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On Reich fixed point theorem of G-contraction mappings on modular function spaces

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Abstract

We define the multivalued Reich (G, ρ)-contraction mappings on a modular function space. Then we obtain sufficient conditions for the existence of fixed points for such mappings. As an application, we introduce a ρ-valued Bernstein operator on the set of functions $f : [0, 1] \to L_\rho$ and then give the modular analogue to Kelisky-Rivlin theorem.

Keywords: Bernstein polynomial, directed graph, Reich fixed point theorem, monotone mapping, multivalued mapping, modular function spaces.
2010 MSC: 47H09, 46B20, 47H10, 47E10.

1. Introduction

Generalizing the Banach contraction principle for multivalued mapping to metric spaces, Nadler [13] obtained the following result:

Theorem 1.1 ([13]). *Let (X, d) be a complete metric space. Denote by $\mathcal{CB}(\mathcal{X})$ the set of all nonempty closed bounded subsets of X. Let $F : X \to \mathcal{CB}(\mathcal{X})$ be a multivalued mapping. If there exists $k \in [0, 1)$ such that*

$$H(F(x), F(y)) \leq k\ d(x, y)$$

for all $x, y \in X$, where H is the Hausdorff-Pompeiu distance on $\mathcal{CB}(\mathcal{X})$, then F has a fixed point in X, that is, there exists $x \in X$ such that $x \in T(x)$.

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A number of extensions and generalizations of Nadler's fixed point theorem were obtained by different authors; see for instance [6, 10] and references cited therein. Rand and Reurings [14] extended the Banach contraction principle to partially ordered metric spaces. Therefore, it was natural to find an extension of Nadler's fixed point theorem to partially ordered metric spaces. Beg and Butt [3] gave the first attempt. But their definition of multivalued monotone mappings was not correct which had the effect that the proof of their version of Nadler's fixed point theorem was wrong (see for example [2]).

Let (X, d) be a metric space. The Hausdorff-Pompeiu distance H is defined by

$$H(A,B) = \max\left\{ \sup_{b\in B} \inf_{a\in A} d(b,a),\ \sup_{a\in A} \inf_{b\in B} d(a,b) \right\}$$

for any $A, B \in \mathcal{CB}(\mathcal{X})$. The following technical result is useful to explain why we only deal with the regular metric distance instead of Hausdorff-Pompeiu distance H.

Lemma 1.2 ([13])**.** *Let (X, d) be a metric space. For any $A, B \in \mathcal{CB}(\mathcal{X})$ and $\varepsilon > 0$, and for any $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Denote by $\mathcal{C}(\mathcal{X})$ the family of all nonempty closed subsets of X. In [11], Mizoguchi and Takahashi obtained a modified version of the following result:

Theorem 1.3. *Let (X, d) be a complete metric space. Let $T : X \to \mathcal{C}(\mathcal{X})$ be a Reich contraction mapping, that is, there exists $k : (0, +\infty) \to [0, 1)$ with $\limsup_{s\to t+} k(s) < 1$, for any $t \in [0, +\infty)$, such that for any $x, y \in X$ and $a \in T(x)$, there exists $b \in T(y)$ such that*

$$d(a, b) \leq k(d(x, y))\ d(x, y).$$

Then T has a fixed point.

The aim of this paper is to extend such a result to the case of modular function spaces endowed with a graph. A number of related results of modular function spaces were obtained by different authors; see for instance [1, 12]. Before giving the results, we need to lie out some definitions, notations and facts about the background spaces. More information can be found in [9].

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended Σ-measurable functions on Ω. By 1_A we denote the characteristic function of the set A. We say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < +\infty\ \rho - a.e.\ \omega \in \Omega\}, \tag{2.1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. For simplicity, we write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.1 ([9])**.** Let ρ be a regular function pseudomodular. We say that ρ is a regular function modular if $\rho(f) = 0$ implies $f = 0\ \rho - a.e.$ for every $f \in \mathcal{M}$. We denote by $\Re$ the class of all nonzero regular function modulars defined on Ω.

Definition 2.2 ([9])**.** Let ρ be a function modular. A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M} \ : \ \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

Definition 2.3 ([9])**.** Let $\rho \in \Re$. We say that ρ satisfies the Δ_2-type condition if there exists $k \in [0, +\infty)$ such that

$$\rho(2f) \leq k \ \rho(f), \ \text{ for every } f \in L_\rho.$$

Definition 2.4 ([9])**.** Let $\rho \in \Re$:

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f \ (\rho)$ if $\rho(f_n - f) \to 0$.
(b) A sequence $\{f_n\}$ in L_ρ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.
(c) A set $B \subset L_\rho$ is called ρ-closed if for every sequence of $f_n \in B$ and every $f \in L_\rho$ the convergence $f_n \to f \ (\rho)$ implies that f belongs to B.

