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A note on strict feasibility and solvability for pseudomonotone equilibrium problems

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Abstract

The aim of this note is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problem with stably pseudomonotone mappings. Our result extends and improves recent results in the literature for monotone equilibrium problems.

Keywords: Equilibrium problem, strict feasibility, stably pseudomonotone mapping.
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1. Introduction

Let K be a nonempty, closed and convex subset of a real reflexive Banach space X and let $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a bifunction. The equilibrium problem [5] is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K, \tag{1.1}$$

and its dual problem is to find $\bar{x} \in K$ such that

$$f(y, \bar{x}) \leq 0, \quad \forall y \in K. \tag{1.2}$$

The solutions sets to (1.1) and (1.2) are denoted by S^P and S^D, respectively.

It is well-known that the equilibrium problem provides a unified model of several classes of problems, including variational inequality problems, complementarity problems, optimization problems. There are

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many papers which have discussed the existence solution to the equilibrium problem (see [2–5] and references therein). In order to solve the problem when K is unbounded, many authors studies the coercive assumption. Strict feasibility condition [9, 10] is one of the most useful tool to characterize the nonemptiness and boundedness for the solution set of variational inequality problem. Recently, the concept of strict feasibility was extended to equilibrium problem by Hu and Fang [11]. They proved that under suitable conditions, the monotone equilibrium problem has a nonempty and bounded solution set, if and only if it is strictly feasible.

Motivated and inspired by Hu and Fang [11] and the idea of stable pseudomonotonicity in [8, 12], the purpose of this paper is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problems with stably pseudomonotone mappings.

2. Preliminaries

Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with dual space X^*. The dual cone K^* of K is defined as

$$K^* := \{\xi \in X^* : \langle \xi, x\rangle \geq 0,\ \forall x \in K\}.$$

The barrier cone, *barr* K of K is defined as

$$barr\,K := \left\{\xi \in X^* : \sup_{x\in K}\langle \xi, x\rangle < +\infty\right\}.$$

It is well-known that $-K^* \subset barr\,K$. The asymptotic cone of K, denoted by K_∞, is defined by

$$K_\infty := \{d \in X : \exists\lambda_k \to 0, \exists x_k \in K \text{ with } \lambda_k x_k \rightharpoonup d\},$$

where "$\rightharpoonup$"means weak convergence. It is known that, for given $x \in K$,

$$K_\infty := \{d \in X : x + td \in K,\ \forall t > 0\}.$$

The following interesting results can be found in [1, 9].

Lemma 2.1. *Let K be a nonempty, closed and convex subset of a reflexive Banach space X. Then K is well-positioned, if and only if int(barr K) $\neq \emptyset$. Furthermore, if K is well-positioned, then there is no sequence $\{x_n\} \subset K$ with $\|x_n\| \to \infty$ such that $x_n/\|x_n\| \rightharpoonup 0$.*

Lemma 2.2. *Let K be a well-positioned, closed, convex subset of a real reflexive Banach space X and $\{A_n\}$ be a decreasing sequence of closed convex subsets of K with $A := \cap_{n=1}^{\infty} A_n$ nonempty and bounded. Then A_n is bounded for some n.*

Lemma 2.3. *Let K be a well-positioned, closed, convex subset of a real reflexive Banach space X, $h : K \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function, and $r_1, r_2 \in \mathbb{R}$ with $r_1 \leq r_2$. If the level set $\{x \in K : h(x) \leq r_1\}$ is nonempty and bounded, then so is the set $\{x \in K : h(x) \leq r_2\}$.*

The concepts of feasibility and strict feasibility for equilibrium problems can be introduced by means of asymptotic cone.

We say that (1.1) (or (1.2)) is feasible [7], if the set $\mathcal{F}_K := \{x \in K : f(x, x + d) \geq 0,\ \forall d \in K_\infty\} \neq \emptyset$. We say that (1.1) (or (1.2)) is strictly feasible [11], if the set $\mathcal{F}_K := \{x \in K : f(x, x + d) > 0,\ \forall d \in K_\infty\backslash\{0\}\}$.

A bifunction $f : K\times K \to \mathbb{R}\cup\{+\infty\}$ is said to be monotone on K, if for all $x, y \in K$, $f(x, y)+f(y, x) \leq 0$, pseudo-monotone on K, if for all $x, y \in K$, $f(x, y) \geq 0$ implies $f(y, x) \leq 0$. A pseudomonotone bifunction f is said to be stably pseudomonotone on K w.r.t. $U \subset X^*$, due to [8, 12], if for all $x, y \in K$ and for all $u^* \in U$, $f(x, y) - \langle \xi, y - x\rangle \geq 0$ implies $f(y, x) - \langle \xi, x - y\rangle \leq 0$. It is well-known that every monotone mapping is a stably pseudomonotone mapping.

