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# An affirmative answer to the open questions on the viscosity approximation methods for nonexpansive mappings in CAT(0) spaces

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**Abstract**

We prove a strong convergence theorem of a two-step viscosity iteration method for nonexpansive mappings in CAT(0) spaces without the nice projection property $\mathbb{N}$ and the restriction of the contraction constant $k \in [0, \frac{1}{2})$. Our result gives an affirmative answer to the open questions raised by Piatek [B. Piatek, Numer. Funct. Anal. Optim., **34** (2013), 1245–1264], and Kaewkhao et al. [A. Kaewkhao, B. Panyanak, S. Suantai, J. Inequal. Appl., **2015** (2015), 9 pages]. 

*Keywords:* Viscosity approximation method, fixed point, strong convergence, multivalued nonexpansive mapping, the nice projection property $\mathbb{N}$, CAT(0) space.
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## 1. Introduction

Let $E$ be a nonempty closed convex subset of a Hilbert space $H$ and $T : E \to E$ be a nonexpansive mapping with a nonempty fixed point set $Fix(T)$. The following scheme is known as the viscosity approximation method or Moudafi's viscosity approximation method: for any given $x_1 \in E$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(x_n), \quad \forall n \geq 1, \tag{1.1}$$

where $f : E \to E$ is a contraction with a constant $k \in (0, 1)$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$. In [10], under some suitable assumptions, the author proved that the sequence $\{x_n\}$ defined by (1.1) converges strongly

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to a point $z \in Fix(T)$ which satisfies the following variational inequality:

$$\langle f(z) - z, z - x\rangle \geq 0,\ \forall x \in Fix(T).$$

We note that the Moudafi viscosity approximation method can be applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations.

The first extension of Moudafi's result to the so-called CAT(0) space was proved by Shi and Chen [14]. However, they assumed that the space CAT(0) must satisfy some addition condition $P$. By using the concept of quasi-linearization introduced by Berg and Nikolaev [1], Wangkeeree and Preechasilp [16] could omit the condition $P$ from Shi and Chen's result. They obtained the following theorems.

**Theorem 1.1** ([16, Theorem 3.1]). *Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, $T : E \to E$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : E \to E$ be a contraction with a constant $k \in (0, 1)$. For each $s \in (0, 1)$, let $x_s$ be given by*

$$x_s = sf(x_s) \oplus (1 - s)T(x_s). \tag{1.2}$$

*Then the net $\{x_s\}$ converges strongly to $\tilde{x}$ as $s \to 0$ such that $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$, which is equivalent to the variational inequality:*

$$\left\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \geq 0\ \forall x \in Fix(T).$$

**Theorem 1.2** ([16, Theorem 3.4]). *Let $E,\ X,\ T,\ f,\ k$ be the same as in Theorem 1.1. Suppose that $x_1 \in E$ is arbitrarily chosen and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n),\ \forall n \geq 1, \tag{1.3}$$

*where $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2-k})$ satisfying:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

*Then $\{x_n\}$ converges strongly to $\tilde{x}$, where $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$ which is equivalent to the variational inequality:*

$$\left\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \geq 0\ \forall x \in Fix(T).$$

Among other things, by using the geometric properties of CAT(0) spaces, Piatek [13] proved the following strong convergence of a two-step viscosity iteration method.

**Theorem 1.3** ([13, Theorem 4.3]). *Let $X$ be a complete CAT(0) space with the nice projection property $\mathbb{N}$ and $C$ be a nonempty closed convex subset of $X$. Let $T : X \to X$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $f : X \to X$ be a contraction with $k \in [0, \frac{1}{2})$. Then there is a unique point $q \in Fix(T)$ such that $q = P_{Fix(T)}(f(q))$. Moreover, for each $u \in X$ and for each couple of sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$.

*For the arbitrary initial point $x_1 = u \in C$, the sequence $\{x_n\}$, generated by*

$$\begin{aligned} y_n &= \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n), \\ x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n)y_n, \quad \forall n \geq 1, \end{aligned} \tag{1.4}$$

*converges to $q$.*

(Concerning the definition of "nice projection property $\mathbb{N}$" please, see, Piatek [13])

In [13], the author provided an example of a CAT(0) space lacking the nice projection property $\mathbb{N}$, and so he raised the following open question.

