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# On the extended multivalued Geraghty type contractions

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**Abstract**

In this paper we present some absolute retract results for modified Geraghty multivalued type contractions in $b$-metric space. Our results, generalize several existing results in the corresponding literature. We also present some examples to support the obtained results. 

*Keywords:* Fixed points, extended multivalued Geraghty type contractions.
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## 1. Introduction and preliminaries

Let $P(X)$ denote the collection of all nonempty subsets of a set $X \neq \emptyset$, and $F : X \to P(X)$ be multifunctions (multivalued mapping). Throughout the paper, set of all nonempty closed and bounded subsets of $X$ will be represented by $P_{b,cl}(X)$ under the assumption that $X$ is equipped with a metric. Further, the set of all fixed point(s) of $F$ will be denoted by $\mathcal{F}_F$, that is,

$$\mathcal{F}_F = \{x \in X : x \in Fx\}.$$

Let $(X, d)$ be a metric space and $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$. For $x \in X$ and $A, B \subseteq X$, we set $D : P(X) \times P(X) \to [0, \infty) \cup \{+\infty\}$, such that

$$D(A, B) = \sup\{D(a, B) : a \in A\} \text{ and } D(B, A) = \sup\{D(b, A) : b \in B\}.$$

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Let $H : P(X) \times P(X) \to [0, \infty) \cup \{+\infty\}$ be defined as

$$H(A,B) = \begin{cases} \max\{D(A,B), D(B,A)\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that $H$ forms a metric and it is called the Hausdorff metric (for more details see e.g. [13, 14] and the references therein).

For non-empty sets $X, Y$, a mapping $\varphi : X \to Y$ is called a selection of $F : X \to P(Y)$, whenever $\varphi(x) \in Fx$ for all $x \in X$. A topological space $X$ is an absolute retract for metric spaces if for each metric space $Y$, $A \in P_{cl}(Y)$ and continuous function $\psi : A \to X$, there exists a continuous function $\varphi : Y \to X$ such that $\varphi|_A = \psi$ (see [12]).

Let $\mathcal{M}$ be the collection of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F : X \to P_{b,cl}(X)$ a lower semi-continuous multifunction. We say that $F$ has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, continuous function $f : Y \to X$ and continuous functional $g : Y \to (0, \infty)$ such that

$$G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$$

for all $y \in Y$, $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of $G$. If $\mathcal{D} = \mathcal{M}$, then we say that $F$ has the selection property and we denote this by $F \in Sp(X)$ (for more details see [13, 14]).

In this paper, we present some new results on absolute retract (see e.g. [4, 10, 12–14]) of the fixed points set of extended multivalued Geraghty type contractions. Our results combine, extend and generalize several existing results on the corresponding literature (see e.g. [1–3, 8, 9, 11, 15, 16] and related references therein).

## 2. Fixed points set of extended multivalued Geraghty type contractions

In the all over this paper let $\Psi$ be the set of all increasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following property: $\psi(ct) \leq c\psi(t)$ for all $c > 1$ and $\psi(0) = 0$. We denote by $\Theta$ the family of all increasing functions $\theta : [0, \infty) \to (0, 1)$.

**Definition 2.1.** Let $F : X \to P_{b,cl}(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a given function. Then $F$ is said to be *$\alpha$-admissible* if

$$(T3) \quad \alpha(x,y) \geq 1 \text{ for all } y \in Fx \Rightarrow \alpha(y,z) \geq 1, \text{ for all } z \in Fy.$$

**Example 2.2.** Let $X = [1, 2]$ and $Fx = [x - \frac{1}{2}, 2]$. Define $\alpha(x, y) = 1$ if $x = y = 2$ and $\alpha(x, y) = 0$ otherwise. Clearly, $F$ is $\alpha$-admissible.

**Definition 2.3.** Let $(X, d)$ be a metric space and $F : X \to P_{b,cl}(X)$ be a multivalued mapping. We say that $F$ is an *extended multivalued Geraghty type contraction* if there exist $\alpha : X \times X \to [0, \infty)$, $a \in [0, 1)$ and some $L \geq 0$ such that

$$\begin{aligned} \eta(a)D(x, F(x)) \leq d(x,y) \Longrightarrow \alpha(x,y)&\psi(H(Fx, Fy)) \\ &\leq \theta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y)) \end{aligned}$$

for all $x, y \in X$, where,

$$M(x,y) = \max\{d(x,y), D(x,Fx), D(y,Fy), \frac{D(x,Fy) + D(y,Fx)}{2}\}$$

and

$$N(x,y) = \min\{D(x,Fx), D(y,Fx)\}$$

and $\eta(a) = \frac{1}{1+a}$, $\theta \in \Theta$ and $\psi, \phi \in \Psi$.

