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Epinormality

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Abstract

A topological space $(X , \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X , \mathcal{T}')$ is T_4. We investigate this property and present some examples to illustrate the relationships between epinormality and other weaker kinds of normality.

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1. Introduction

In this paper, we investigate a new topological property called *epinormality* which was presented by Arhangel'skiĭ. We prove that submetrizability and T_4 imply epinormality but the converse is not true in general. We present some examples to show that epinormality and some weaker versions of normality, such as mild normality, almost normality, and π-normality, are independent from each other. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. A space X is *normal* if any two disjoint closed sets can be separated by two disjoint open sets. A T_4 space is a T_1 normal space and a Tychonoff ($T_{3\frac{1}{2}}$) space is a T_1 completely regular space. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω_0 and the first uncountable ordinal is ω_1.

2. Epinormality

Definition 2.1 (Arhangel'skiĭ, 2012)**.** A topological space $(X , \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X , \mathcal{T}')$ is T_4.

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Recall that a topological space (X , $\mathcal{T}$) is called *submetrizable* if there exists a metric d on X such that the topology $\mathcal{T}_d$ on X generated by d is coarser than $\mathcal{T}$, i.e., $\mathcal{T}_d \subseteq \mathcal{T}$, see [5]. Obviously, any T_4 space is epinormal, just by taking $\mathcal{T}' = \mathcal{T}$, but the converse is not true in general. For example, the square of the Sorgenfrey line, see [12], is epinormal which is not normal. It is epinormal because it is submetrizable, since, by definitions, any submetrizable space is epinormal. The converse of the last statement is not true in general. For example, $\omega_1 + 1$ is epinormal being T_2-compact, hence T_4. But it is not submetrizable, because if $\omega_1 + 1$ was submetrizable, then there would be a metric d on $\omega_1 + 1$ such that the topology $\mathcal{T}_d$ on $\omega_1 + 1$ generated by d is coarser than the usual ordered topology. This means that $(\omega_1 + 1, \mathcal{T}_d)$ is perfectly normal. So, the closed set $\{\omega_1\}$ is a G_δ-set in $(\omega_1 + 1, \mathcal{T}_d)$, i.e., $\{\omega_1\} = \cap_{n\in\mathbb{N}} U_n$, where $U_n \in \mathcal{T}_d$, hence U_n is open in the usual ordered topology on $\omega_1 + 1$, which is a contradiction. Epinormality and normality do not imply each other. For example, the Niemytzki plane is epinormal being submetrizable, but it is not normal by Jones' lemma. Any indiscrete space which has more than one element is an example of a normal space which is not epinormal. In Example 2.10 below, we give a Tychonoff space which is not epinormal. Observe that if $\mathcal{T}'$ and $\mathcal{T}$ are topologies on X such that $\mathcal{T}'$ is coarser than $\mathcal{T}$ and $(X, \mathcal{T}')$ is T_i, $i \in \{0, 1, 2\}$, then so is $(X, \mathcal{T})$. So, we can conclude the following.

Theorem 2.2. *Every epinormal space is Hausdorff.*

We can separate in epinormal spaces disjoint compact subsets by a continuous real-valued function. It is the Urysohn's version for epinormality.

Theorem 2.3. *Let (X , $\mathcal{T}$) be an epinormal space. Then for each disjoint compact subsets A and B of (X , $\mathcal{T}$) there exists a continuous function $f : X \longrightarrow I$, where $I = [0, 1]$ is considered with its usual topology, such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$.*

Proof. Let (X , $\mathcal{T}$) be any epinormal space, then there is a coarser topology $\mathcal{T}'$ than $\mathcal{T}$ such that (X , $\mathcal{T}'$) is T_4. Let A and B be any disjoint compact subsets of (X , $\mathcal{T}$). Then A and B are both closed in (X , $\mathcal{T}'$). Apply Urysohn's lemma, then there exists a continuous function $f : (X, \mathcal{T}') \longrightarrow I$ such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$. Since $\mathcal{T}'$ is coarser than $\mathcal{T}$, then $f : (X, \mathcal{T}) \longrightarrow I$ is still continuous. □

Theorem 2.4. *Epinormality is a topological property.*

Proof. Let (X , $\mathcal{T}$) be any epinormal space. Assume that $(X, \mathcal{T}) \cong (Y, \mathcal{S})$. Let $\mathcal{T}'$ be a coarser topology on X such that (X , $\mathcal{T}'$) is T_4. Let $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ be a homeomorphism. Define a topology $\mathcal{S}'$ on Y by $\mathcal{S}' = \{ f(U) : U \in \mathcal{T}' \}$. Then $\mathcal{S}'$ is a coarser than $\mathcal{S}$ and (Y , $\mathcal{S}'$) is T_4. □

Theorem 2.5. *Epinormality is an additive property.*

Proof. Let (X_α , $\mathcal{T}_\alpha$) be an epinormal space for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, let $\mathcal{T}'_\alpha$ be a topology on X_α, coarser than $\mathcal{T}_\alpha$ such that (X_α , $\mathcal{T}'_\alpha$) is T_4. Since T_1 and normality are both additive [4], then $\oplus_{\alpha\in\Lambda}(X_\alpha, \mathcal{T}'_\alpha)$ is T_4, and its topology is coarser than the topology on $\oplus_{\alpha\in\Lambda}(X_\alpha, \mathcal{T}_\alpha)$. □

Since metrizability is countable multiplicative and T_2-compactness is multiplicative, then we conclude the following statement about the multiplicity of epinormality.