**Open question 1.** *Does Theorem* 1.3 *still hold without the nice projection property* $\mathbb{N}$ *and* $k \in [0, 1)$*?*

By combining the ideas of [16] and [13] intensively, Kaewkhao-Panyanak-Suantai [7] omit the property $\mathbb{N}$ from Theorem 1.3, and proved the following result.

**Theorem 1.4** ([7])**.** *Let $C$ be a nonempty, closed, and convex subset of a complete CAT(0) space $X$, $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : C \to C$ be a contraction with $k \in [0, \frac{1}{2})$. For the arbitrary initial point $u \in C$, let $\{x_n\}$ be generated by*

$$\begin{aligned} x &= u, \\ y_n &= \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n), \\ x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n)y_n, \quad \forall n \geq 1, \end{aligned} \tag{1.5}$$

*where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$*.*

*Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$ and $\tilde{x}$ also satisfies*

$$\left\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \geq 0 \quad \forall x \in Fix(T).$$

Although Theorem 1.4 gives a partial answer to Open question 1 mentioned above, but it remains an open problem. Therefore the authors also raised the following.

**Open question 2.** *Whether Theorem* 1.3 *and Theorem* 1.4 *hold for* $k \in [0, 1)$*?*

The purpose of this paper is by using a different method to prove a strong convergence theorem of a two-step viscosity iteration for nonexpansive mappings in CAT(0) spaces without the nice projection property $\mathbb{N}$ and the restriction of the contraction constant $k \in [0, \frac{1}{2})$. Our result not only gives an affirmative answer to the Open questions 1 and 2 mentioned above, but also extends and improves the main results of Wangkeeree and Preechasilp [16], Piatek [13], Kaewkhao-Panyanak-Suantai [7] and Nilsrakoo-Saejung [11].

## 2. Preliminaries and Lemmas

Recall that a metric space $(X, d)$ is called a CAT(0) space, if it is geodesically connected and if every geodesic triangle in $X$ is at least as 'thin' as its comparison triangle in the Euclidean plane. It is known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces (see [2]), R-trees (see [8]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [6]), and many others. A complete CAT(0) space is often called Hadamard space. A subset $K$ of a CAT(0) space $X$ is convex if, for any $x, y \in K$, $[x, y] \subset K$, where $[x, y]$ is the uniquely geodesic joining $x$ and $y$.

In this paper, we write $(1 - t)x \oplus ty$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \tag{2.1}$$

It is well known that a geodesic space $(X, d)$ is a CAT(0) space if and only if the following inequality

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y) \tag{2.2}$$

is satisfied for all $x, y, z \in X$ and $t \in [0,1]$. In particular, if $x, y, z$ are points in a CAT(0) space $(X, d)$ and $t \in [0,1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z). \tag{2.3}$$

The concept of quasi-linearization was introduced by Berg and Nikolaev [1]. Let $(X, d)$ be a metric space. We denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasi-linearization is a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right) \quad \forall a, b, c, d \in X. \tag{2.4}$$

It is easy to see that $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{cd}, \overrightarrow{ab} \right\rangle, \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = -\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle$ and $\left\langle \overrightarrow{ax}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{xb}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle$ for all $a, b, c, d \in X$.

We say that $(X, d)$ satisfies the Cauchy-Schwarz inequality if

$$\left| \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \right| \le d(a,b)d(c,d) \quad \forall a, b, c, d \in X. \tag{2.5}$$

It is well known [1] that $(X, d)$ is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

**Lemma 2.1** ([4], [5]). *Let $C$ be a nonempty convex subset of a complete CAT(0) space $(X, d)$, $x \in X$ and $u \in C$. Then $u = P_C(x)$ (the metric projection of $x$ to $C$) if and only if*

$$\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \ge 0, \quad \forall y \in C.$$

**Lemma 2.2** ([17]). *Let $X$ be a complete CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for any $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \le t \langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;
(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \le t \langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ and $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \le t \langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Recall that a continuous linear functional $\mu$ on $l^\infty$, the Banach space of bounded real sequences, is called a *Banach limit* if $||\mu|| = \mu(1,1,1,\cdots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in l^\infty$.