Furthermore, we say that $F$ is *generalized multivalued Geraghty type contraction* if

$$\alpha(x,y)\psi(H(Fx,Fy)) \leq \theta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y)) \tag{2.1}$$

for all $x, y \in X$, where, $L, M(x,y), N(x,y), \alpha(x,y), \theta, \psi, \phi$ are defined as above.

*Remark* 2.4. The functions belonging to $\Theta$ are strictly smaller than 1. Then, the expression $\theta(\psi(M(x,y)))$ in (2.1) satisfies

$$\theta(\psi(M(x,y))) < 1 \text{ for any } x, y \in X \text{ with } x \neq y.$$

**Theorem 2.5.** *Let $(X,d)$ be a complete metric space and $F : X \to P_{b,cl}(X)$ be a extended multivalued Geraghty type contraction such that*

(i) *$F$ is $\alpha$-admissible;*

(ii) *there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $\alpha(x_0,x_1) \geq 1$;*

(iii) *$F$ is continuous.*

*Then $F$ has a fixed point.*

*Proof.* By condition (ii), there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $\alpha(x_0,x_1) \geq 1$. If $x_1 = x_0$, as $x_1 \in Fx_1$, then $x_1$ is a fixed point of $F$ and we have nothing to prove. First, we note that

$$\begin{aligned} M(x_0,,x_1) &= \max\{d(x_0,,x_1), D(x_0,Fx_0), D(x_1,Fx_1), \frac{D(x_0,Fx_1)+D(x_1,Fx_0)}{2}\} \\ &= \max\{d(x_0,,x_1), D(x_1,Fx_1)\}. \end{aligned}$$

Since $\eta(a)D(x_0,Fx_0) \leq d(x_0,x_1)$, if $M(x_0,,x_1) = D(x_1,Fx_1)$, then

$$\begin{aligned} \psi(D(x_1,Fx_1)) &\leq \alpha(x_0,x_1)\psi(H(Fx_0,Fx_1)) \leq \theta(\psi(D(x_1,Fx_1)))\psi(D(x_1,Fx_1)) + L\phi(0) \\ &< \psi(D(x_1,Fx_1)), \end{aligned}$$

which is a contradiction. It follows that $M(x_0,,x_1) = d(x_0,,x_1)$. Let $q = \frac{1}{\sqrt{\theta(\psi(d(x_0,x_1)))}} > 1$, then there exists $x_2 \in Fx_1$ such that

$$\psi(d(x_1,x_2)) \leq q\alpha(x_0,x_1)\psi(H(Fx_0,Fx_1)). \tag{2.2}$$

Using (2.1) with $x = x_0$ and $y = x_1$, by (2.2) we get

$$\psi(d(x_1,x_2)) \leq \sqrt{\theta(\psi(d(x_0,x_1)))}\psi(d(x_0,x_1)). \tag{2.3}$$

Now, by the properties of the function $\psi$, we deduce

$$\psi(\frac{d(x_1,x_2)}{\sqrt{\theta(\psi(d(x_0,x_1)))}}) \leq \frac{1}{\sqrt{\theta(\psi(d(x_0,x_1)))}}\psi(d(x_1,x_2)) < \psi(d(x_0,x_1))$$

and so $d(x_1,x_2) < \sqrt{\theta(\psi(d(x_0,x_1)))}d(x_0,x_1) < d(x_0,x_1)$. If $x_2 \in Fx_2$, then $x_2$ is a fixed point of $F$. Assume that $x_1 \neq x_2 \notin Fx_2$. We have:

$$M(x_1,x_2) = \max\{d(x_1,x_2), D(x_2,Fx_2)\}, N(x_1,x_2) = 0$$

and $\eta(a)D(x_1,Fx_1) \leq d(x_1,x_2)$. If $M(x_1,x_2) = D(x_2,Fx_2)$, then

$$0 < \psi(D(x_2,Fx_2)) \leq \alpha(x_1,x_2)\psi(H(Fx_1,Fx_2))$$

$$\begin{aligned}&\leq \theta(\psi(D(x_2,Fx_2)))\psi(D(x_2,Fx_2))\\&<\psi(D(x_2,Fx_2)),\end{aligned}$$

which is a contradiction and hence $M(x_1,x_2)=d(x_1,x_2)$.

Put $q_1=\frac{\sqrt{\theta(\psi(d(x_0,x_1)))}\psi(d(x_0,x_1))}{\psi(d(x_1,x_2))}>1$ (by (2.3)). Then there exists $x_3\in Fx_2$ such that

$$\psi(d(x_2,x_3))<q_1\alpha(x_1,x_2)\psi(H(Fx_1,Fx_2)).$$

Since $\eta(a)D(x_2,Fx_2)\leq d(x_2,x_3)$, by (2.1) with $x=x_2$ and $y=x_3$, we have

$$\begin{aligned}\psi(d(x_2,x_3))&<q_1\alpha(x_1,x_2)\psi(H(Fx_1,Fx_2))\\&\leq q_1\theta(\psi(M(x_1,x_2)))\psi(M(x_1,x_2))+q_1LN(x_1,x_2)\\&=q_1\theta(\psi(d(x_1,x_2)))\psi(d(x_1,x_2))\\&\leq\sqrt{\theta(\psi(d(x_0,x_1)))}\sqrt{\theta(\psi(d(x_0,x_1)))}\psi(d(x_0,x_1))\\&\leq(\sqrt{\theta(\psi(d(x_0,x_1)))})^2\psi(d(x_0,x_1)).\end{aligned}$$

