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# Iterative computation of fixed points of quasi-asymp -totic pseudo-contractions

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**Abstract**

An iterative algorithm is presented to find the fixed points of a quasi-asymptotic pseudo-contraction in Hilbert spaces. It is shown that the proposed algorithm converges strongly to the fixed point of a quasi-asymptotic pseudo-contraction. 

*Keywords:* Quasi-asymptotic pseudo-contraction, fixed point, iterative algorithm, Hilbert spaces.
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## 1. Introduction

Throughout, we assume that $H$ is a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\emptyset \neq C \subset H$ be a closed and convex set.

**Definition 1.1.** An operator $T : C \to C$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.

**Definition 1.2.** An operator $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ and for all $n \geq 1$.

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The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] in 1972. They proved the following result.

**Proposition 1.3.** *Let $X$ be a uniformly convex Banach space. Let $\emptyset \neq C \subset X$ be a bounded, closed and convex set. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Then, the set $F(T)$ of fixed points of $T$ is nonempty, closed and convex.*

Since then, a large number of authors have devoted to study the weak and strong convergence problems of the iterative algorithms for such a class of mappings (see, e.g., [1, 5, 8]and [7, 10, 15–17]). In particular, the following two algorithms have been studied extensively in the literature.

**Algorithm 1.4.** For arbitrary $x_0 \in C$, compute the sequence $\{x_n\}$ by the manner

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1.$$

**Algorithm 1.5.** For arbitrary $x_0 \in C$, compute the sequence $\{x_n\}$ by the manner

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1. \end{cases} \tag{1.2}$$

**Definition 1.6.** An operator $T : C \to C$ is called asymptotically pseudo-contractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n\|x - y\|^2 \tag{1.3}$$

for all $x, y \in C$ and for all $n \geq 1$.

*Remark* 1.7. It is easy to check that (1.3) equals

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(x - T^n x) - (y - T^n y)\|^2 \tag{1.4}$$

for all $x, y \in C$ and for all $n \geq 1$.

The class of asymptotic pseudo-contractions was introduced by Schu [6] in 1991. We know that the class of asymptotic pseudo-contractions contains properly the class of asymptotically nonexpansive mappings as a subclass.

To compute the fixed point of asymptotic pseudo-contractions, Schu [6] demonstrated the following convergence theorem.

**Theorem 1.8.** *Let $H$ be a real Hilbert space and $\emptyset \neq C \subset H$ a closed and convex set. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotic pseudo-contraction with $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$ and $\{x_n\}$ a sequence generated by* (1.2)*. Suppose the following conditions are satisfied:*

(i) *$C$ is bounded and $T$ is completely continuous;*
(ii) *$\sum_{n=1}^{\infty}(k_n - 1) < \infty$;*
(iii) *$0 < \kappa_1 \leq \alpha_n \leq \beta_n \leq \kappa_2 < \frac{\sqrt{1+L^2}-1}{L^2}$ for all $n \geq 1$.*

*Then the sequence $\{x_n\}$ generated by* (1.2) *converges strongly to some fixed point of $T$.*

Further, Chidume and Zegeye [2] introduced the following algorithm.

**Algorithm 1.9.** For given $x_1 \in C$, compute the sequence $\{x_n\}$ by the manner

$$x_{n+1} = \lambda_n\theta_n x_1 + (1 - \lambda_n - \lambda_n\theta_n)x_n + \lambda_n T^n x_n, \forall n \geq 1, \tag{1.5}$$

where the sequences $\{\lambda_n\} \subset (0, 1)$ and $\{\theta_n\} \subset (0, 1)$ satisfy

(i) $\lambda_n(1+\theta_n) \leq 1$ and $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$;
(ii) $\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = \lim_{n\to\infty} \frac{k_n-1}{\theta_n} = \lim_{n\to\infty} \frac{\theta_n-\theta_{n-1}}{\lambda_n\theta_n^2} = \lim_{n\to\infty} \frac{k_n-k_{n-1}}{\lambda_n\theta_n^2} = 0.$

Furthermore, Chidume and Zegeye showed the strong convergence of the above algorithm (1.5) under some more assumptions on the mapping $T$ in Banach spaces.

**Definition 1.10.** An operator $T : C \to C$ is said to be quasi-asymptotically pseudo-contractive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - p, x - p\rangle \leq k_n \|x-p\|^2 \tag{1.6}$$

for all $x \in C, p \in F(T)$ and for all $n \geq 1$.