The following example illustrates that the converse inclusion may not be true.

Example 2.4. Let $X = \mathbb{R}$ with $X^* \equiv \mathbb{R}$ and $K = [3, 4]$. Define $f : K \times K \to \mathbb{R} \cap \{+\infty\}$ by

$$f(x, y) = y(x - y), \quad \forall x, y \in K.$$

Then f is stably pseudomonotone on K w.r.t. $[1, 2]$ but not monotone.

Proof. We first show that f is stably pseudomonotone on K w.r.t. $[1, 2]$. For any $x, y \in K$ and $\xi \in [1, 2]$, If

$$\begin{aligned} 0 \le f(x, y) - \langle \xi, y - x\rangle &= y(y - x) - \xi(y - x) \\ &= (y - \xi)(y - x). \end{aligned}$$

Then $y - x \ge 0$, since $z - \xi > 0$ for all $z \in K$ and $\xi \in [1, 2]$. It then follows that

$$\begin{aligned} f(y, x) - \langle \xi, x - y\rangle &= x(x - y) - \xi(x - y) \\ &= (x - \xi)(x - y) \le 0. \end{aligned}$$

□

Hence, f is stably pseudomonotone on K w.r.t. $[1, 2]$.

Finally, it is easily seen that f is not monotone. Indeed, for any $x, y \in K$

$$f(x, y) + f(y, x) = y(y - x) + x(x - y) = (x - y)^2 \ge 0.$$

We collect the following well-known KKM-Fan lemma.

Lemma 2.5 ([6])**.** *Let M be a nonempty, closed and convex subset of X and $F : M \to 2^M$ be a set-valued map. Suppose that for any finite set $\{x_1, \ldots, x_m\} \subseteq M$, one has*

(i) *$conv\{x_1, \ldots, x_m\} \subset \bigcup_{i=1}^m F(x_i)$ (i.e., F is a KKM map on M);*

(ii) *$F(x)$ is closed for every $x \in M$; and*

(iii) *$F(x)$ compact for some $x \in M$.*

Then $\bigcap_{x\in M} F(x) \neq \emptyset$.

3. Main results

In this section, we present the characterization of the solution for (1.1) and (1.2).

Before proving our results, we list the following assumptions hold.

(f_1) For any $x \in K$, $f(x, x) \ge 0$.

(f_2) For any $x, y \in K$ and $\alpha \in [0, 1]$, $\lim_{\alpha\to+0} \dfrac{f(x + t(y - x), x)}{t} = -f(x, y)$.

(f_3) For any $x \in K$, the map $y \mapsto f(x, y)$ is convex and lower semicontinuous.

(f_4) f is stably pseudomonotone on K w.r.t. K^*.

The following lemma shows that under suitable conditions, the solution set of (1.1) and (1.2) are the same.

Lemma 3.1. *Let K be a nonempty, closed, and convex subset of a real reflexive Banach space X. Suppose that f satisfies the conditions (f_1)–(f_4). Then for any given $\xi \in K^*$, the solution of the problem $x \in K$ is a solution of the problem*

$$\text{find } x \in K \text{ such that } f(x, y) - \langle \xi, y - x\rangle \ge 0, \quad \forall y \in K, \tag{3.1}$$

if and only if it is a solution of the problem

$$\text{find } x \in K \text{ such that } f(y, x) - \langle \xi, x - y\rangle \le 0, \quad \forall y \in K. \tag{3.2}$$

In addition, if K is bounded, then S^P and S^D are nonempty.

Proof. Let x be a solution of (3.1). Then $f(x,y) - \langle \xi, y-x\rangle \geq 0$, for all $y \in K$. By condition (f_4), x is a solution of (3.2). Conversely, let x be a solution of (3.2). Then $f(y,x) - \langle \xi, x-y\rangle \geq 0$, for all $y \in K$. For any $z \in K$, set $z_t := x + t(z-x)$, for any $t \in (0,1)$. Then $z_t \in K$, because of the convexity of K. It follows from (f_1) that

$$f(z_t, z) \leq \langle \xi, x - z_t\rangle = t\langle \xi, x - z\rangle.$$

Condition (f_2) gives that $-f(z,x) \leq \langle \xi, x-z\rangle$. Hence, x solves (3.1).