Since

$$\psi(\frac{d(x_2,x_3)}{(\sqrt{\theta(\psi(d(x_0,x_1)))})^2})\leq\frac{\psi(d(x_2,x_3))}{(\sqrt{\theta(\psi(d(x_0,x_1)))})^2}<\psi(d(x_0,x_1))$$

and $\psi$ is increasing, then

$$d(x_2,x_3)<(\sqrt{\theta(\psi(d(x_0,x_1)))})^2d(x_0,x_1)<d(x_0,x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}$ in $X$ such that $x_n\neq x_{n-1}$ and $d(x_n,x_{n+1})<(\sqrt{\theta(\psi(d(x_0,x_1)))})^n d(x_0,x_1)$ for all $n\in\mathbb{N}$.

Let $t=\sqrt{\theta(\psi(d(x_0,x_1)))}$, then $0<t<1$. By the triangle inequality for $n<m$, we have

$$\begin{aligned}d(x_n,x_m)&\leq\sum_{k=n}^{m-1}d(x_k,x_{k+1})\leq(t^n\sum_{k=0}^{m-n-1}t^k)d(x_0,x_1)\\&\leq\frac{t^n}{1-t}d(x_0,x_1).\end{aligned}$$

The previous inequality shows that $\{x_n\}$ is a Cauchy sequence in $(X,d)$. Since $(X,d)$ is a complete metric space, so there exists $x^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$. The continuity of $F$ implies that

$$0\leq D(x^\star,Fx^\star)=\lim_{n\to\infty}D(x_{n+1},Fx^\star)\leq\lim_{n\to\infty}H(Fx_n,Fx^\star)=0$$

and so $x^\star\in Fx^\star$. □

**Example 2.6.** Let $X=[-1,\infty)$, $d(x,y)=|x-y|$ and for any $A,B\subset X$

$$D(A,B)=\sup\{D(a,B):a\in A\},$$

$$H(A,B)=\max\{\sup_{x\in A}D(x,B),\sup_{y\in B}D(y,A)\}.$$

Define a multivalued mapping $F:X\to P_{b,cl}(X)$ by $F(x)=[-1,\frac{x}{4}]$ for every $x\in X$. It is easy to see that $(X,d)$ is a complete metric space. We have

$$\eta(a)D(x,F(x))\leq d(x,y),\quad \eta(a)=\frac{1}{1+a},a\in[0,1),$$

whenever $x, y \in [-1, 0]$. Hence, if we set $\psi(t) = t$, $\theta(t) = \frac{t+1}{t+2}$, and

$$\alpha(x,y) = \begin{cases} 2 \text{ if } x \geq y, \\ \frac{1}{2} \text{ if } x < y, \end{cases}$$

because

$$H(Fx, Fy) = \begin{cases} \frac{x-y}{4} \text{ if } x \geq y, \\ \frac{y-x}{4} \text{ if } x < y, \end{cases}$$

and $M(x, y) = |x - y|, N(x, y) = 0$, therefore

$$\alpha(x,y)\psi(H(Tx,Ty)) \leq \theta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y)).$$

It is straightforward that conditions of Theorem 2.5 are satisfied and so $F$ has a fixed point. For this example we have $\mathcal{F}_F = [-1, 0]$.

## 3. Extended multivalued Geraghty type contractions in the setting of $b$-metric spaces

In this section, first we recall the notion of $b$-metric and introduce the notion of a extended multivalued Geraghty type contractions in the setting of $b$-metric spaces. After then, we state and prove our main results.

**Definition 3.1** ([6])**.** Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to [0, \infty)$ is said to be a *b-metric* if for all $x, y, z \in X$ the following conditions are satisfied:

($bM_1$) $d(x, y) = 0$ if and only if $x = y$;

($bM_2$) $d(x, y) = d(y, x)$;

($bM_3$) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair $(X, d)$ is called a *b-metric space* (with constant $s$).

For $s = 1$, $b$-metric turns into standard metric. That is why $b$-metric spaces attracted the attention of researchers on this fields (see e.g. [5, 7]). Let $(X, d)$ be a $b$-metric space. We consider next the following family of subsets given by

$$\mathcal{P}(X) := \{Y | Y \subset X \text{ and } Y \neq \emptyset\}.$$

In this case $D$ is a generalized functional on a $b$-metric space $(X, d)$ defined by $D : P(X) \times P(X) \to [0, \infty) \cup \{+\infty\}$,

$$D(A,B) = \begin{cases} \inf\{d(a,b) | a \in A, b \in B\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{otherwise.} \end{cases}$$

In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

The following basic lemmas will be useful in the proof of main results.