*Remark* 1.11. It is easy to check that (1.6) equals

$$\|T^n x - p\|^2 \leq (2k_n - 1)\|x-p\|^2 + \|x - T^n x\|^2 \tag{1.7}$$

for all $x \in C, p \in F(T)$ and for all $n \geq 1$.

Thus, every asymptotically pseudo-contractive mapping with $F(T) \neq \emptyset$ is quasi-asymptotically pseudo-contractive, but the converse may not be true.

Recently, Zhou and Su [19] established a demi-closedness principle for quasi-asymptotic pseudo-contract -ions in Hilbert spaces. By utilizing the demi-closedness principle, they suggested a $CQ$ algorithm and proved its strong convergence. Some related work, please refer to [11–14].

Inspired by the results in the literature, the main purpose of this article is to construct an iterative method to find the fixed points of quasi-asymptotically pseudo-contractive mappings. We suggest an algorithm based on the algorithms (1.2) and (1.5). Under some mild conditions, we prove that the suggested algorithm converges strongly to the fixed point of quasi-asymptotically pseudo-contractive mapping $T$.

## 2. Lemmas

In Hilbert spaces, the following results are well known.

**Lemma 2.1.** *Let $H$ be a Hilbert space, then we have*

$$\|u + u^\dagger\|^2 \leq \|u\|^2 + 2\langle u^\dagger, u + u^\dagger\rangle \tag{2.1}$$

*and*

$$\|\delta u + (1-\delta)u^\dagger\|^2 = \delta\|u\|^2 + (1-\delta)\|u^\dagger\|^2 - \delta(1-\delta)\|u - u^\dagger\|^2 \tag{2.2}$$

*for all $u, u^\dagger \in H$ and $\delta \in [0,1]$.*

**Lemma 2.2** ([18])**.** *Let $C$ be a nonempty bounded and closed convex subset of a real Hilbert space $H$. Let $T\colon C \to C$ be a uniformly L-Lipschitzian and asymptotically pseudo-contraction. Then $I - T$ is demiclosed at zero.*

**Lemma 2.3** ([4])**.** *Let $\{\sigma_n\}_{n\geq 1}$ be a sequence of real numbers. Assume $\{\sigma_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{\sigma_{n_k}\}$ of $\{\sigma_n\}$ such that $\sigma_{n_k} \leq \sigma_{n_k+1}$ for all $k \geq 0$. For every $n \geq N$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : \sigma_{n_i} < \sigma_{n_i+1}\}.$$

*Then $\tau(n) \to \infty$ as $n \to \infty$, and for all $n \geq N$*

$$\max\{\sigma_{\tau(n)}, \sigma_n\} \leq \sigma_{\tau(n)+1}.$$

**Lemma 2.4** ([9]). *Let $\{\zeta_n\} \subset [0,\infty)$, $\{\varsigma_n\} \subset (0,1)$ and $\{\varrho_n\}$ be three sequences such that*

$$\zeta_{n+1} \leq (1-\varsigma_n)\zeta_n + \varrho_n, \quad \forall n \geq 1.$$

*Assume the following restrictions are satisfied*

(i) $\sum_{n=1}^{\infty} \varsigma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\varrho_n}{\varsigma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\varrho_n| < \infty$.

*Then* $\lim_{n\to\infty} \zeta_n = 0$.

## 3. Main results

In the sequel, let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$, let $T : C \to C$ be a uniformly $L$-Lipschitzian and quasi-asymptotic pseudo-contraction with coefficient $\{k_n\}$ and $f : C \to C$ a $\rho$-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real number sequences in $[0,1]$.

**Algorithm 3.1.** For $x_1 \in C$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[(1-\beta_n)x_n + \beta_n T^n y_n], \forall n \geq 1. \end{cases} \tag{3.1}$$

Next, we prove our main result as follows.

**Theorem 3.2.** *Suppose $F(T) \neq \emptyset$ and $I - T$ is demiclosed at zero. Assume the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \leq \gamma_n$, $0 < \liminf_{n\to\infty} \beta_n$ *and* $\sum_{n=1}^{\infty} (k_n - 1) < \infty$;

(iii) $0 < a \leq \gamma_n \leq b < \frac{1}{\sqrt{k_n^2 + L^2} + k_n}$ *for all* $n \geq 1$.

*Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $u = P_{F(T)}f(u)$, which is the unique solution of the variational inequality $\langle (I-f)u, x-u \rangle \geq 0, \forall x \in F(T)$.*

*Proof.* Note that $u = P_{F(T)}f(u)$ is unique due to the mapping $f$ being contractive. From (3.1), we have

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n T^n y_n\right] - u\| \\ &= \|\alpha_n (f(x_n) - u) + (1-\alpha_n)\left[(1-\beta_n)(x_n - u) + \beta_n (T^n y_n - u)\right]\| \\ &\leq \alpha_n \|f(x_n) - u\| + (1-\alpha_n)\|(1-\beta_n)(x_n - u) + \beta_n (T^n y_n - u)\|. \end{aligned} \tag{3.2}$$

By (2.2), we get

$$\begin{aligned} \|(1-\beta_n)(x_n - u) + \beta_n (T^n y_n - u)\|^2 = (1-\beta_n)\|x_n - u\|^2 + \beta_n \|T^n y_n - u\|^2 \\ - \beta_n (1-\beta_n)\|x_n - T^n y_n\|^2. \end{aligned} \tag{3.3}$$

Choosing $y = u$ in (1.7) to derive

$$\|T^n x - u\|^2 \leq (2k_n - 1)\|x - u\|^2 + \|x - T^n x\|^2 \tag{3.4}$$

for all $x \in C$.

From (3.1), (3.4) and (2.2), we obtain

$$\begin{aligned}
\|T^n y_n - u\|^2 &\le (2k_n-1)\,\|y_n-u\|^2 + \|y_n - T^n y_n\|^2\\
&= (2k_n-1)\,\|(1-\gamma_n)x_n + \gamma_n T^n x_n - u\|^2 + \|(1-\gamma_n)x_n + \gamma_n T^n x_n - T^n y_n\|^2\\
&= (2k_n-1)\,\|(1-\gamma_n)(x_n-u) + \gamma_n(T^n x_n - u)\|^2\\
&\quad + \|(1-\gamma_n)\,(x_n - T^n y_n) + \gamma_n(T^n x_n - T^n y_n)\|^2\\
&= (2k_n-1)\left[(1-\gamma_n)\|x_n-u\|^2 + \gamma_n\|T^n x_n - u\|^2 - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2\right]\\
&\quad + (1-\gamma_n)\|x_n - T^n y_n\|^2 + \gamma_n\|T^n x_n - T^n y_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2\\
&\le (2k_n-1)\big[(1-\gamma_n)\|x_n-u\|^2 + \gamma_n(2k_n-1)\|x_n-u\|^2 + \gamma_n\|x_n - T^n x_n\|^2\\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2\big] + (1-\gamma_n)\|x_n - T^n y_n\|^2 + \gamma_n\|T^n x_n - T^n y_n\|^2\\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2.
\end{aligned} \tag{3.5}$$

Observe that

$$\|x_n - y_n\| = \gamma_n\|x_n - T^n x_n\|. \tag{3.6}$$

Since $T$ is uniformly $L$-Lipschitzian, from (3.5) and (3.6), we deduce

$$\begin{aligned}
\|T^n y_n - u\|^2 &\le (2k_n-1)[(1-\gamma_n)\|x_n-u\|^2 + \gamma_n(2k_n-1)\|x_n-u\|^2 + \gamma_n\|x_n - T^n x_n\|^2\\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2] + (1-\gamma_n)\|x_n - T^n y_n\|^2\\
&\quad + \gamma_n L^2\|x_n - y_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2\\
&= (2k_n-1)[(1-\gamma_n)\|x_n-u\|^2 + \gamma_n(2k_n-1)\|x_n-u\|^2 + \gamma_n\|x_n - T^n x_n\|^2\\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2] + (1-\gamma_n)\|x_n - T^n y_n\|^2 + \gamma_n^3 L^2\|x_n - T^n x_n\|^2\\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2\\
&= [1 + 2(2k_n\gamma_n - \gamma_n + 1)(k_n-1)]\,\|x_n-u\|^2 + (1-\gamma_n)\|x_n - T^n y_n\|^2\\
&\quad - \gamma_n(1 - 2k_n\gamma_n - \gamma_n^2 L^2)\|x_n - T^n x_n\|^2.
\end{aligned} \tag{3.7}$$

