Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4722–4736
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Certain new summation formulas for the series $_4F_3(1)$ with applications

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Communicated by Y. J. Cho

Abstract

The main objective of this paper is to provide thirteen (presumably) new summation formulas for the series $_4F_3$ of unit argument expressed in terms of Gamma functions. As special cases of our main results, we also present twenty four summation formulas for the terminating $_4F_3(1)$, whose further special cases are derived to give thirty two known summation formulas for the terminating $_3F_2(1)$. The results presented here are established with the help of a general result recorded in the book of Prudnikov et al. and the generalization of Watson's summation theorem obtained earlier by Lavoie et al..

Keywords: Gamma function, Pochhammer symbol, generalized hypergeometric function, Watson's summation theorem, generalized Watson's summation theorem.
2010 MSC: 33B20, 33C20, 33B15, 33C05.

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1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive and non-positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The natural generalization of the Gauss's hypergeometric function $_2F_1$ is called the generalized hypergeometric series $_pF_q$ $(p,\, q \in \mathbb{N}_0)$ defined by (see [1], [8, p. 73] and [9, pp. 71–75]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\, \ldots,\, \alpha_p; \\ \beta_1,\, \ldots,\, \beta_q;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\, \ldots,\, \alpha_p;\, \beta_1,\, \ldots,\, \beta_q;\, z), \end{aligned} \tag{1.1}$$

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Received 2016-06-11

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [9, p. 2 and p. 5]):

$$\begin{aligned}(\lambda)_n := &\begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\ = &\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-),\end{aligned}$$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here an empty product is interpreted as 1, and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is,

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q).$$

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, among an extensive literature, one may refer to [1, 3, 8–10].

It is worthy of note that whenever the generalized hypergeometric function ${}_pF_q$ (including ${}_2F_1$) with its specified argument (for example, unit argument) can be summed to be expressed in terms of the Gamma functions, the result may be very important from both theoretical and applicable points of view. Here, the classical summation theorems for the hypergeometric series ${}_2F_1$ such as those of Gauss and Gauss second, Kummer, and Bailey; Watson's, Dixon's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$ and others play important roles in theory and application. During 1992–1996, in a series of works [4–6], Lavoie et al. have generalized the above mentioned classical summation theorems for ${}_3F_2$ of Watson, Dixon, and Whipple and presented a large number of special and limiting cases of their results. Those results have also been obtained and verified with the help of computer programs (for example, Mathematica).

In our present investigation, we recall the following classical Watson's summation theorem (see, e.g., [9, p. 351]):

$${}_3F_2\left[\begin{matrix} a, \quad b, \quad c\,; \\ \frac{1}{2}(a+b+1),\ 2c\,; \end{matrix}\ 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{1.2}$$

provided $\Re(2c-a-b) > -1$.

Lavoie et al. [4] established a generalization of (1.2), which contains twenty five identities closely related to (1.2), recorded in the following single form:

$$\begin{aligned}{}_3F_2&\left[\begin{matrix} a, \quad b, \quad c\,; \\ \frac{1}{2}(a+b+i+1),\ 2c+j\,; \end{matrix}\ 1\right] = \mathcal{A}_{j,i}\, 2^{a+b+i-2} \\ &\times \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\ \Gamma\left(c+[j/2]+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}(a+b+|i+j|-j-1)\right)}{\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(a\right)\ \Gamma\left(b\right)} \\ &\times \left\{\mathcal{B}_{j,i}\, \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1-(-1)^i\right)\right)\ \Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}+[j/2]-\frac{1}{4}(-1)^j\ \left(1-(-1)^i\right)\right)\ \Gamma\left(c-\frac{1}{2}b+\frac{1}{2}+[j/2]\right)}\right. \\ &\left.+\,\mathcal{C}_{j,i}\, \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+[(j+1)/2]+\frac{1}{4}(-1)^j\ \left(1-(-1)^i\right)\right)\ \Gamma\left(c-\frac{1}{2}b+[(j+1)/2]\right)}\right\}\end{aligned} \tag{1.3}$$

for $i,\ j = 0,\ \pm1,\ \pm2$. Here, $[x]$ denotes the greatest integer less than or equal to x and $|x|$ is the absolute value of x. The coefficients $\mathcal{A}_{j,i}$, $\mathcal{B}_{j,i}$ and $\mathcal{C}_{j,i}$ are given in the tables below.