**Lemma 3.2** ([7])**.** *Let $(X, d)$ be a b-metric space. Then, we have*

$$D(x, A) \leq s[d(x, y) + D(y, A)] \quad \text{for all } x, y \in X \text{ and } A \subset X.$$

**Lemma 3.3** ([7])**.** *Let $(X, d)$ be a b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Then*

$$d(x_n, x_0) \leq sd(x_0, x_1) + ... + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n).$$

We denote by $\mathcal{F}$ the family of all functions $\beta : [0, \infty) \to [0, \frac{1}{s^2})$ for some $s > 1$.

**Definition 3.4.** Let $(X, d)$ be a complete $b$-metric space and $F : X \to P_{b,cl}(X)$ be a multivalued mapping. We say that $F$ is a extended multivalued Geraghty type contraction in $b$-metric space with $(s > 1)$, whenever there exist $\alpha : X \times X \to [0, \infty)$, $a \in [0, 1)$ and some $L \geq 0$ such that for

$$M(x,y) = \max\{d(x,y), D(x,Fx), D(y,Fy), \frac{D(x,Fy)+D(y,Fx)}{2s}\}$$

and

$$N(x,y) = \min\{D(x,Fx), D(y,Fx)\},$$

we have

$$\begin{aligned}\eta(a)D(x,F(x)) \leq d(x,y) \Longrightarrow \alpha(x,y)\psi(s^3H(Fx,Fy))\\ \leq \beta(\psi(M(x,y)))\psi(M(x,y)) + L\phi(N(x,y))\end{aligned} \tag{3.1}$$

for all $x, y \in X$, where $\eta(a) = \frac{1}{1+a}$, $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$.

**Theorem 3.5.** *Let $(X,d)$ be a complete b-metric space with $(s > 1)$, and $F : X \to P_{b,cl}(X)$ be a extended multivalued Geraghty type contraction such that*

(i) *$F$ is $\alpha$-admissible;*

(ii) *there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $\alpha(x_0,x_1) \geq 1$;*

(iii) *$F$ is continuous.*

*Then $F$ has a fixed point.*

*Proof.* By condition (ii), there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $\alpha(x_0,x_1) \geq 1$. If $x_1 = x_0$, as $x_1 \in Fx_1$, then $x_1$ is a fixed point of $F$ and we have nothing to prove. First, we note that

$$\begin{aligned}M(x_0,,x_1) &= \max\{d(x_0,,x_1), D(x_0,Fx_0), D(x_1,Fx_1), \frac{D(x_0,Fx_1)+D(x_1,Fx_0)}{2s}\}\\ &= \max\{d(x_0,,x_1), D(x_1,Fx_1)\}.\end{aligned}$$

Since $\eta(a)D(x_0,Fx_0) \leq d(x_0,x_1)$, if $M(x_0,,x_1) = D(x_1,Fx_1)$, then

$$\begin{aligned}\psi(D(x_1,Fx_1)) &\leq \alpha(x_0,x_1)\psi(s^3H(Fx_0,Fx_1)) \leq \beta(\psi(D(x_1,Fx_1)))\psi(D(x_1,Fx_1)) + L\phi(0)\\ &< \psi(D(x_1,Fx_1)),\end{aligned}$$

which is a contradiction. It follows that $M(x_0,,x_1) = d(x_0,,x_1)$. Let us take a real $q$ such that $1 < q < s$. Then

$$0 < \psi(D(x_1,Fx_1)) \leq \alpha(x_0,x_1)\psi(H(Fx_0,Fx_1)) < q\alpha(x_0,x_1)\psi(s^3H(Fx_0,Fx_1)).$$

Hence, there exists $x_2 \in Fx_1$ such that

$$\psi(d(x_1,x_2)) < q\alpha(x_0,x_1)\psi(s^3H(Fx_0,Fx_1)). \tag{3.2}$$

Using (3.1) with $x = x_0$ and $y = x_1$, by (3.2) we get

$$\psi(d(x_1,x_2)) < \frac{q}{s^2}\psi(d(x_0,,x_1)). \tag{3.3}$$

Now, by the properties of the function $\psi$ and regarding the fact that $\frac{q}{s^2} < 1$, we deduce

$$\psi(\frac{s^2}{q}d(x_1,x_2)) \leq \frac{s^2}{q}\psi(d(x_1,x_2)) < \psi(d(x_0,x_1)),$$

$$d(x_1,x_2) \leq \frac{q}{s^2}d(x_0,x_1) < d(x_0,x_1).$$

If $x_2 \in Fx_2$, then $x_2$ is a fixed point of $F$. Assume that $x_1 \neq x_2 \notin Fx_2$. We have:

