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Extension of Furuta inequality with nonnegative powers for multi-operator

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Abstract

We prove an extension of Furuta inequality with nonnegative powers for multi-operator. Then we show its application to Pedersen-Takesaki type operator equation.

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1. Introduction

A capital letter, such as T, stands for a bounded linear operator on a Hilbert space $\mathscr{H}$. $T \geq 0$ and $T > 0$ mean that T is positive and T is strictly positive, respectively.

As an extension of Löwner-Heinz inequality ($A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$), T. Furuta in 1987 obtained the following famous inequality, which is called Furuta inequality.

Theorem 1.1 ([2], Furuta inequality). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}} \tag{1.1}$$

and

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{1.2}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

In 1995, T. Furuta proved a grand form of Theorem 1.1, which is called grand Furuta inequality as follows.

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Theorem 1.2 ([3], grand Furuta inequality). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.3}$$

holds for $s \geq 1$ and $r \geq t$.

In 2008, T. Furuta proved an extension of grand Furuta inequality as follows.

Theorem 1.3 ([4], extension of grand Furuta inequality). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n} \geq 1$, the following inequality*

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \ldots A^{\frac{t}{2}} \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} \left(A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}} \right)^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{t}{2}} \ldots A^{\frac{-t}{2}} \right]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\phi[2n]}} \tag{1.4}$$

holds for $r \geq t$, where $\phi[2n] = \{\ldots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\cdots-t\}p_{2n}+r$.

Furthermore, C. Yang and Y. Wang proved an extension of grand Fututa inequality for multi-operator, which is called Further extension of grand Furuta inequality.

Theorem 1.4 ([12], Further extension of grand Furuta inequality). *If $A_{2n+1} \geq A_{2n} \geq A_{2n-1} \geq \cdots \geq A_3 \geq A_2 \geq A_1 \geq 0$ with $A_2 > 0$, $t_1, t_2, \ldots, t_{n-1}, t_n \in [0,1]$, $p_1, p_2, \ldots, p_{2n-1}, p_{2n} \geq 1$, then the following operator inequality*

$$\begin{aligned} A_{2n+1}^{1-t_n+r} \geq \Big\{ & A_{2n+1}^{\frac{r}{2}} \Big[A_{2n}^{-\frac{t_n}{2}} \Big\{ A_{2n-1}^{\frac{t_{n-1}}{2}} \cdots A_5^{\frac{t_2}{2}} \Big[A_4^{-\frac{t_2}{2}} \Big\{ A_3^{\frac{t_1}{2}} (A_2^{-\frac{t_1}{2}} A_1^{p_1} A_2^{-\frac{t_1}{2}})^{p_2} \\ & A_3^{\frac{t_1}{2}} \Big\}^{p_3} A_4^{-\frac{t_2}{2}} \Big]^{p_4} A_5^{\frac{t_2}{2}} \cdots A_{2n-1}^{\frac{t_{n-1}}{2}} \Big\}^{p_{2n-1}} A_{2n}^{-\frac{t_n}{2}} \Big]^{p_{2n}} A_{2n+1}^{\frac{r}{2}} \Big\}^{\frac{1-t_n+r}{\psi[2n]-t_n+r}} \end{aligned} \tag{1.5}$$

holds for $r \geq t_n$, where $\psi[2n] = \{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$.

The powers of grand Furuta inequality, extension of grand Furuta inequality, further extension of grand Furuta inequality includes negative powers. Recently, T. Furuta obtained an extension of Furuta inequality with nonnegative powers as follows.

Theorem 1.5 ([5]). *If $A \geq B \geq 0$, $r_1, r_2, \cdots, r_n \geq 0$, then*

$$\begin{aligned} A^{1+r_1+r_2+\cdots+r_n} \geq \Big\{ & A^{\frac{r_n}{2}} \Big[A^{\frac{r_{n-1}}{2}} \Big(A^{\frac{r_{n-2}}{2}} \cdots \big[A^{\frac{r_2}{2}} \big(A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}} \big)^{p_2} A^{\frac{r_2}{2}} \big]^{p_3} \\ & \cdots A^{\frac{r_{n-2}}{2}} \Big)^{p_{n-1}} A^{\frac{r_{n-1}}{2}} \Big]^{p_n} A^{\frac{r_n}{2}} \Big\}^{\frac{1+r_1+r_2+\cdots+r_n}{\phi[n]}} \end{aligned} \tag{1.6}$$

holds, where $p_1 \geq 1, p_2 \geq \frac{1+r_1}{p_1+r_1}, \cdots, p_n \geq \frac{1+r_1+r_2+\cdots+r_{n-1}}{\phi[n-1]}$, and

$$\phi[k] = \big\{ \big[\big(\cdots [(p_1+r_1)p_2+r_2]p_3+\cdots \big) p_{k-2} + r_{k-2} \big] p_{k-1} + r_{k-1} \big\} p_k + r_k.$$

There are many applications of the above-mentioned operator inequalities. See [7, 8, 9, 10, 11] for details.