By condition (iii), we know that $\gamma_n \le b < \frac{1}{\sqrt{k_n^2+L^2+k_n}}$ for all $n \ge 1$. Then, we deduce that $1 - 2k_n\gamma_n - \gamma_n^2 L_n^2 > 0$ for all $n \ge 1$. Thus, compute (3.7) to deduce

$$\|T^n y_n - u\|^2 \le \big[1 + 2(2k_n\gamma_n - \gamma_n + 1)(k_n-1)\big]\,\|x_n-u\|^2 + (1-\gamma_n)\|x_n - T^n y_n\|^2. \tag{3.8}$$

Substituting (3.8) into (3.3) to get

$$\begin{aligned}
\|(1-\beta_n)(x_n-u) + \beta_n(T^n y_n - u)\|^2 &\le \beta_n[1 + 2(2k_n\gamma_n - \gamma_n + 1)(k_n-1)]\|x_n-u\|^2\\
&\quad + (1-\gamma_n)\beta_n\|x_n - T^n y_n\|^2 + (1-\beta_n)\|x_n-u\|^2\\
&\quad - \beta_n(1-\beta_n)\|x_n - T^n y_n\|^2\\
&= [1 + 2\beta_n(2k_n\gamma_n - \gamma_n + 1)(k_n-1)]\|x_n-u\|^2\\
&\quad + \beta_n(\beta_n - \gamma_n)\|x_n - T^n y_n\|^2\\
&\le [1 + 2\beta_n(2k_n\gamma_n - \gamma_n + 1)(k_n-1)]\|x_n-u\|^2.
\end{aligned}$$

So,

$$\begin{aligned}
\|(1-\beta_n)(x_n-u) + \beta_n(T^n y_n - u)\| &\le \sqrt{1 + 2\beta_n(2k_n\gamma_n - \gamma_n + 1)(k_n-1)}\,\|x_n-u\|\\
&\le [1 + 2\beta_n(2k_n\gamma_n - \gamma_n + 1)(k_n-1)]\,\|x_n-u\|.
\end{aligned} \tag{3.9}$$

Since $k_n \to 1$, without loss of generality, we assume that $k_n \le 2$ for all $n \ge 1$. It follows from (3.2) and (3.9) that

$$\|x_{n+1} - u\| \le \alpha_n\|f(x_n) - u\| + (1-\alpha_n)\,[1 + 2(2k_n\gamma_n - \gamma_n + 1)(k_n-1)]\,\|x_n-u\|$$

$$\begin{aligned}
&\leq \alpha_n\|f(x_n)-f(u)\|+\alpha_n\|f(u)-u\|+(1-\alpha_n)\left[1+2(2k_n\gamma_n-\gamma_n+1)(k_n-1)\right]\|x_n-u\|\\
&\leq \alpha_n\rho\|x_n-u\|+\alpha_n\|f(u)-u\|+(1-\alpha_n)\left[1+2(2k_n\gamma_n-\gamma_n+1)(k_n-1)\right]\|x_n-u\|\\
&\leq \alpha_n\|f(u)-u\|+\left[1-(1-\rho)\alpha_n\right]\|x_n-u\|+2\left(2k_n\gamma_n-\gamma_n+1\right)(k_n-1)\|x_n-u\|\\
&\leq (1-\rho)\alpha_n\frac{\|f(u)-u\|}{1-\rho}+\left[1-(1-\rho)\alpha_n\right]\|x_n-u\|+10(k_n-1)\|x_n-u\|.
\end{aligned}$$

An induction induces that

$$\begin{aligned}
\|x_{n+1}-u\| &\leq [1+10(k_n-1)]\max\left\{\|x_n-u\|,\frac{\|f(u)-u\|}{1-\rho}\right\}\\
&\leq \prod_{i=1}^{n}[1+10(k_i-1)]\max\left\{\|x_0-u\|,\frac{\|f(u)-u\|}{1-\rho}\right\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded by the condition $\sum_{n=1}^{\infty}(k_n-1)<\infty$.