$$M(x_1, x_2) = \max\{d(x_1, x_2), D(x_2, Fx_2)\}, N(x_1, x_2) = 0$$

and $\eta(a)D(x_1, Fx_1) \leq d(x_1, x_2)$. If $M(x_1, x_2) = D(x_2, Fx_2)$, then

$$\begin{aligned} 0 < \psi(D(x_2, Fx_2)) &\leq \alpha(x_1, x_2)\psi(s^3 H(Fx_1, Fx_2)) \\ &\leq \theta(\psi(D(x_2, Fx_2)))\psi(D(x_2, Fx_2)) \\ &< \psi(D(x_2, Fx_2)), \end{aligned}$$

which is a contradiction and hence $M(x_1, x_2) = d(x_1, x_2)$. Put

$$q_1 = \frac{\frac{q}{s^2}\psi(d(x_0, x_1))}{\psi(d(x_1, x_2))}.$$

By (3.3), we have $q_1 > 1$. Hence, there exists $x_3 \in Fx_2$ such that

$$\psi(d(x_2, x_3)) < q_1\alpha(x_1, x_2)\psi(s^3 H(Fx_1, Fx_2)).$$

Since $\eta(a)D(x_, Fx_2) \leq d(x_2, x_3)$, by (3.1) with $x = x_2$ and $y = x_3$, we have

$$\begin{aligned} \psi(d(x_2, x_3)) &< q_1\alpha(x_1, x_2)\psi(s^3 H(Fx_1, Fx_2)) \\ &\leq q_1\beta(\psi(M(x_1, x_2)))\psi(M(x_1, x_2)) + q_1 L\phi(N(x_1, x_2)) \\ &< \frac{q_1}{s^2}\psi(d(x_1, x_2)). \end{aligned}$$

So

$$\psi(d(x_2, x_3)) \leq \frac{q_1}{s^2}\psi(d(x_1, x_2)) \leq (\frac{q}{s^2})^2\psi(d(x_0, x_1)).$$

By properties of $\psi$ we obtain

$$d(x_2, x_3) \leq (\frac{q}{s^2})^2 d(x_0, x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}$ in $X$ such that $x_n \in Fx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) < (\frac{q}{s^2})^n d(x_0, x_1)$ for all $n \in \mathbb{N}$. By the triangle inequality for $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} s^{k-n+1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{\infty} s^{k-n+1} (\frac{q}{s^2})^k d(x_0, x_1) \\ &= [\frac{s(\frac{q}{s^2})^n}{1 - s(\frac{q}{s^2})}] d(x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

We deduce that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete $b$-metric space, so there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. The mapping $F$ is continuous, so

$$D(x^\star, Fx^\star) = \lim_{n\to\infty} D(x_{n+1}, Fx^\star) \leq \lim_{n\to\infty} H(Fx_n, Fx^\star) = 0$$

and so $x^\star \in Fx^\star$. □

**Example 3.6.** Put $X = \{1\} \cup \{m + \frac{1}{n+2} : m, n \in \mathbb{N}\}$ and define a metric $d$ on $X$ by

$$d(x, y) =\mid x - y \mid .$$

Define a mapping $F$ on $X$ by

$$F(x) = \begin{cases} 1 & x = 1, \\ 7m + \frac{1}{n+2} & x = m + \frac{1}{n}. \end{cases}$$

Then $F$ satisfies in the assumptions of Theorem 3.5.

*Proof.* It is obvious that $(X, d)$ is a complete metric space and 1 is a unique fixed point of $F$. if $n < m$, we have

$$\eta(a)D(m + \frac{1}{n+2}, F(m + \frac{1}{n+2})) < d(m + \frac{1}{n+2}, n + \frac{1}{n+2})$$
$$\Updownarrow$$
$$\eta(a)d(m + \frac{1}{n+2}, 7m + \frac{1}{n+2}) < d(m + \frac{1}{n+2}, n + \frac{1}{n+2})$$
$$\Updownarrow$$
$$\eta(a) \mid m + \frac{1}{n+2} - 7m - \frac{1}{n+2} \mid <\mid m + \frac{1}{n+2} - n - \frac{1}{n+2} \mid$$
$$\Updownarrow$$
$$\frac{1}{2} \mid -6m \mid\leq \eta(a) \mid -6m \mid <\mid m + \frac{1}{n+2} - n - \frac{1}{n+2} \mid .$$
$$\Updownarrow$$
$$3m < m - n < m.$$

This is a contradiction. Therefore $F$ satisfies in the assumptions of Theorem 3.5. □

**Example 3.7.** Let $X$ be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |x(t)|dt < 1$. Define $d : X \times X :\to [0,\infty)$ by

$$d(x, y) = \int_0^1 |x(t) - y(t)|^2 dt.$$

Then, $d$ is a $b$-metric on $X$, with $s = 2$. The multivalued mapping $T : X \to 2^X$ is defined by

$$Tx(t) = \begin{cases} 3x + 4, & \text{if } x(t) < -1, \\ [-x, 1], & \text{if } -1 \leq x(t) < 0, \\ \frac{1}{8}\ln(1 + x(t)), & \text{if } x(t) \geq 0. \end{cases}$$