**Lemma 2.3** ([15]). *Let $\alpha$ be a real number and let $(a_1,\ a_2, \cdots) \in l^\infty$ be such that $\mu_n(a_n) \le \alpha$ for all Banach limits $\mu$ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$. Then $\limsup_{n\to\infty} a_n \le \alpha$.*

**Lemma 2.4** ([5, 17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space $(X, d)$ and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_n \beta_n \le \limsup_n < 1$. Suppose that $x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n$ for all $n \ge 1$ and*

$$\limsup_{n\to\infty}(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0. \tag{2.6}$$

*Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

**Lemma 2.5** ([18]). *Let $\{c_n\}$ be a sequence of non-negative real numbers satisfying the property $c_{n+1} \le (1-\gamma_n)c_n + \gamma_n\eta_n, \quad n \ge 1$, where $\{\gamma_n\} \subset (0,1)$ and $\{\eta_n\} \subset \mathbb{R}$ such that*

(i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \eta_n \le 0$ *or* $\Sigma_{n=1}^{\infty} |\gamma_n\eta_n| < \infty$.

*Then $\{c_n\}$ converges to zero as $n \to \infty$.*

**Lemma 2.6** ([12, Theorem 3.1]). *Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : E \to E$ be a nonexpansive mapping, and $f : E \to E$ be a contraction with $k \in (0,1)$. Then the following statements hold:*

(i) *the net* $\{x_s\}$ *defined by*

$$x_s = sf(x_s) \oplus (1-s)T(x_s), \quad s \in (0,1) \tag{2.7}$$

*converges strongly to* $\tilde{x}$ *as* $s \to 0$ *where* $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$*;*

(ii) *if* $\{x_n\}$ *is a bounded sequence in* $E$ *such that* $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$*, then*

$$\mu_n \left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),x_n)\right) \le 0, \tag{2.8}$$

*for all Banach limits* $\mu$*.*

## 3. Main Results

We are now in a position to give the main results of the paper.

**Theorem 3.1.** *Let* $E$ *be a nonempty closed convex subset of a complete CAT(0) space* $X$*,* $T : E \to E$ *be a nonexpansive mapping with* $Fix(T) \neq \emptyset$*. Let* $f : E \to E$ *be a contraction with* $k \in (0,1)$*. For the arbitrary initial point* $u \in C$*, let* $\{x_n\}$ *be generated by*

$$\begin{cases} x = u, \\ y_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

*where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_n \beta_n \le \limsup_n \beta_n < 1$.

*Then* $\{x_n\}$ *converges strongly to* $\tilde{x}$ *such that* $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$ *and* $\tilde{x}$ *also satisfies*

$$\left\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \ge 0 \quad \forall x \in Fix(T).$$

*Proof.* We divide the proof into four steps.

**step 1.** We show that $\{x_n\}$, $\{y_n\}$, $\{T(x_n)\}$, and $\{f(x_n)\}$ are bounded sequences in $E$.

Let $p \in Fix(T)$. By inequality (2.3), we have

$$\begin{aligned} d(x_{n+1},p) &\le \beta_n d(x_n,p) + (1-\beta_n)d(y_n,p) \\ &\le \beta_n d(x_n,p) + (1-\beta_n)\left[d(\alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n),p)\right] \\ &\le \beta_n d(x_n,p) + (1-\beta_n)\left\{\alpha_n\left[d(f(x_n),f(p)) + d(f(p),p)\right] + (1-\alpha_n)d(x_n,p)\right\} \\ &\le \left[1-\alpha_n(1-k) + (1-k)\alpha_n\beta_n\right]d(x_n,p) + (1-\beta_n)\alpha_n d(f(p),p) \\ &\le max\left\{d(x_n,p), \frac{d(f(p),p)}{1-k}\right\}. \end{aligned}$$

By induction, we have

$$d(x_n,p) \le max\left\{d(x_1,p), \frac{d(f(p),p)}{1-k}]\right\}, \quad \forall n \ge 1.$$

Hence, $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{T(x_n)\}$ and $\{y_n\}$.