Table 1: Table for $\mathcal{A}_{j,i}$

$j \diagdown i$	2	1	0	-1	-2
-2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{2(c-1)}$	$\frac{1}{c-1}$	$\frac{1}{2(c-1)}$
-1	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{a-b}$	1	1	1
0	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{a-b}$	1	2	1
1	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b)}$	1	2	2
2	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$	$\frac{1}{2(c+1)(a-b)}$	$\frac{1}{2(c+1)}$	$\frac{2}{c+1}$	$\frac{2}{c+1}$

Table 2: Table for $\mathcal{B}_{j,i}$

$j \diagdown i$	2	1	0	-1	-2
-2	$\mathcal{B}_{-2,2}$	$c-b-1$	$\mathcal{B}_{-2,0}$	$\mathcal{B}_{-2,-1}$	$\mathcal{B}_{-2,-2}$
-1	$a-b+1$	1	1	$2c-a+b-2$	$\mathcal{B}_{-1,-2}$
0	$\mathcal{B}_{0,2}$	1	1	1	$\mathcal{B}_{0,-2}$
1	$\mathcal{B}_{1,2}$	$2c-a+b$	1	1	$a+b-1$
2	$\mathcal{B}_{2,2}$	$\mathcal{B}_{2,1}$	$\mathcal{B}_{2,0}$	$c-b+1$	$\mathcal{B}_{2,-2}$

Here

$$\mathcal{B}_{-2,2} := c(a+b-1)-(a+1)(b+1)+2;$$

$$\mathcal{B}_{-2,0} := (c-a-1)(c-b-1)+(c-1)(c-2);$$

$$\mathcal{B}_{-2,-1} := 2(c-1)(c-2)-(a-b)(c-b-1);$$

$$\begin{aligned}\mathcal{B}_{-2,-2} := {} & 2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\} \\ & -(a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)+ab\};\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{2,2} := {} & 2c(c+1)\{(2c+1)(a+b-1)-a(a-1)-b(b-1)\} \\ & -(a-b-1)(a-b+1)\{(c+1)(2c-a-b+1)+ab\};\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{-1,-2} &:= 2(c-1)(a+b-1)-(a-b)^2+1;\\
\mathcal{B}_{0,2} &:= a(2c-a)+b(2c-b)-2c+1;\\
\mathcal{B}_{0,-2} &:= a(2c-a)+b(2c-b)-2c+1;\\
\mathcal{B}_{2,1} &:= 2c(c+1)-(a-b)(c-b+1);\\
\mathcal{B}_{2,0} &:= (c-a+1)(c-b+1)+c(c+1);\\
\mathcal{B}_{2,-2} &:= c(a+b-1)-(a-1)(b-1).
\end{aligned}$$

Table 3: Table for $\mathcal{C}_{j,i}$

$j \diagdown i$	2	1	0	-1	-2
-2	-4	$-(c-a-1)$	4	$\mathcal{C}_{-2,-1}$	$\mathcal{C}_{-2,-2}$
-1	$-(4c-a-b-3)$	-1	1	$2c+a-b-2$	$\mathcal{C}_{-1,-2}$
0	-8	-1	0	1	8
1	$\mathcal{C}_{1,2}$	$-(2c+a-b)$	-1	1	$4c-a-b+1$
2	$\mathcal{C}_{2,2}$	$\mathcal{C}_{2,1}$	-4	$c-a+1$	4