Consider the mapping $\alpha : X \times X \to [0,\infty)$ by the following

$$\alpha(x, y) = \begin{cases} 2, & \text{if } y \leq x \leq -3, \\ 1, & \text{if } x \geq y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We take $\beta : [0,\infty) \to [0, \frac{1}{4})$ and $\psi : [0,\infty) \to [0,\infty)$ as

$$\psi(t) = t \quad \text{and} \quad \beta(t) = \frac{t^2 + 1}{4t^2 + 8}.$$

Evidently, $\psi \in \Psi$ and $\beta \in \mathcal{F}$. Moreover, $T$ is $\alpha$-admissible, $\alpha(1, T1) \geq 1$ and $T$ is continuous. Now, we prove that $T$ is a generalized $\alpha - \psi$-Suzuki-Geraghty multivalued type contraction. For $x(t) \geq 0$, we have

$$
\begin{aligned}
\alpha(x(t), y(t))\psi(s^3 d(Tx(t), Ty(t))) &\leq 2^3(\int_0^1 |Tx(t) - Ty(t)|^2 dt) \\
&= 2^3 \int_0^1 |\frac{1}{8}\ln(1 + x(t)) - \frac{1}{8}\ln(1 + y(t))|^2 dt \\
&= 2^{-3} \int_0^1 |\ln(\frac{1 + x(t)}{1 + y(t)})|^2 dt = 2^{-3} \int_0^1 |\ln(1 + \frac{x(t) - y(t)}{1 + y(t)})|^2 dt \\
&\leq 2^{-3} \int_0^1 |\ln(1 + |x(t) - y(t)|)|^2 dt \leq 2^{-3} \int_0^1 |x(t) - y(t)|^2 dt \\
&= 2^{-3} d(x, y) \leq \frac{d(x, y)^2 + 1}{4d(x, y)^2 + 8} d(x, y) = \beta(d(x, y)d(x, y).
\end{aligned}
$$

For $x(t) < 0$, by definition of $Tx(t)$ and $\alpha(x(t), y(t))$ the condition of (3.1) is satisfied. Thus, $T$ is a generalized $\alpha - \psi$-Suzuki-Geraghty multivalued type contraction. By Theorem 3.5, $T$ has a fixed point. Here $0, -2$ are fixed points.

If in (3.2), $\mathcal{F}$ is a family of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ for some $s \geq 1$, we can deduce the following theorem.

**Theorem 3.8.** *Let $(X, d)$ be a complete b-metric space and absolute retract for b-metric spaces, $F : X \to P_{b,cl}(X)$ an extended multivalued Geraghty type contraction, $F$ is continuous, and $F \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y \in F(x)$, then $\mathcal{F}_F$ is an absolute retract for b-metric spaces.*

*Proof.* Let $Y$ be a $b$-metric space, $A \in P_{cl}(Y)$ and $\xi : A \to \mathcal{F}_F$ a continuous function. Since $X$ is an absolute retract for $b$-metric spaces, there exists a continuous function $\varphi_0 : Y \to X$ such that $\varphi_0|_A = \xi$. Define the function $g_0 : Y \to (0, \infty)$ by

$$g_0(y) = \sup\{d(\varphi_0(y), z)| z \in F(\varphi_0(y))\} + 1$$

for all $y \in Y$. It is not difficult to see that $g_0$ is continuous and

$$F(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F(\varphi_0(y))$$

for all $y \in A$ (see [14]). Also we observe that the function $\xi : A \to \mathcal{F}_F$ has the property $\xi(y) \in F(\varphi_0(y))$ $(y \in A)$, so is a continuous selection of the multivalued mapping. Since $F \in Sp(X)$, there exists a continuous function $\varphi_1 : Y \to X$ such that $\varphi_1|_A = \xi$ and $\varphi_1(y) \in F(\varphi_0(y))$ for all $y \in Y$. First, we note that

$$
\begin{aligned}
M(\varphi_0(y), \varphi_1(y)) = \max\{&d(\varphi_0(y), \varphi_1(y)), D(\varphi_0(y), F\varphi_0(y)), D(\varphi_1(y), F\varphi_1(y)) \\
&, \frac{D(\varphi_0(y), F\varphi_1(y)) + D(\varphi_1(y), F\varphi_0(y))}{2s}\} \\
= \max\{&d(\varphi_0(y), , \varphi_1(y)), D(\varphi_1(y), F\varphi_1(y))\}.
\end{aligned}
$$