In this paper, we shall prove an extension of Furuta inequality with nonnegative powers for multi-operator. Then we show its application to Pedersen-Takesaki type operator equation.

2. Main Result

In this section, we shall show the main result.

Theorem 2.1. *If $A_n \geq A_{n-1} \geq \cdots \geq A_2 \geq A_1 \geq 0$, $t_1, t_2, \cdots, t_{n-1} \geq 0$, then*

$$A_n^{1+t_{n-1}} \geq \left\{ A_n^{\frac{t_{n-1}}{2}} \left(A_{n-1}^{\frac{t_{n-2}}{2}} \cdots \big[A_3^{\frac{t_2}{2}} \big(A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}} \big)^{p_2} A_3^{\frac{t_2}{2}} \big]^{p_3} \cdots A_{n-1}^{\frac{t_{n-2}}{2}} \right)^{p_{n-1}} A_n^{\frac{t_{n-1}}{2}} \right\}^{\frac{1+t_{n-1}}{\beta_{n-1}}} \tag{2.1}$$

holds for $p_1 \geq 1, p_2 \geq \frac{1}{\beta_1}, p_3 \geq \frac{1}{\beta_2}, \cdots, p_{n-1} \geq \frac{1}{\beta_{n-2}}$, where $\beta_1 = p_1 + t_1$, $\beta_k = \beta_{k-1} p_k + t_k$, $k = 1, 2, \cdots, n-1$.

Proof. If $n = 2$, Theorem 2.1 is that $A_2 \geq A_1 \geq 0$ ensures $A_2^{1+t} \geq (A_2^{\frac{t}{2}} A_1^p A_2^{\frac{t}{2}})^{\frac{1+t}{p+t}}$ for $p \geq 1$ and $t \geq 0$, which is just Furuta inequality.

If the theorem holds for $n = k$, then we have

$$A_k^{1+t_{k-1}} \geq \left\{ A_k^{\frac{t_{k-1}}{2}} \left(A_{k-1}^{\frac{t_{k-2}}{2}} \cdots [A_3^{\frac{t_2}{2}} (A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}})^{p_2} A_3^{\frac{t_2}{2}}]^{p_3} \cdots A_{k-1}^{\frac{t_{k-2}}{2}} \right)^{p_{k-1}} A_k^{\frac{t_{k-1}}{2}} \right\}^{\frac{1+t_{k-1}}{\beta_{k-1}}}. \tag{2.2}$$

For $0 < \frac{1}{1+t_{k-1}} \leq 1$, applying Löwner-Heinz inequality to (2.2), we have

$$A_k \geq \left\{ A_k^{\frac{t_{k-1}}{2}} \left(A_{k-1}^{\frac{t_{k-2}}{2}} \cdots [A_3^{\frac{t_2}{2}} (A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}})^{p_2} A_3^{\frac{t_2}{2}}]^{p_3} \cdots A_{k-1}^{\frac{t_{k-2}}{2}} \right)^{p_{k-1}} A_k^{\frac{t_{k-1}}{2}} \right\}^{\frac{1}{\beta_{k-1}}}. \tag{2.3}$$

Let $A = A_{k+1}$, $B = \left\{ A_k^{\frac{t_{k-1}}{2}} \left(A_{k-1}^{\frac{t_{k-2}}{2}} \cdots [A_3^{\frac{t_2}{2}} (A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}})^{p_2} A_3^{\frac{t_2}{2}}]^{p_3} \cdots A_{k-1}^{\frac{t_{k-2}}{2}} \right)^{p_{k-1}} A_k^{\frac{t_{k-1}}{2}} \right\}^{\frac{1}{\beta_{k-1}}}$. Notice that $A = A_{k+1} \geq A_k \geq B$, $p_k\beta_{k-1} \geq 1$, $t_k \geq 0$. By Furuta inequality, we have

$$A^{1+t_k} \geq (A^{\frac{t_k}{2}} B^{p_k\beta_{k-1}} A^{\frac{t_k}{2}})^{\frac{1+t_k}{p_k\beta_{k-1}+t_k}}. \tag{2.4}$$

Equation (2.4) is just

$$A_k^{1+t_k} \geq \left\{ A_{k+1}^{\frac{t_k}{2}} \left\{ A_k^{\frac{t_{k-1}}{2}} \cdots [A_3^{\frac{t_2}{2}} (A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}})^{p_2} A_3^{\frac{t_2}{2}}]^{p_3} \cdots A_k^{\frac{t_{k-1}}{2}} \right\}^{p_k} A_{k+1}^{\frac{t_k}{2}} \right\}^{\frac{1+t_k}{\beta_k}}. \tag{2.5}$$

Equation (2.5) means that theorem holds for $n = k + 1$. □

Corollary 2.2 ([13])**.** *If $A_3 \geq A_2 \geq A_1 \geq 0$, $t_1, t_2 \geq 0$, then*

$$A_3^{1+t_2} \geq [A_3^{\frac{t_2}{2}} (A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}})^{p_2} A_3^{\frac{t_2}{2}}]^{\frac{1+t_2}{(p_1+t_1)p_2+t_2}} \tag{2.6}$$

holds for $p_1 \geq 1, p_2 \geq \frac{1}{p_1+t_1}$.