Here

$$\begin{aligned}
\mathcal{C}_{-2,-1} &:= 2(c-1)(c-2)+(a-b)(c-a-1);\\
\mathcal{C}_{-2,-2} &:= 4(2c-a+b-3)(2c+a-b-3);\\
\mathcal{C}_{-1,-2} &:= 8c^2-2(c-1)(a+b+7)-(a-b)^2-7;\\
\mathcal{C}_{1,2} &:= -8c^2+2c(a+b-1)+(a-b)^2-1;\\
\mathcal{C}_{2,2} &:= -4(2c+a-b+1)(2c-a+b+1);\\
\mathcal{C}_{2,1} &:= -2c(c+1)-(a-b)(c-a+1).
\end{aligned}$$

It is noted that the special case of (1.3) when $i=j=0$ yields the classical Watson's summation theorem (1.2).

We outline the contents of this paper. In Section 2, we obtain thirteen new summation formulas for the series

$$ {}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+i+1),\; 2c+j,\; d; \end{matrix}\; 1\right]$$

for i, $j = 0, \pm 1, \pm 2, \pm 3$. The results are derived with the help of a general formula given in [7, p. 439, Entry 14] with $p = 4$ and $q = 3$:

$$ {}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+i+1),\, 2c+j,\, d; \end{matrix}\; 1\right] = {}_3F_2\left[\begin{matrix} a, \quad b, \quad c\,; \\ \frac{1}{2}(a+b+i+1),\, 2c+j\,; \end{matrix}\; 1\right] + \frac{2abc}{d(a+b+i+1)(2c+j)} \times {}_3F_2\left[\begin{matrix} a+1, \quad b+1, \quad c+1\,; \\ \frac{1}{2}(a+b+i+3),\, 2c+j+1\,; \end{matrix}\; 1\right] \tag{1.4} $$

for $i,\, j = 0, \pm 1, \pm 2, \ldots$ and the generalized Watson's summation theorem (1.2).

In Section 3, we present twenty four (presumably) new summation formulas for terminating ${}_4F_3$ which are certain to have not been recorded in the existing literature.

In Section 4, we recall thirty two formulas for terminating ${}_3F_2$ which are derived here in a different method from that in Lavoie et al. [4].

2. Summation formulas for ${}_4F_3(1)$

Here we establish thirteen (presumably) new summations formulas for ${}_4F_3(1)$ asserted in the following theorem.

Theorem 2.1. *Each of the following summation formulas holds true:*

1. ($\Re(2c-a-b) > 1$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+1),\, 2c,\, d; \end{matrix}\; 1\right] = \left(1 + \frac{ab}{d(2c-a-b-1)}\right) \times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} + \frac{2}{d}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b\right)}; \tag{2.1} $$

2. ($\Re(2c-a-b) > -1$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+1),\, 2c+1,\, d; \end{matrix}\; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} + \left(\frac{2c}{d} - 1\right) \times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}; \tag{2.2} $$

3. ($\Re(2c-a-b) > -1$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+1),\, 2c+2,\, d; \end{matrix}\; 1\right] = \left(\frac{(c-a+1)(c-b+1)+c(c+1)}{2(c+1)} - \frac{abc}{2d(c+1)}\right) \times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{3}{2}\right)} + \frac{2(c-d)}{d(c+1)}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}; \tag{2.3} $$

4. ($\Re(2c-a-b)>2$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$\begin{aligned}{}_4F_3\left[\begin{array}{c} a,\quad b,\quad c,\quad d+1\,;\\ \frac{1}{2}(a+b),\, 2c,\, d;\end{array} 1\right] &= \left(1+\frac{a(2c-a+b)}{d(2c-a-b-2)}\right)\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)}\\ &\quad+\left(1+\frac{b(2c+a-b)}{d(2c-a-b-2)}\right)\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)};\end{aligned}\tag{2.4}$$