From (2.1) and (3.1), we have

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \|(1-\alpha_n)(x_n-u)-\beta_n(1-\alpha_n)(x_n-T^ny_n)+\alpha_n(f(x_n)-u)\|^2\\
&\leq \|(1-\alpha_n)(x_n-u)-\beta_n(1-\alpha_n)(x_n-T^ny_n)\|^2+2\alpha_n\langle f(x_n)-u,x_{n+1}-u\rangle\\
&= (1-\alpha_n)^2\|x_n-u\|^2-2\beta_n(1-\alpha_n)^2\langle x_n-T^ny_n,x_n-u\rangle\\
&\quad +\beta_n^2(1-\alpha_n)^2\|x_n-T^ny_n\|^2+2\alpha_n\langle f(x_n)-u,x_{n+1}-u\rangle.
\end{aligned} \tag{3.10}$$

From (3.7), we deduce

$$\begin{aligned}
2\langle x_n-T^ny_n,x_n-u\rangle &\geq \gamma_n\|x_n-T^ny_n\|^2+\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2\\
&\quad -2(2k_n\gamma_n-\gamma_n+1)(k_n-1)\|x_n-u\|^2\\
&\geq \gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2+\gamma_n\|x_n-T^ny_n\|^2.
\end{aligned} \tag{3.11}$$

By condition (ii), we have $\gamma_n\geq\beta_n\geq 0$ for all $n\geq 1$. Hence, by (3.10) and (3.11), we get

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\leq (1-\alpha_n)^2\|x_n-u\|^2-\beta_n\gamma_n(1-\alpha_n)^2\|x_n-T^ny_n\|^2+\beta_n^2(1-\alpha_n)^2\|x_n-T^ny_n\|^2\\
&\quad -\beta_n(1-\alpha_n)^2\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2+2\alpha_n\langle f(x_n)-u,x_{n+1}-u\rangle\\
&\leq (1-\alpha_n)\|x_n-u\|^2+2\alpha_n\langle f(x_n)-u,x_{n+1}-u\rangle\\
&\quad -\beta_n(1-\alpha_n)^2\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2.
\end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned}
&\|x_{n+1}-u\|^2-\|x_n-u\|^2+\beta_n(1-\alpha_n)^2\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2\\
&\quad\leq \alpha_n\left(2\langle f(x_n)-u,x_{n+1}-u\rangle-\|x_n-u\|^2\right).
\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, there exists $M>0$ such that

$$\sup_n\left\{2\langle f(x_n)-u,x_{n+1}-u\rangle-\|x_n-u\|^2\right\}\leq M.$$

So,

$$\beta_n(1-\alpha_n)^2\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2+\|x_{n+1}-u\|^2-\|x_n-u\|^2\leq\alpha_nM. \tag{3.13}$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m>0$ such that $\{\|x_n-u\|\}$ is decreasing for all $n\geq m$.

In this case, we know that $\lim_{n\to\infty}\|x_n-u\|$ exists. From (3.13), we deduce

$$\beta_n(1-\alpha_n)^2\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)\|x_n-T^nx_n\|^2\leq\|x_n-u\|^2-\|x_{n+1}-u\|^2+M\alpha_n. \tag{3.14}$$

By conditions (ii) and (iii), we have $\liminf_{n\to\infty}\beta_n(1-\alpha_n)^2\gamma_n(1-2k_n\gamma_n-\gamma_n^2L^2)>0$. Thus, from (3.14), we get

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \tag{3.15}$$

It follows from (3.6) and (3.15) that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.16}$$

Since $T$ is uniformly $L$-Lipschitzian, we have $\|T^ny_n-T^nx_n\|\le L\|x_n-y_n\|$. This together with (3.16) imply that

$$\lim_{n\to\infty}\|T^ny_n-T^nx_n\|=0. \tag{3.17}$$

Note that

$$\|x_n-T^ny_n\|\le\|x_n-T^nx_n\|+\|T^nx_n-T^ny_n\|. \tag{3.18}$$

Combining (3.15), (3.17) and (3.18), we have

$$\lim_{n\to\infty}\|x_n-T^ny_n\|=0. \tag{3.19}$$

From (3.1), we have

$$\|x_{n+1}-x_n\|\le\alpha_n\|f(x_n)-x_n\|+(1-\alpha_n)\beta_n\|T^ny_n-x_n\|.$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.20}$$

Since $T$ is uniformly $L$-Lipschitzian, we can derive

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\|&\le\|x_{n+1}-T^{n+1}x_{n+1}\|+\|T^{n+1}x_{n+1}-T^{n+1}x_n\|+\|T^{n+1}x_n-Tx_{n+1}\|\\&\le\|x_{n+1}-T^{n+1}x_{n+1}\|+L\|x_{n+1}-x_n\|+L\|T^nx_n-x_{n+1}\|\\&\le\|x_{n+1}-T^{n+1}x_{n+1}\|+2L\|x_{n+1}-x_n\|+L\|T^nx_n-x_n\|.\end{aligned} \tag{3.21}$$