Since $\eta(a)D(\varphi_0(y), F(\varphi_0(y))) \leq d(\varphi_0(y), \varphi_1(y))$, if $M(\varphi_0(y), \varphi_1(y)) = D(\varphi_1(y), F\varphi_1(y))$, then

$$
\begin{aligned}
\psi(D(\varphi_1(y), F\varphi_1(y))) &\leq \alpha(\varphi_0(y), \varphi_1(y))\psi(s^3 H(F(\varphi_0(y), F(\varphi_1(y))))) \\
&\leq \beta(\psi(D(\varphi_1(y), F\varphi_1(y))))\psi(D(\varphi_1(y), F\varphi_1(y))) + L\phi(0) \\
&< \psi(D(\varphi_1(y), F\varphi_1(y))),
\end{aligned}
$$

which is contradiction. It follows that $M(\varphi_0(y), \varphi_1(y)) = d(\varphi_0(y), \varphi_1(y))$. Let $1 < q < s$ and $r \in (1, \frac{s}{q})$, then

$$
\begin{aligned}
\psi(D(\varphi_1(y), F\varphi_1(y))) &\leq \alpha(\varphi_0(y), \varphi_1(y))\psi(s^3 H(F(\varphi_0(y), F(\varphi_1(y))))) \\
&\leq \beta(\psi(d(\varphi_0(y), \varphi_1(y)))\psi(d(\varphi_0(y), \varphi_1(y))) \\
&< \frac{q}{s}\psi(d(\varphi_0(y), \varphi_1(y))).
\end{aligned}
$$

Now, by the property of $\psi \in \Psi$ and regarding the fact that $\frac{q}{s} < 1$ we have

$$\psi(\frac{s}{q}D(\varphi_1(y), F\varphi_1(y))) \leq \frac{s}{q}\psi(D(\varphi_1(y), F\varphi_1(y))) < \psi(d(\varphi_0(y), \varphi_1(y))).$$

Since $\psi$ is increasing, therefore

$$D(\varphi_1(y), F\varphi_1(y))) \leq \frac{q}{s}d(\varphi_0(y), \varphi_1(y)) < \frac{q}{s}d(\varphi_0(y), \varphi_1(y)) + r^{-1}. \tag{3.4}$$

Hence, $G_2(y) := F(\varphi_1(y)) \cap B(\varphi_1(y), \frac{q}{s^2}(d(\varphi_0(y), \varphi_1(y))) + r^{-1}) \neq \emptyset$ for all $y \in Y$. Since we know that $F \in Sp(X)$, there exists a continuous function $\varphi_2 : Y \to X$ such that $\varphi_2|_A = \xi$ and $\varphi_2(y) \in \overline{G_2(y)}$ for all $y \in Y$. Thus, $\varphi_2(y) \in F(\varphi_1(y))$ for all $y \in Y$ and

$$d(\varphi_1(y), \varphi_2(y)) < \frac{q}{s}(d(\varphi_0(y), \varphi_1(y))) + r^{-1}.$$

Similarly we have

$$M(\varphi_1(y), \varphi_2(y)) = \max\{d(\varphi_1(y), , \varphi_2(y)), D(\varphi_2(y), F\varphi_2(y))\}, N(\varphi_1(y), \varphi_2(y)) = 0.$$

If $M(\varphi_1(y), \varphi_2(y)) = D(\varphi_2(y), F\varphi_2(y))$, then

$$\begin{aligned} 0 < \psi(D(\varphi_2(y), F\varphi_2(y))) &\leq \alpha(\varphi_1(y), \varphi_2(y))\psi(s^3 H(F(\varphi_1(y), F(\varphi_2(y))))) \\ &\leq \beta(\psi(D(\varphi_2(y), F\varphi_2(y)))\psi(D(\varphi_2(y), F\varphi_2(y))) \\ &< \frac{q}{s}\psi(D(\varphi_2(y), F\varphi_2(y))) \\ &< \psi(D(\varphi_2(y), F\varphi_2(y))), \end{aligned}$$

which is contradiction. It follows that $M(\varphi_1(y), \varphi_2(y)) = d(\varphi_1(y), \varphi_2(y))$.

Now, by the property of $\psi$ we have

$$\psi(\frac{s}{q}D(\varphi_2(y), F\varphi_2(y))) \leq \frac{s}{q}\psi(D(\varphi_2(y), F\varphi_2(y))) < \psi(d(\varphi_1(y), \varphi_2(y))).$$

Since $\psi$ is increasing, therefore

$$D(\varphi_2(y), F\varphi_2(y)) \leq \frac{q}{s}d(\varphi_1(y), \varphi_2(y)) < \frac{q}{s}d(\varphi_1(y), \varphi_2(y)) + r^{-1}.$$

By (3.4) we have

$$D(\varphi_2(y), F\varphi_2(y)) < (\frac{q}{s})^2 d(\varphi_0(y), \varphi_1(y)) + r^{-2}.$$