Next, we prove that S^P and S^D are nonempty provided that K is bounded. Define $F, G : K \to 2^K$ by

$$F(y) = \{x \in K : f(x,y) \geq 0\}, \ \forall y \in K \text{ and } G(y) = \{x \in K : f(y,x) \leq 0\}, \quad \forall y \in K.$$

Then, $G(y)$ is closed, convex and compact for all $x \in K$. It follows from Lemma 2.5 that

$$\bigcap_{y\in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

Then there exists $\bar{x} \in K$ which solves (1.1) and (1.2). □

Theorem 3.2. *Let K be a nonempty, closed, and convex subset of a real reflexive Banach space X with $int\, K^* \neq \emptyset$. Suppose that f satisfies the conditions (f_1)–(f_4). Then the following statements are equivalent:*

(i) *S^P is a nonempty and bounded.*

(ii) *S^D is a nonempty and bounded.*

(iii) *$\mathcal{F}_K^+ \neq \emptyset$.*

Proof. It follows from Lemma 3.1,
(i) $\Leftrightarrow$ (ii): in the case where $\xi = 0$. Define a function $g : K \times K^* \to \mathbb{R}$ by

$$g(x,\xi) := \sup_{y\in K} \frac{f(y,x) - \langle \xi, x-y\rangle}{\max\{1, \|y\|\}}, \quad \forall x \in K, \ \xi \in K^*.$$

It follows from (f_3) that $g(\cdot,\xi)$ is convex and lower semicontinuous function and $g(x) \geq 0$, for all $x \in K$ and $\xi \in K^*$. Define the set $A := \{x \in K : g(x) \leq 0\}$. Then by Lemma 2.5, A is nonempty, closed and convex set. Clearly, A is the solution set of (1.2). By Lemma 3.1, A is also the solution set of (1.1). Let $\xi' \in int\, K^*$. For every positive integer n, define

$$A_n := \left\{ x \in K : g(x,0) \leq \frac{1}{n}\langle \xi', x\rangle \right\}.$$

Then $\{A_n\}$ is a decreasing sequence as $n \to +\infty$ of closed and convex subsets of K and $A = \bigcap_{i=1}^{\infty} A_i$. Notice that $barr\, K$ has a nonempty interior, then K is well-positioned.

(ii) $\Rightarrow$ (iii): Suppose that A is nonempty and bounded. It follows from Lemma 2.2 that there exists n_0 such that A_{n_0} is nonempty and bounded. Set $r := g\left(z_0, \frac{1}{n}\xi'\right)$ for some $z_0 \in A_{n_0}$. We then have $r \geq 0$. We now consider the following set $C_0 := \left\{ x \in K : g(x,0) \leq \frac{1}{n_0}\langle \xi, x\rangle + r\right\}$. Then by Lemma 2.3, C_0 is nonempty and bounded. After calculating we have, for any $y \in K$,

$$\frac{f(y,x)}{\max\{1,\|y\|\}} - \frac{f(y,x) - \left\langle \frac{1}{n_0}\xi, x-y\right\rangle}{\max\{1,\|y\|\}} = \frac{1}{n_0}\frac{\langle \xi, x-y\rangle}{\max\{1,\|y\|\}} \leq \frac{1}{n_0}\langle \xi, x\rangle,$$

because of $\xi' \in int\ K^*$. Thus $g(x,0) - g\left(x, \frac{1}{n_0}\xi'\right) \leq \frac{1}{n_0}\langle \xi', x\rangle$, and so the set

$$C := \{x \in K : g\left(x, \frac{1}{n_0}\xi'\right) \leq r\} \subset C_0,$$

is nonempty, closed, convex and bounded. Set

$$K_i = \{x \in K : \|x\| \leq i\}, \quad i = 1, 2, \cdots,$$

and

$$\hat{g}_i(x) := \sup_{y \in K_i} \frac{f(y,x) - \frac{1}{n_0}\langle \xi', x - y\rangle}{\max\{1, \|y\|\}}, \quad \forall x \in X.$$

Then $\hat{g}_i(x)$ is convex and lower semicontinuous on bounded subset K_i for all $i \in \mathbb{N}$ and $\hat{g}_i(x) \geq 0$ for all $x \in K_i$. For every $i \in \mathbb{N}$, it follows from the proof in Lemma 3.1 that there are $x_i \in K_i$ such that $f(y, x_i) - \frac{1}{n_0}\langle \xi', x_i - y\rangle \leq 0,\ \forall y \in K_i$. This implies that $\hat{g}_i(x_i) = 0$. Define $D_i := \{x \in K : \hat{g}_i(x) \leq r\}$. It is not hard to check that $\{D_i\}_{i=1}^{\infty}$ is a decreasing sequence of nonempty, closed and convex subsets of K and $C = \bigcap_{i=1}^{\infty} D_i$. Since C is nonempty and bounded, it follows from Lemma 2.2 that there exists $i_0 \in \mathbb{N}$ such that D_{i_0} is nonempty and bounded. Then there exists a positive integer L such that $\sup_{x \in D_{i_0}} \|x\| < L$ and $i_0 \leq L$. Since $\hat{g}_L(x_L) = 0$, $x_L \in D_L \subset D_{i_0}$, and so $x_L \in K_L$.