By (3.15), (3.20) and (3.21), we have immediately that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.22}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$x_{n_k}\rightharpoonup\tilde{x}\in C,$$

and

$$\limsup_{n\to\infty}\langle f(u)-u,x_n-u\rangle=\lim_{k\to\infty}\langle f(u)-u,x_{n_k}-u\rangle.$$

By the assumption of Lemma 2.2 and (3.22), we obtain

$$\tilde{x}\in F(T).$$

So,

$$\begin{aligned}\limsup_{n\to\infty}\langle f(u)-u,x_n-u\rangle&=\lim_{k\to\infty}\langle f(u)-u,x_{n_k}-u\rangle\\&=\langle f(u)-u,\tilde{x}-u\rangle\\&\le 0.\end{aligned}$$

Returning to (3.12) to obtain

$$\|x_{n+1}-u\|^2\le(1-\alpha_n)\|x_n-u\|^2+2\alpha_n\langle f(x_n)-u,x_{n+1}-u\rangle$$

$$\begin{aligned}&= (1-\alpha_n)\|x_n-u\|^2 + 2\alpha_n\langle f(x_n)-f(u), x_{n+1}-u\rangle \\ &\quad + 2\alpha_n\langle f(u)-u, x_{n+1}-u\rangle \\ &\le (1-\alpha_n)\|x_n-u\|^2 + 2\alpha_n\rho\|x_n-u\|\|x_{n+1}-u\| \\ &\quad + 2\alpha_n\langle f(u)-u, x_{n+1}-u\rangle \\ &\le (1-\alpha_n)\|x_n-u\|^2 + \alpha_n\rho(\|x_n-u\|^2+\|x_{n+1}-u\|^2) \\ &\quad + 2\alpha_n\langle f(u)-u, x_{n+1}-u\rangle.\end{aligned}$$

It follows that

$$\|x_{n+1}-u\|^2 \le [1-(1-\rho)\alpha_n]\|x_n-u\|^2 + \frac{2\alpha_n}{1-\alpha_n\rho}\langle f(u)-u, x_{n+1}-u\rangle. \tag{3.23}$$

Applying Lemma 2.4 to deduce $x_n \to u$.

Case 2. Assume there exists an integer $n_0$ such that $\|x_{n_0}-u\| \le \|x_{n_0+1}-u\|$. In this case, we set $\omega_n = \{\|x_n-u\|\}$. Then, we have $\omega_{n_0} \le \omega_{n_0+1}$. Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \le l \le n, \omega_l \le \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty}\tau(n) = \infty$$

and

$$\omega_{\tau(n)} \le \omega_{\tau(n)+1}$$

for all $n \ge n_0$. From (3.22), we get

$$\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

This implies that $\omega_w(x_{\tau(n)}) \subset F(T)$. Thus, we obtain

$$\limsup_{n\to\infty}\langle f(u)-u, x_{\tau(n)}-u\rangle \le 0. \tag{3.24}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from (3.23) that

$$\omega_{\tau(n)}^2 \le \omega_{\tau(n)+1}^2 \le [1-(1-\rho)\alpha_{\tau(n)}]\omega_{\tau(n)}^2 + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\rho}\langle f(u)-u, x_{\tau(n)+1}-u\rangle.$$

It follows that

$$\omega_{\tau(n)}^2 \le \frac{2}{(1-\alpha_{\tau(n)}\rho)(1-\rho)}\langle f(u)-u, x_{\tau(n)+1}-u\rangle. \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\limsup_{n\to\infty}\omega_{\tau(n)} \le 0,$$

and hence

$$\lim_{n\to\infty}\omega_{\tau(n)} = 0. \tag{3.26}$$

From (3.23), we obtain

$$\|x_{\tau(n)+1}-u\|^2 \le [1-(1-\rho)\alpha_\tau(n)]\|x_{\tau(n)}-u\|^2 + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\rho}\langle f(u)-u, x_{\tau(n)+1}-u\rangle.$$

It follows that

$$\limsup_{n\to\infty}\omega_{\tau(n)+1} \le \limsup_{n\to\infty}\omega_{\tau(n)}.$$

This together with (3.26) imply that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.3 to get

$$0 \le \omega_n \le \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to u$. The proof is completed. □

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