**step 2.** Next, we show that

$$\lim_{n\to\infty} d(x_n,y_n) = 0; \ \lim_{n\to\infty} d(x_n,T(x_n)) = 0; \ \lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{3.2}$$

In fact, we have

$$\begin{aligned}
d(y_{n+1}, y_n) &\le d(\alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1})T(x_{n+1}), \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n)) \\
&\le d(\alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1})T(x_{n+1}), \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1})T(x_n)) \\
&\quad + d(\alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1})T(x_n), \alpha_{n+1} f(x_n) \oplus (1-\alpha_{n+1})T(x_n)) \\
&\quad + d(\alpha_{n+1} f(x_n) \oplus (1-\alpha_{n+1})T(x_n), \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n)) \\
&\le (1-\alpha_{n+1}) d(T(x_{n+1}), Tx_n) + \alpha_{n+1} d(f(x_{n+1}), f(x_n)) + |\alpha_{n+1}-\alpha_n| d(f(x_n), Tx_n) \\
&\le (1-\alpha_{n+1}) d(x_{n+1}, x_n) + \alpha_{n+1} k d(x_{n+1}, x_n) + |\alpha_{n+1}-\alpha_n|\, d(f(x_n), Tx_n).
\end{aligned}$$

This implies that

$$d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \le (\alpha_{n+1} k - \alpha_{n+1})\, d(x_{n+1}, x_n) + |\alpha_{n+1}-\alpha_n|\, d(f(x_n), Tx_n).$$

Hence we have,

$$\limsup_{n\to\infty} \{d(y_{n+1}, y_n) - d(x_{n+1}, x_n)\} \le 0.$$

By Lemma 2.4, we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.3}$$

It follows from (3.3) and (3.1) that

$$\begin{aligned}
&d(x_n, T(x_n)) \le d(x_n, y_n) + d(y_n, Tx_n) \le d(x_n, y_n) + \alpha_n d(f(x_n), Tx_n) \to 0 \quad (as\ n \to \infty), \\
&d(x_{n+1}, y_n) \le \beta_n d(x_n, y_n) \to 0, \\
&d(x_{n+1}, x_n) \le d(x_{n+1}, y_n) + d(y_n, x_n) \to 0.
\end{aligned}$$

**step 3.** Next, we prove that

$$\limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right\} \le 0, \tag{3.4}$$

where $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$.

In fact, since $\{x_n\}$ is bounded and $d(x_n, Tx_n) \to 0$, by Lemma 2.6 (ii), for all Banach limits $\mu$, we have

$$\mu_n \left( d^2(f(\tilde{x}), \tilde{x}) - \mu_n d^2(f(\tilde{x}), x_n) \right) \le 0. \tag{3.5}$$

Since $d(x_{n+1}, x_n) \to 0$, we have

$$\limsup_{n\to\infty} \left\{ (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_{n+1}) - (d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n)) \right\} \le 0. \tag{3.6}$$

It follows from (3.5), (3.6) and Lemma 2.3 that

$$\limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n) \right\} \le 0. \tag{3.7}$$

From (3.2) and (3.7), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right\} \\
&\le \limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n) \right\} + \limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), x_n) - d^2(f(\tilde{x}), Tx_n) \right\} \\
&\le \limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n) \right\} \\
&\quad + \limsup_{n\to\infty} \{ d(f(\tilde{x}), x_n) + d(f(\tilde{x}), T(x_n))\ d(f(\tilde{x}), x_n) - d(f(\tilde{x}), T(x_n)) \} \\
&\le \limsup_{n\to\infty} \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), x_n) \right\} \\
&\quad + \limsup_{n\to\infty} \{ d(f(\tilde{x}), x_n) + d(f(\tilde{x}), T(x_n))\ d(x_n, T(x_n)) \} \le 0.
\end{aligned} \tag{3.8}$$

**step 4.** Finally, we show that $\{x_n\}$ converges strongly to a point $\tilde{x} \in Fix(T)$ where $\tilde{x} = P_{Fix(T)}(f(\tilde{x}))$.

