\alpha_n} := \{(p, q) \in \Omega : |p|+|q| \ge \alpha_n\}$. By Corollary 3.4, Ω_{α_n} satisfies conditions $(C1)$ and $(C2)$. Thus, by Theorem 2.1, there exist a unique additive mapping $A_n : \mathbb{R} \to \mathbb{R}$ and a unique quadratic mapping $Q_n : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A_n(x) - Q_n(x) - f(0)| \le \frac{100}{3n}, \tag{3.8}$$

and

$$|h(x) - Q_n(x)| \le \frac{3}{2n} \tag{3.9}$$

for all $x \in \mathbb{R}$. Replacing $n \in \mathbb{N}^*$ by $m \in \mathbb{N}^*$ in (3.9) and using the triangle inequality, we have

$$|Q_m(x) - Q_n(x)| \le \frac{3}{2n} + \frac{3}{2m} \le 3 \tag{3.10}$$

for all $m, n \in \mathbb{N}^*$ and $x \in \mathbb{R}$. For every $x \in \mathbb{R}$ and $k \in \mathbb{N}^*$, we have

$$|Q_m(x) - Q_n(x)| = \frac{1}{k^2}|Q_m(kx) - Q_n(kx)| \le \frac{3}{k^2}. \tag{3.11}$$

Letting $k \to \infty$ in (3.11), we get $Q_m = Q_n$. Replacing $n \in \mathbb{N}^*$ by $m \in \mathbb{N}^*$ in (3.8) and using the inequality (3.10), we have

$$|A_m(x) - A_n(x)| \le \frac{100}{3m} + \frac{100}{3n} + 3 \le \frac{209}{3}$$

for all $m, n \in \mathbb{N}^*$ and $x \in \mathbb{R}$. For every $x \in \mathbb{R}$ and $k \in \mathbb{N}^*$, we have

$$|A_m(x) - A_n(x)| = \frac{1}{k}|A_m(kx) - A_n(kx)| \le \frac{209}{3k}. \tag{3.12}$$

Letting $k \to \infty$ in (3.12), we get $A_m = A_n$. Now, letting $n \to \infty$ in (3.8) and (3.9), we obtain the result. □

Remark 3.7. If we define $\Omega \subset \mathbb{R}^{2n}$ as an appropriate rotation of $2n$-product L^{2n} of L, then Ω has $2n$-dimensional measure 0 and satisfies the conditions $(C1)$ and $(C2)$. Consequently, we obtain the following theorem.

Theorem 3.8. *Suppose that $f, h : \mathbb{R}^n \to Y$ (Y is a Banach space) satisfy the Pexider quadratic functional inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2h(y)\| \le \epsilon$$

for all $(x, y) \in \Omega$ and some constant $\epsilon \ge 0$. Then there exist a unique additive mapping $A : \mathbb{R}^n \to Y$ and a unique quadratic mapping $Q : \mathbb{R}^n \to Y$ such that

$$\|f(x) - A(x) - Q(x) - f(0)\| \le \frac{100}{3}\epsilon,$$

and

$$\|h(x) - Q(x)\| \le \frac{3}{2}\epsilon$$

for all $x \in \mathbb{R}^n$.

Now, we give some corollaries, which are particular cases of Theorem 2.1.

Corollary 3.9. *Suppose that* $f : X \to Y$ *satisfies the Drygas functional inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \epsilon$$

for all $(x, y) \in \Omega$ *and some constant* $\epsilon \geq 0$. *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - A(x) - Q(x) - f(0)\| \leq \frac{100}{3}\epsilon$$

for $x \in X$.

Proof. Letting $2h(x) = f(x) + f(-x)$ in Theorem 2.1, we get the desired result. □

Corollary 3.10. *Suppose that* $f : X \to Y$ *satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon$$

for all $(x, y) \in \Omega$ *and* $\epsilon \geq 0$ *is some constant. Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{3}{2}\epsilon$$

for $x \in X$.

Proof. Letting $h(x) = f(x)$ in Theorem 2.1, we get the desired result. □

Corollary 3.11. *Suppose that* $f : X \to Y$ *satisfies*

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \epsilon$$

for all $(x, y) \in \Omega$ *and some constant* $\epsilon \geq 0$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \leq \frac{209}{6}\epsilon$$

for $x \in X$.

Proof. Letting $h(x) = 0$ in Theorem 2.1, we get the desired result. □

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