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general that $\lambda f_n \to \lambda f$ for every $\lambda > 1$.

Proposition 2.5 ([9])**.** *Let $\rho \in \Re$. Then we have the following:*

(i) *L_ρ is ρ-complete, that is, every ρ-Cauchy sequence in L_ρ is ρ-convergent.*
(ii) *If $\{f_n\}_{n \geq 1}$ is a monotone increasing (resp. decreasing) ρ-Cauchy sequence in L_ρ, then there exists $f \in L_\rho$ such that $\rho(f_n - f) \to 0$ and $f_n \leq f$ ρ-a.e. (resp. $f \leq f_n$ ρ-a.e.).*

The following technical lemma is essential to prove our main result.

Lemma 2.6 ([4])**.** *Let $\rho \in \Re$ be convex and satisfies the Δ_2-type condition. Let $\{f_n\}$ be a sequence in L_ρ such that*

$$\rho(f_{n+1} - f_n) \leq K \ \alpha^n, \ \ n = 1, \ldots,$$

where $K > 0$ and $\alpha \in (0, 1)$ are arbitrary constants. Then $\{f_n\}$ is ρ-Cauchy.

We conclude this section with the graph theory terminology that needed throughout. A directed graph G is an ordered triple $(V(G), E(G), I_G)$ where $V(G)$ is a nonempty set called the set of vertices of G, $E(G)$ is a possibly empty set, called the set of edges of G and I_G is an incidence map that associates with each edge of G an ordered pair of vertices of G. If e is an edge of G, and $I_G(e) = (u, u)$ for some $u \in V(G)$, then e is called a loop. If $E(G)$ contains all the loops, then G is reflexive.

As Jachymski did in [7], we introduce the following property:

Property 1. For any sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C \subseteq L_\rho$, if f_n ρ-converges to f and $(f_n, f_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there exists a subsequence $\{f_{\varphi(n)}\}$ of $\{f_n\}$ such that $(f_{\varphi(n)}, f) \in E(G)$, for every $n \in \mathbb{N}$.

We close this section by defining the Reich (G, ρ)-contraction mapping on a modular metric space L_ρ.

Definition 2.7. Let $\rho \in \Re$ and C be a nonempty subset of L_ρ. The multivalued map $T : C \to \mathcal{C}(\mathcal{C})$ is said to be Reich (G, ρ)-contraction if for every $f, g \in C$ such that $f \neq g$ and $(f, g) \in E(G)$, we have:

- if $u \in T(f)$, there exists $v \in T(g)$ such that $(u, v) \in E(G)$ and $\rho(u - v) \leq \alpha(\rho(f - g)) \ \rho(f - g)$,

where $\alpha : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t+} \alpha(s) < 1$, for any $t \in [0, +\infty)$.

Definition 2.8. Let $T : C \to \mathcal{C}(\mathcal{C})$. f is called a fixed point of T if $f \in T(f)$.

3. Results and discussions

We are now ready to state our main result of this work. In the sequel, we denote by $\mathcal{C}_\rho(C)$ the collection of all nonempty ρ-closed subsets of C.

Theorem 3.1. *Let $\rho \in \Re$, $C \subseteq L_\rho$ be nonempty ρ-closed and G a reflexive directed graph defined on C. Assume that ρ is convex and satisfies the Δ_2-type condition. Let $T : C \to \mathcal{C}_\rho(C)$ be a Reich (G,ρ)-contraction mapping and $C_T := \{f \in C;\ (f,g) \in E(G)\ for\ some\ g \in T(f)\}$. If C has Property 1, then T has a fixed point provided that $C_T \neq \emptyset$.*

Proof. Assume $C_T \neq \emptyset$. Let $f_0 \in C_T$. Then there exists $f_1 \in T(f_0)$ such that $(f_0, f_1) \in E(G)$. If $f_1 = f_0$, then f_0 is a fixed point of T. Assume $f_0 \neq f_1$, then there exists $f_2 \in T(x_1)$ such that

$$\rho(f_1 - f_2) \leq \alpha(\rho(f_0 - f_1))\rho(f_0 - f_1).$$

By induction, we construct a sequence $\{f_n\}$ in C such that $f_n \neq f_{n+1}$, $f_{n+1} \in T(f_n)$, $(f_n, f_{n+1}) \in E(G)$ and

$$\rho(f_n - f_{n+1}) \leq \alpha(\rho(f_{n-1} - f_n))\ \ \rho(f_{n-1} - f_n)$$

for any $n \geq 1$. Since $\alpha(t) < 1$, for any $t \in [0, +\inf)$, we conclude that $\{\rho(f_n - f_{n+1})\}$ is a decreasing sequence of positive numbers. Let

$$t = \lim_{n\to+\infty} \rho(f_n - f_{n+1}) = \inf_{n\in\mathbb{N}} \rho(f_n - x_{n+1}).$$