Hence, $G_3(y) := F(\varphi_2(y)) \cap B(\varphi_2(y)), (\frac{q}{s})^2 d(\varphi_0(y), \varphi_1(y)) + r^{-2}) \neq \emptyset$. Since $F \in Sp(X)$, there exists a continuous function $\varphi_3 : Y \to X$ such that $\varphi_3|_A = \xi$ and $\varphi_3(y) \in \overline{G_3(y)}$ for all $y \in Y$. Also, we have $d(\varphi_2(y), \varphi_3(y)) < (\frac{q}{s})^2 d(\varphi_0(y), \varphi_1(y)) + r^{-2}$ and $\varphi_3(y) \in F(\varphi_2(y))$ for all $y \in Y$. By continuing this process, we obtain $\{\varphi_n : Y \to X\}_{n \geq 0}$ a sequence of continuous functions such that $\varphi_n|_A = \xi$ and $d(\varphi_{n-1}(y), \varphi_n(y)) < (\frac{q}{s})^{n-1} d(\varphi_0(y), \varphi_1(y)) + r^{-(n-1)}$ and $\varphi_n(y) \in F(\varphi_{n-1}(y))$ for all $y \in Y$ and $n \geq 1$. Now, for each $\lambda > 0$ we put

$$Y_\lambda := \{y \in Y : d(\varphi_0(y), \varphi_1(y)) < \lambda\}.$$

Since $\varphi_1(y) \in F(\varphi_0(y))$ and

$$F(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F(\varphi_0(y)),$$

$\varphi_1(y) \in B(\varphi_0(y), g_0(y))$. Hence, $d(\varphi_0(y), \varphi_1(y)) < \lambda_y := g_0(y)$. Thus, $y \in Y_{\lambda_y}$. Since $Y_\lambda$ is open for each $\lambda > 0$, the family of sets $\{Y_\lambda | \lambda > 0\}$ is an open covering of $Y$ and we have

$$d(\varphi_{n-1}(y), \varphi_n(y)) \leq (\frac{q}{s})^{n-1} d(\varphi_0(y), \varphi_1(y)) + r^{-(n-1)}$$

for all $n \geq 1$ and $y \in Y$. Since $\frac{q}{s} < 1$, $r > 1$, and $X$ is complete, the sequence $\{\varphi_n\}_{n\geq 0}$ converges uniformly on $Y_\lambda$ for all $\lambda > 0$. Let $\varphi : Y \to X$ be the pointwise limit of $\{\varphi_n\}_{n\geq 0}$ and note that $\varphi$ is continuous and $\varphi|_A = \xi$ because $\varphi_n|_A = \xi$ for all $n \geq 0$. Since $F$ is continuous, hence $\varphi(y) \in F(\varphi(y))$ for all $y \in Y$. Therefore, $\varphi : Y \to B$ is a continuous extension of $\xi$, that is, $B = \{x \in X : x \in F(x)\}$ is an absolute retract for $b$-metric spaces. □

## 4. Corollaries

By letting $\alpha(x, y) = 1$ for all $x, y \in X$, we get the following consequences:

**Corollary 4.1.** *Let $(X, d)$ be a complete b-metric space and absolute retract for b-metric spaces, $F : X \to P_{b,cl}(X)$, also there exists $a \in [0, 1)$ and some $L \geq 0$ such that,*

$$\begin{aligned} \eta(a)D(x, F(x)) \leq d(x, y) \Longrightarrow \psi(s^3 d(Tx, Ty)) \\ \leq \beta(\psi(d(x, y)))\psi(d(x, y)) + L\phi(N(x, y)) \end{aligned} \tag{4.1}$$

*for all $x, y \in X$, where $\eta(a) = \frac{1}{1+a}$, $\beta \in \mathcal{F}$, and $\psi, \phi \in \Psi$ and*

$$N(x, y) = \min\{d(x, Tx), d(y, Tx)\},$$

*$F$ is continuous and $F \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y \in F(x)$, then $\mathcal{F}_F$ is an absolute retract for b-metric spaces.*

If in (4.1), we let $L = 0$ then we obtain the following sequence.

**Corollary 4.2.** *Let $(X, d)$ be a complete b-metric space and absolute retract for b-metric spaces, $F : X \to P_{b,cl}(X)$, also there exist $a \in [0, 1)$ such that,*

$$\eta(a)D(x, F(x)) \leq d(x, y) \Longrightarrow \psi(s^3 d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

*for all $x, y \in X$, where $\eta(a) = \frac{1}{1+a}$, $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$, $F$ is continuous, and $F \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y \in F(x)$, then $\mathcal{F}_F$ is an absolute retract for b-metric spaces.*

## 5. Consequences

As it is expected, the main results of the paper yield several existing results in the literature by choosing the auxiliary functions $\alpha, \eta, \psi, \phi$ in a proper way. To list more results it is sufficient to take $d(x, y)$ instead of $M(x, y)$, and /or take $L = 0$. Notice also that, one can replace the single valued mapping instead of multivalued mapping to cover more results in the literature. Furthermore, by relaxing $b$-metric with metric, we observe more results as a consequence of our main results.

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