In fact, it follows from (2.2) and (3.1) that

$$\begin{aligned} d^2(x_{n+1}, \tilde{x}) &= d^2(\beta_n x_n \oplus (1-\beta_n) y_n, \tilde{x}) \\ &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}) - \beta_n(1-\beta_n) d^2(x_n, y_n) \\ &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} d^2(y_n, \tilde{x}) &= d^2(\alpha_n f(x_n) \oplus (1-\alpha_n) T(x_n), \tilde{x}) \\ &\le \alpha_n d^2(f(x_n), \tilde{x}) + (1-\alpha_n) d^2(Tx_n, \tilde{x}) - \alpha_n(1-\alpha_n) d^2(f(x_n), Tx_n) \\ &= (1-\alpha_n) d^2(Tx_n, \tilde{x}) + \alpha_n(d^2(f(x_n), \tilde{x}) - d^2(f(x_n), , Tx_n)) + \alpha_n^2 d^2(f(x_n), Tx_n) \\ &\le (1-\alpha_n) d^2(x_n, \tilde{x}) + \alpha_n(d^2(f(x_n), \tilde{x}) - d^2(f(x_n), Tx_n)) + \alpha_n^2 d^2(f(x_n), Tx_n) \end{aligned} \tag{3.10}$$

By using (2.4), Lemma 2.2, the Cauchy-Schwarz inequality (2.5) and for any $n \ge 1$, we have

$$\begin{aligned} &\alpha_n \left(d^2(f(x_n), \tilde{x}) - d^2(f(x_n), Tx_n)\right) \\ &= 2\alpha_n \left\{ \left\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)\tilde{x}} \right\rangle - d^2(Tx_n, \tilde{x}) \right\} \\ &= 2\alpha_n \left\{ \left\langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{T(x_n)\tilde{x}} \right\rangle, + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{T(x_n)\tilde{x}} \right\rangle - d^2(Tx_n, \tilde{x}) \right\} \\ &\le 2\alpha_n \left\{ k d(x_n, \tilde{x}) d(Tx_n, \tilde{x}) + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{T(x_n)\tilde{x}} \right\rangle - d^2(Tx_n, \tilde{x}) \right\} \\ &\le \alpha_n k \left\{ d^2(x_n, \tilde{x}) + d^2(Tx_n, \tilde{x}) \right\} + 2\alpha_n \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{T(x_n)\tilde{x}} \right\rangle - 2\alpha_n d^2(Tx_n, \tilde{x}) \\ &= \alpha_n k d^2(x_n, \tilde{x}) + \alpha_n(k-2) d^2(Tx_n, \tilde{x}) + \alpha_n \left\{ d^2(f(\tilde{x}), \tilde{x}) + d^2(Tx_n, \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right\} \\ &= \alpha_n k d^2(x_n, \tilde{x}) + \alpha_n(k-1) d^2(Tx_n, \tilde{x}) + \alpha_n \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right\} \\ &\le \alpha_n k d^2(x_n, \tilde{x}) + \alpha_n \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right\} \ (since\ \alpha_n(k-1) \le 0). \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), and after simplifying, we have

$$\begin{aligned} d^2(y_n, \tilde{x}) \le &(1 - \alpha_n(1-k)) d^2(x_n, \tilde{x}) \\ &+ \alpha_n \left\{ d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right\} + \alpha_n^2 d^2(f(x_n), Tx_n). \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.9) and simplifying, for any $n \ge 1$, we have

$$\begin{aligned} d^2(x_{n+1}, \tilde{x}) &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) \Big\{ (1 - \alpha_n(1-k))\ d^2(x_n, \tilde{x}) \\ &\quad + \alpha_n \left( d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right) + \alpha_n^2 d^2(f(x_n), Tx_n) \Big\} \\ &\le (1 - (1-\beta_n)(1-k)\alpha_n)\, d^2(x_n, \tilde{x}) \\ &\quad + (1-\beta_n)\alpha_n \left( d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right) + \alpha_n^2 d^2(f(x_n), Tx_n). \end{aligned} \tag{3.13}$$

Putting, in Lemma 2.5, $c_n = d^2(x_n, \tilde{x})$, $\gamma_n = (1-\beta_n)(1-k)\alpha_n$ and

$$\eta_n = \frac{(1-\beta_n) \left( d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), Tx_n) \right) + \alpha_n d^2(f(x_n), Tx_n)}{(1-k)(1-\beta_n)},$$

then (3.13) can be written as

$$c_{n+1} \le (1-\gamma_n) c_n + \gamma_n \eta_n, \quad \forall n \ge 1. \tag{3.14}$$

By virtue of the conditions (i), (ii), (iii), and by using (3.4), we know that

(i) $\gamma_n \in (0,1)$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \eta_n \le 0$.