Theorem 2.6. *For each $\alpha \in \Lambda$, let (X_α , $\mathcal{T}_\alpha$) be an epinormal space such that the witness of epinormality (X_α , $\mathcal{T}'_\alpha$) is T_2-compact for each $\alpha \in \Lambda$, or metrizable for each $\alpha \in \Lambda$, where Λ is countable. Then $\prod_{\alpha\in\Lambda}(X_\alpha, \mathcal{T}_\alpha)$ is epinormal.*

We still do not know if epinormality is multiplicative or not.

There are various methods of generating a new topological space from a given one. In 1929, Alexandroff introduced his method by constructing the double circumference space [1]. In 1968, Engelking generalized this construction to an arbitrary space as follows [3]: Let X be any topological space and $X' = X \times \{1\}$.

Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in X' by x' and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of* X. In [2], the following theorem was proved.

Theorem 2.7. *If* $(X, \mathcal{S})$ *is epinormal, then so is its Alexandroff duplicate* $(A(X), \mathcal{T})$.

Recall that a subset E of a space X is called *closed domain* [4] (called also, *regularly closed, κ-closed* [11]) if $E = \overline{\text{int}E}$. It is clear that the closure of an open set is a closed domain.

Theorem 2.8. *If* X *is pseudocompact epinormal space of cardinality less than continuum, then* X *is* T_4.

Proof. Let $(X, \mathcal{T})$ be a pseudocompact epinormal space of cardinality less than continuum and let $\mathcal{T}'$ be a topology on X such that $(X, \mathcal{T}')$ is T_4 and $\mathcal{T}' \subseteq \mathcal{T}$. Observe that we have $(X, \mathcal{T}')$ is also pseudocompact. Since any pseudocompact normal space is countably compact, [4, 3.10.21], then $(X, \mathcal{T}')$ is countably compact and Hausdorff, since it is of cardinality less than continuum, then $(X, \mathcal{T}')$ is also first countable.

We show that $\mathcal{T}' \supseteq \mathcal{T}$. Let C be any closed set in $(X, \mathcal{T})$. Consider the family $\mathcal{W} = \{W = \overline{U}^{\mathcal{T}} : U \in \mathcal{T}$ and $C \subseteq U\}$. Clearly we have $C = \bigcap\{W : W \in \mathcal{W}\}$. Since pseudocompactness is hereditary with respect to the closed domains, see [4], W is pseudocompact in $(X, \mathcal{T})$ for each $W \in \mathcal{W}$. Hence W is pseudocompact in $(X, \mathcal{T}')$ for each $W \in \mathcal{W}$. Since pseudocompact subsets of first countable Tychonoff space are closed, [13], then W is closed in $(X, \mathcal{T}')$ for each $W \in \mathcal{W}$. Thus C is closed in $(X, \mathcal{T}')$ and hence $\mathcal{T}' = \mathcal{T}$. Therefore, $(X, \mathcal{T})$ is T_4. □

From the above proof, we conclude the following.

Corollary 2.9. *If* X *is pseudocompact epinormal first countable space, then* X *is* T_4.

Here is an example of a Tychonoff separable first countable pseudocompact space which is not epinormal.

Example 2.10. For simplicity, we will denote the first infinite ordinal ω_0 just by ω. Two countably infinite sets are called *almost disjoint* if their intersection is finite. Call a subfamily of $[\omega]^\omega = \{A \subset \omega : A \text{ is infinite}\}$ a *mad family on* ω if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Let $\mathcal{A}$ be a pairwise almost disjoint subfamily of $[\omega]^\omega$. The *Mrówka space* $\Psi(\mathcal{A})$ is defined as follows: The underlying set is $\omega \cup \mathcal{A}$, the points of ω are isolated and a basic open neighborhood of $A \in \mathcal{A}$ has the form $\{A\} \cup (A \setminus F)$, with $F \in [\omega]^{<\omega}$, i.e., F is a finite subset of ω, [14]. It is well-known, see [14], that there exists an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ such that $|\mathcal{A}| > \omega$ and $\Psi(\mathcal{A})$ is a Tychonoff, first countable, separable, and locally compact space which is not countably compact nor normal. Moreover, $\mathcal{A}$ is a mad family if and only if $\Psi(\mathcal{A})$ is pseudocompact.

Now, the Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A} \subset [\omega]^\omega$ is mad is first countable and pseudocompact which is not normal. So, by Corollary 2.9, it cannot be epinormal.