Remark 2.3. Corollary 2.2 is a known result which is proved in [13] by the method of operator monotonic function.

3. Application

In this section, we shall show the main result's application to Pedersen-Takesaki type operator equation. In order to prove the result, we list a lemma first.

Lemma 3.1 ([1], Douglas Theorem)**.** *The following statements are equivalent.*

(I) *$BB^* \leq \lambda^2 AA^*$, that is, $\|B^*x\| \leq \lambda \|A^*x\|$ for some $\lambda \geq 0$ and all $x \in \mathscr{H}$;*

(II) *There exists C such that $B = AC$.*

Moreover, $\|C\|^2 = inf\{u : BB^* \leq uAA^*\}$.

Next, we shall show the application.

Theorem 3.2. *If there is a nonnegative integer k such that $(k + 1)(1 + t_{n-1}) = \beta_{n-1}$, $t_1, t_2, \cdots, t_{n-1} \geq 0$, $p_1 \geq 1, p_2 \geq \frac{1}{\beta_1}, p_3 \geq \frac{1}{\beta_2}, \cdots, p_{n-1} \geq \frac{1}{\beta_{n-2}}$, where $\beta_1 = p_1 + t_1$, $\beta_k = \beta_{k-1}p_k + t_k$, $k = 1, 2, \cdots, n-1$, then there exists a unique X, $X \geq 0$ with $\|X\| \leq 1$ satisfies the following equation,*

$$\begin{aligned} &\left(A_{n-1}^{\frac{t_{n-2}}{2}} \cdots [A_3^{\frac{t_2}{2}} (A_2^{\frac{t_1}{2}} A_1^{p_1} A_2^{\frac{t_1}{2}})^{p_2} A_3^{\frac{t_2}{2}}]^{p_3} \cdots A_{n-1}^{\frac{t_{n-2}}{2}} \right)^{p_{n-1}} \\ &\qquad = A_n^{\frac{1}{2}} (X A_n^{1+t_{n-1}})^k X A_n^{\frac{1}{2}} = A_n^{\frac{1}{2}} X (A_n^{1+t_{n-1}} X)^k A_n^{\frac{1}{2}}, \end{aligned} \tag{3.1}$$

where $A_n \geq A_{n-1} \geq \cdots \geq A_2 \geq A_1 > 0$.

Proof. Applying Douglas Theorem to Theorem 2.1, there exists a unique operator S, such that $\|S\| \leq 1$ and

$$\left\{A_n^{\frac{t_{n-1}}{2}}\left(A_{n-1}^{\frac{t_{n-2}}{2}}\cdots\left[A_3^{\frac{t_2}{2}}\left(A_2^{\frac{t_1}{2}}A_1^{p_1}A_2^{\frac{t_1}{2}}\right)^{p_2}A_3^{\frac{t_2}{2}}\right]^{p_3}\cdots A_{n-1}^{\frac{t_{n-2}}{2}}\right)^{p_{n-1}}A_n^{\frac{t_{n-1}}{2}}\right\}^{\frac{1}{2(1+k)}} = A_n^{\frac{1+t_{n-1}}{2}}S = S^*A_n^{\frac{1+t_{n-1}}{2}}. \tag{3.2}$$

Let $X = SS^*$, then X is unique, $\|X\| \leq 1$, and X satisfies

$$\begin{aligned} A_n^{\frac{t_{n-1}}{2}}\left(A_{n-1}^{\frac{t_{n-2}}{2}}\cdots\left[A_3^{\frac{t_2}{2}}\left(A_2^{\frac{t_1}{2}}A_1^{p_1}A_2^{\frac{t_1}{2}}\right)^{p_2}A_3^{\frac{t_2}{2}}\right]^{p_3}\cdots A_{n-1}^{\frac{t_{n-2}}{2}}\right)^{p_{n-1}}A_n^{\frac{t_{n-1}}{2}} &=(A_n^{\frac{1+t_{n-1}}{2}}XA_n^{\frac{1+t_{n-1}}{2}})^{k+1} \\ &=A_n^{\frac{1+t_{n-1}}{2}}(XA_n^{1+t_{n-1}})^kXA_n^{\frac{1+t_{n-1}}{2}} \\ &=A_n^{\frac{1+t_{n-1}}{2}}X(A_n^{1+t_{n-1}}X)^kA_n^{\frac{1+t_{n-1}}{2}}. \end{aligned} \tag{3.3}$$

Deleting $A_n^{\frac{1+t_{n-1}}{2}}$ from both sides of above equation, then we can obtain (3.1). □

Remark 3.3. If $n = 2$, Theorem 3.2 is just the main result in [6].

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