Since $\limsup_{s\to t+} \alpha(s) < 1$, there exist $k < 1$ and $n_0 \geq 1$ such that $\alpha(\rho(f_n - f_{n+1})) \leq k$, for any $n \geq n_0$. Then, we have

$$\rho(f_n - f_{n+1}) \leq \prod_{i=n_0}^{i=n} \alpha(\rho(f_i - f_{i+1}))\ \rho(f_{n_0} - f_{n_0+1}) \leq k^{n-n_0}\ \rho(f_{n_0} - f_{n_0+1})$$

for any $n \geq n_0$. Lemma 2.6 implies that $\{f_n\}$ is a ρ-Cauchy sequence. Since L_ρ is complete and C is ρ-closed, $\{f_n\}$ ρ-converges to some point $f \in C$. Let us prove that f is a fixed point of T. By Property 1 there exists a subsequence $\{f_{\varphi(n)}\}$ of $\{f_n\}$ such that $(f_{\varphi(n)}, f) \in E(G)$, for every $n \in \mathbb{N}$. Since T is Reich (G,ρ)-contraction, there exists $g_n \in T(f)$ such that

$$\rho(f_{\varphi(n)+1} - g_n) \leq \alpha(\rho(f_{\varphi(n)} - f))\ \rho(f_{\varphi(n)} - f) < \rho(f_{\varphi(n)} - f)$$

for every $n \in \mathbb{N}$ with $(f_{\varphi(n)+1}, g_n) \in E(G)$. Now, from the convexity of ρ,

$$\begin{aligned}
\rho\left(\frac{g_n - f}{2}\right) &= \rho\left(\frac{1}{2}(g_n - f_{\varphi(n)+1}) + \frac{1}{2}(f_{\varphi(n)+1} - f)\right) \\
&\leq \frac{1}{2}\rho(g_n - f_{\varphi(n)+1}) + \frac{1}{2}\rho(f_{\varphi(n)+1} - g) \\
&\leq \rho(g_n - f_{\varphi(n)+1}) + \rho(f_{\varphi(n)+1} - f) \\
&< \rho(f_{\varphi(n)} - f) + \rho(f_{\varphi(n)+1} - f)
\end{aligned}$$

for every $n \geq 1$. Since $\{f_n\}$ ρ-converges to f, we conclude that $\lim_{n\to\infty} \rho((g_n - f)/2) = 0$. The Δ_2- type condition satisfied by ρ implies that $\lim\limits_{n\to\infty} \rho(g_n - f) = 0$, that is, $\{g_n\}$ ρ-converges to f. Since $T(f)$ is ρ-closed, we conclude that $f \in T(f)$, that is, f is a fixed point of T.

□

Remark 3.2.

1. Once Theorem 3.1 is established, it is easy to extend it to the case of uniformly locally contractive mappings in the sense of Edelstein [5] with or without a graph.
2. If we assume G is such that $E(G) := C \times C$, then clearly G is connected and our Theorem 3.1 gives Mizoguchi-Takahashi theorem [11] and Nadler's theorem [13] as a consequence in the case that $\alpha(s)$ is constant. Moreover if T is single-valued, then we get the Reich's extension of Banach contraction principle [15].

4. Application: Bernstein operator in modular function spaces

In [8], Kelisky and Rivlin investigated the behavior of the iterates of the Bernstein polynomial of degree $n \geq 1$ defined by

$$B_n(f)(t) := \sum_{k=0}^{k=n} f\left(\frac{k}{n}\right)\left(\begin{array}{c} n \\ k \end{array}\right) t^k(1-t)^{n-k}$$

for every $f \in C([0,1])$ and $t \in [0,1]$, where $C([0,1])$ is the space of continuous functions defined on $[0,1]$. In particular, they proved that for any $f \in C([0,1])$, we have

$$\lim_{j\to+\infty} {B_n}^j(f)(t) = f(0)(1-t) + f(1)t, \qquad 0 \leq t \leq 1, \tag{KRB}$$

where ${B_n}^j$ means the j-th iterate/power of B_n. Their proof uses the techniques of matrix algebra. Rus [16] was the first one to notice that a proof of (KRB), of metric nature, exists. In fact, his proof inspired Jachymski [7] to rephrase it in identically using the graph language. Our aim here is to extend the classical Bernstein operator B_n to the modular case. Indeed, let L_ρ be a modular function space. Fix $n \geq 1$. For any $f : [0,1] \to L_\rho$ define the Bernstein operator

$$B_{n,\rho}(f)(t) := \sum_{k=0}^{k=n} \left(\begin{array}{c} n \\ k \end{array}\right) t^k(1-t)^{n-k}\ f\left(\frac{k}{n}\right)$$

for every $t \in [0,1]$. In this case, we have a conclusion similar to the result of Kelisky and Rivlin for modular function spaces.