3. Epinormality and other weaker versions of normality.

A topological space X is called *C-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$, [7]. It is clear that any epinormal space is C-normal, but the converse is not true in general. For example, $\mathbb{R}$ with the left ray topology, [12], is C-normal, because it is normal, but it is not epinormal because it is not T_2 by Theorem 2.2. Another example is the above Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A} \subset [\omega]^\omega$ is mad. It is C-normal because it is locally compact [7].

A topological space X is called *L-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each Lindelöf subspace $C \subseteq X$, [9]. It was proved in [9] that *"If X is T_3 separable L-normal and of countable tightness, then X is* *normal* (T_4)*"*. Now, the Sorgenfrey line is submetrizable, thus the Sorgenfrey line square is submetrizable, hence epinormal. Since the Sorgenfrey line square is first countable Tychonoff separable non-normal space, see [12], then it is not L-normal. $\mathbb{R}$ with the left ray topology is L-normal, because it is normal, but it is not epinormal because it is not T_2.

A topological space X is called *mildly normal* [10] (called also, κ-*normal* [11]) if for any two disjoint closed domains E and F of X, there exist two disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$. In general, epinormality and mild normality do not imply each other. $(\mathbb{R}, \mathcal{T}_p)$ is not epinormal, as it is not T_2, but it is mildly normal as the only closed domains are $\mathbb{R}$ and the empty set. Here is an example of an epinormal space which is not mildly normal.

Example 3.1. Let $\mathbb{P}$ and $\mathbb{Q}$ denote the irrationals and the rationals, respectively. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $p_n = \langle p, \frac{1}{n} \rangle \in \mathbb{R}^2$. For each $p \in \mathbb{P}$, fix a sequence $(p_n^\star)_{n\in\mathbb{N}}$ of rationals such that $p'_n = \langle p_n^\star, 0\rangle \longrightarrow \langle p, 0\rangle$, where the convergence is taken in $\mathbb{R}^2$ with its usual topology $\mathcal{U}$. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $A_n(\langle p, 0\rangle) = \{p'_k : k \geq n\}$ and $B_n(\langle p, 0\rangle) = \{p_k : k \geq n\}$. Now, for each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $U_n(\langle p, 0\rangle) = \{\langle p, 0\rangle\} \cup A_n(\langle p, 0\rangle) \cup B_n(\langle p, 0\rangle)$. Let $X = \{\langle x, 0\rangle \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{\langle p, \frac{1}{n}\rangle = p_n : p \in \mathbb{P}$ and $n \in \mathbb{N}\}$. For each $q \in \mathbb{Q}$, let $\mathcal{B}(\langle q, 0\rangle) = \{\{\langle q, 0\rangle\}\}$. For each $p \in \mathbb{P}$, let $\mathcal{B}(\langle p, 0\rangle) = \{U_n(\langle p, 0\rangle) : n \in \mathbb{N}\}$. For each $p \in \mathbb{P}$ and each $n \in \mathbb{N}$, let $\mathcal{B}(p_n) = \{\{p_n\}\}$. Denote by $\mathcal{T}$ the unique topology on X that has $\{\mathcal{B}(\langle x, 0\rangle), \mathcal{B}(p_n) : x \in \mathbb{R}, p \in \mathbb{P}, n \in \mathbb{N}\}$ as its neighborhood system. Let $Z = \{\langle x, 0\rangle : x \in \mathbb{R}\}$. That is, Z is the x-axis. Then $(Z, \mathcal{T}_Z) \cong (\mathbb{R}, \mathcal{RS})$, where $\mathcal{RS}$ is the rational sequence topology, see [12]. X is submetrizable, hence epinormal. But it is not mildly normal, see [7].

A space X is called *almost normal* [8] if for any two disjoint closed subsets A and B of X one of which is closed domain, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A subset A of a space X is called π-*closed* [6] if A is a finite intersection of closed domains. A space X is called π-*normal* [6] if for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A space X is called *quasi-normal* [15] if for any two disjoint π-closed subsets A and B of X, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [6].

Epinormality and π-normality are independent from each other. For example, the half-disc topological space is not π-normal, see [6], but it is epinormal because it is submetrizable. The real numbers $\mathbb{R}$ with the countable complement topology is π-normal being weakly extremally disconnected, see [6], but it is not epinormal since it is not Hausdorff.

Epinormality and almost normality do not imply each other. For example, the Michael product is not almost normal, see [8], but it is epinormal because it is submetrizable. $(\mathbb{R}, \mathcal{T}_p)$, where $\mathcal{T}_p$ is the particular point topology, $p \in \mathbb{R}$ [8], is almost normal being weakly extremally disconnected, see [8], but it is not epinormal as it is not Hausdorff.

Quasi-normality does not imply epinormality. For example, $\mathbb{R}$ with the finite complement topology is quasi-normal as the only π-closed subsets are $\mathbb{R}$ and the empty set. But it is not epinormal because it is not Hausdorff.

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