For any $y \in K$ and $t \in (0,1)$, by setting $y_t := (1-t)x_L + ty \in K_L$, we have $y_t \to x_L$ as $t \to +0$. It follows from linearity that, for all $y \in K$

$$f(y_t, x_L) - \frac{1}{n_0}\langle \xi', x_L - y_t\rangle \leq 0 \Rightarrow \frac{f(y_t, x_L)}{t} - \frac{1}{n_0}\langle \xi', x_L - y\rangle \leq 0.$$

By (f_3), we have $f(x_L, y) \geq \frac{1}{n_0}\langle \xi', y - x_L\rangle$ for all $y \in K$.

For any $d \in K_\infty \backslash \{0\}$, we known that $x_L + d \in K$, and so $f(x_L, x_L + d) \geq \frac{1}{n_0}\langle \xi', d\rangle > 0$, since $\xi' \in int\, K^*$. Therefore, $x_L \in \mathcal{F}_K^+$.

(iii) $\Rightarrow$ (i): Suppose that $\mathcal{F}_K^+ \neq \emptyset$. Then there exists $x_0 \in K$ such that $f(x_0, x_0 + d) > 0$, for all $d \in K_\infty \backslash \{0\}$. Set $D := \{x \in K : f(x_0, x) \leq 0\}$. It is not hard to check that D is nonempty, closed and convex. We claim that D is bounded. If not, there exists $x_n \in D$ with $\|x_n\| \to +\infty$. Without loss of generality, we can suppose that $x_n/\|x_n\| \rightharpoonup d \in K_\infty$. Since K is well-positioned, from Lemma 2.1 we have $d \neq 0$. It follows from $x_n \in D$ that $f(x_0, x_n) \leq 0$. Since f is convex and lower semicontinuous at second variable, it follows that

$$f(x_0, x_0 + d) \leq \liminf_{n \to \infty} f\left(x_0, x_0 + \frac{1}{\|x_n\|}(x_n - x_0)\right) \leq 0,$$

which contradicts $x_0 \in \mathcal{F}_K^+$. Thus D is nonempty, convex and weak compact. Define

$$D_z := \{x \in D : f(x, z) \geq 0\}, \quad \forall z \in K.$$

Thus

$$\bar{x} \in S^P \Leftrightarrow \bar{x} \in \cap_{z \in K} D_z.$$

Define

$$D'_z = \{x \in D : f(z, x) \leq 0,\ \forall z \in K\}.$$

Then D'_z is nonempty, closed and convex. By a similar argument as in the proof of Lemma 3.1, we can show that

$$\cap_{z \in K} D'_z = \cap_{z \in K} D_z.$$

For any finite set $\{z_i : i = 1, 2, \dots, n\} \subset K$, let $M = conv\{D \cup \{z_1, z_2, \dots, z_n\}\}$. It is known that M is weakly compact. By Lemma 3.1, there exists $\hat{x} \in M$ such that

$$f(y, \hat{x}) \leq 0, \quad \forall y \in M.$$

From $x_0 \in D \subset M$, we have $f(x_0, \hat{x}) \leq 0$ and so $\hat{x} \in D$. Furthermore, we have

$$f(z_i, \hat{x}) \leq 0, \quad i = 1, 2, \cdots, n,$$

because $z_i \in M$. Therefore $\hat{x} \in \cap_{i=1}^{n} D'_{z_i}$, and so $\{D'_z : z \in K\}$ has the finite intersection property. Since D is weakly compact, we get $S^P = \cap_{z \in K} D_z = \cap_{z \in K} D'_z \neq \emptyset$. Then S^P is nonempty. Since D is bounded and $S^P = \cap_{z \in K} D_z \subset D$, we have S^P is bounded. The proof is complete. □

Remark 3.3. Theorem 3.2 discusses the characterization of nonemptiness and boundedness for S^P and S^D, which is more general than the result in [11, Theorem 3.1] in the case that f is relaxed to the stably pseudomonotone mapping.

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