Therefore all conditions in Lemma 2.5 are satisfied. We have $c_n \to 0$ as $n \to \infty$. This implies that $x_n$ converges strongly to $\tilde{x}$, where $\tilde{x} = P_{Fix(T)} f(\tilde{x})$.

The proof of Theorem 3.1 is completed.

□

*Remark* 3.2. Theorem 3.1 not only gives an affirmative answer to the Open questions 1 and 2 raised by Piatek [13] and Kaewkhao-Panyanak-Suantai [7], respectively, but also extends and improves the corresponding results of Wangkeeree and Preechasilp [16], Piatek [13], Kaewkhao-Panyanak-Suantai [7] and Nilsrakoo-Saejung [11], Kumam et al. [9] and many others.

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## References

[1] I. D. Berg, I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicada, **133** (2008), 195–218. 1, 2, 2
[2] M. R. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, (1999). 2
[3] K. S. Brown, *Buildings*, Springer-Verlag, New York, (1989). 2
[4] S. Dhompongsa, W. A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal., **8** (2007), 35–45. 2.1
[5] S. Dhompongsa, B. Panyanak, *On* $\Delta$*-convergence theorems in* CAT(0) *spaces*, Comput. Math. Appl., **56** (2008), 2572–2579. 2.1, 2.4
[6] K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Monogr. Textb. Pure Appl. Math., Marcel Dekker, New York, (1984). 2
[7] A. Kaewkhao, B. Panyanak, S. Suantai, *Viscosity iteration method in* CAT(0) *spaces without the nice projection property*, J. Inequal. Appl., **2015** (2015), 9 pages. 1, 1.4, 1, 3.2
[8] W. A. Kirk, *Fixed point theory in* CAT(0) *spaces and R-trees*, Fixed Point Theory Appl., **4** (2004), 309–316. 2
[9] P. Kumam, G. S. Saluja, H. K. Nashine, *Convergence of modified S-iteration process for two asymptotically nonexpansive mappings in the intermediate sense in* CAT(0) *spaces*, J. Inequal. Appl., **2014** (2014), 15 pages. 3.2
[10] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241** (2000), 46–55. 1
[11] W. Nilsrakoo, S. Saejung, *Equilibrium problems and Moudafi's viscosity approximation methods in Hilbert spaces*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **17** (2010), 195–213. 1, 3.2
[12] B. Panyanak, S. Suantai, *Viscosity approximation methods for multivalued nonexpansive mappings in geodesic spaces*, Fixed Point Theory Appl., **2015** (2015), 14 pages. 2.6
[13] B. Piatek, *Viscosity iteration in* CAT($\kappa$) *spaces*, Numer. Funct. Anal. Optim., **34** (2013), 1245–1264. 1, 1.3, 1, 1, 1, 3.2
[14] L. Y. Shi, R. D. Chen, *Strong convergence of viscosity approximation methods for nonexpansive mappings in* CAT(0) *spaces*, J. Appl. Math., **2012** (2012), 11 pages. 1
[15] N. Shioji, W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., **125** (1997), 3641–3645. 2.3
[16] R. Wangkeeree, P. Preechasilp, *Viscosity approximation methods for nonexpansive mappings in* CAT(0) *spaces*, J. Inequal. Appl., **2013** (2013), 15 pages. 1, 1.1, 1.2, 1, 1, 3.2
[17] R. Wangkeeree, P. Preechasilp, *Viscosity approximation methods for nonexpansive semigroups in* CAT(0) *spaces*, Fixed point Theory Appl., **2013** (2013), 16 pages. 2.2, 2.4
[18] H. K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116** (2003), 659–678. 2.5