Theorem 4.1. *Let ρ be a convex modular function satisfying δ_2-type condition. Then for every $f : [0,1] \to L_\rho$, we have*

$$\lim_{j\to+\infty}\ \sup_{t\in[0,1]} \rho\Big({B_{n,\rho}}^j(f)(t) - (1-t)f(0) - tf(1)\Big) = 0,$$

provided $\sup_{t\in[0,1]} \rho\Big(f(t) - (1-t)f(0) - tf(1)\Big) < +\infty.$

Proof. Let us first notice that

$$\sum_{k=0}^{k=n} \left(\begin{array}{c} n \\ k \end{array}\right) t^k(1-t)^{n-k} = 1, \qquad \textit{and} \qquad \sum_{k=0}^{k=n} \frac{k}{n}\left(\begin{array}{c} n \\ k \end{array}\right) t^k(1-t)^{n-k} = t$$

for any $t \in [0,1]$. Define the reflexive directed graph G on functions from $[0,1]$ into L_ρ such that $(f,g) \in E(G)$ if and only if $f(0) = g(0)$ and $f(1) = g(1)$. Set $h(t) = (1-t)f(0) + tf(1)$, for $t \in [0,1]$. Obviously $h : [0,1] \to L_\rho$. We have $B_n(h) = h$. Since $f(0) = h(0)$ and $f(1) = h(1)$, we have $(f,h) \in E(G)$ and

$$B_n(f)(t) - B_n(h)(t) = \sum_{k=1}^{n-1} \left(\begin{array}{c} n \\ k \end{array}\right) t^k(1-t)^{n-k}\ \left(f\left(\frac{k}{n}\right) - h\left(\frac{k}{n}\right)\right)$$

for every $t \in [0,1]$. Hence, by convexity of ρ, we have

$$\rho(B_n(f)(t) - B_n(h)(t)) \leq \sum_{k=1}^{n-1} \left(\begin{array}{c} n \\ k \end{array}\right) t^k(1-t)^{n-k}\ \rho\left(f\left(\frac{k}{n}\right) - h\left(\frac{k}{n}\right)\right)$$

for every $t \in [0,1]$, which implies

$$\sup_{t\in[0,1]} (\rho(B_n(f)(t) - B_n(h)(t))) \leq \left(1 - \frac{1}{2^{n-1}}\right)\ \sup_{t\in[0,1]} (\rho(f(t) - h(t))).$$

Since $B_n(h) = h$, we get

$$\sup_{t\in[0,1]} (\rho(B_n(f)(t) - h(t))) \leq \left(1 - \frac{1}{2^{n-1}}\right) \sup_{t\in[0,1]} (\rho(f(t) - h(t))).$$

By induction, we obtain

$$\sup_{t\in[0,1]} (\rho(B_n^j(f)(t) - h(t))) \leq \left(1 - \frac{1}{2^{n-1}}\right)^j \sup_{t\in[0,1]} (\rho(f(t) - h(t)))$$

for every $j \in \mathbb{N}$. This clearly implies the conclusion of Theorem 4.1 as claimed. □

Motivated by Sultana and Vetrivel example [17], we introduce the following Bernstein operator B'_n defined by

$$B'_n(f)(t) := \sum_{k=0}^{k=n} \binom{n}{k} t^k (1-t)^{n-k}\, T\left(f\left(\frac{k}{n}\right)\right), \qquad t \in [0,1], \tag{4.1}$$

where $T : L_\rho \to L_\rho$. Since $B'_n(f) = B_n(T \circ f)$, we obtain the following result:

Corollary 4.2. *Let ρ be a convex modular function satisfying δ_2-type condition. Then for every $f \in ([0,1], L_\rho)$, we have*

$$\lim_{j\to+\infty} (B'_n)^j(f)(t) = (1-t)T\Big(f(0)\Big) + t\, T\Big(f(1)\Big), \qquad 0 \leq t \leq 1,$$

provided $\sup_{t\in[0,1]} \rho\Big(T(f(t)) - (1-t)T(f(0)) - t\, T(f(1))\Big) < +\infty.$

Notice that our generalized Bernstein operator B_n is Reich's (G,ρ)-contraction map with constant $\alpha = 1 - \frac{1}{2^{n-1}}$.

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