5. ($\Re(2c-a-b)>0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$\begin{aligned}{}_4F_3\left[\begin{array}{c} a,\quad b,\quad c,\quad d+1\,;\\ \frac{1}{2}(a+b),\, 2c+1,\, d;\end{array} 1\right] &= \frac{1}{2c-a}\left(2c-a-b+\frac{2bc}{d}\right)\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\\ &\quad+\frac{1}{2c-b}\left(2c-a-b+\frac{2ac}{d}\right)\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)};\end{aligned}\tag{2.5}$$

6. ($\Re(2c-a-b)>0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$\begin{aligned}{}_4F_3\left[\begin{array}{c} a,\quad b,\quad c,\quad d+1\,;\\ \frac{1}{2}(a+b+2),\, 2c,\, d;\end{array} 1\right] &= \frac{2(d-b)}{d(a-b)}\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\\ &\quad+\frac{2(a-d)}{d(a-b)}\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)};\end{aligned}\tag{2.6}$$

7. ($\Re(2c-a-b)>0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$\begin{aligned}{}_4F_3\left[\begin{array}{c} a,\quad b,\quad c,\quad d+1\,;\\ \frac{1}{2}(a+b+2),\, 2c+1,\, d;\end{array} 1\right] &= \frac{2\left(2c-a+b-\frac{2bc}{d}\right)}{(2c-a)(a-b)}\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\\ &\quad-\frac{2\left(2c+a-b-\frac{2ac}{d}\right)}{(2c-b)(a-b)}\\ &\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)};\end{aligned}\tag{2.7}$$

8. ($\Re(2c-a-b)>0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$
{}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+2),\, 2c+2,\, d; \end{matrix}\ 1\right] = \mathcal{D}_{1,2}^{(1)}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c+\frac{3}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a+1\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{3}{2}\right)} - \mathcal{D}_{1,2}^{(1)}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c+\frac{3}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b\right)\,\Gamma\left(c-\frac{1}{2}a+\frac{3}{2}\right)\,\Gamma\left(c-\frac{1}{2}b+1\right)}, \tag{2.8}
$$

where

$$
\mathcal{D}_{1,2}^{(1)} := \frac{2c(c+1)-(a-b)(c-b+1)-\frac{bc(2c+a-b+2)}{d}}{(c+1)(a-b)},
$$

and

$$
\mathcal{D}_{1,2}^{(2)} := \frac{2c(c+1)+(a-b)(c-a+1)-\frac{ac(2c-a+b+2)}{d}}{(c+1)(a-b)};
$$

9. ($\Re(2c-a-b)>3$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+1),\, 2c-1,\, d; \end{matrix}\ 1\right] \\
&\quad = \mathcal{D}_{0,-1}^{(1)}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} \\
&\qquad + \mathcal{D}_{0,-1}^{(2)}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)\,\Gamma\left(c-\frac{1}{2}a\right)\,\Gamma\left(c-\frac{1}{2}b\right)},
\end{aligned} \tag{2.9}
$$

where

$$
\mathcal{D}_{0,-1}^{(1)} := \frac{1}{2}(2c-a-b-3)+\frac{ab}{d},
$$

and

$$
\mathcal{D}_{0,-1}^{(2)} := \frac{1}{2}(2c-a-b-3)+\frac{(c-a-1)(c-b-1)+c(c-1)}{d};
$$

10. ($\Re(2c-a-b)>2$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1\,; \\ \frac{1}{2}(a+b+2),\, 2c-1,\, d; \end{matrix}\ 1\right] \\
&\quad = \mathcal{D}_{1,-1}^{(1)}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-1\right)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a\right)\,\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} \\
&\qquad + \mathcal{D}_{1,-1}^{(2)}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b\right)},
\end{aligned} \tag{2.10}
$$

where

$$
\mathcal{D}_{1,-1}^{(1)} := \frac{2c-a-b-2-\frac{2b(c-a-1)}{d}}{a-b},
$$

and

$$
\mathcal{D}_{1,-1}^{(2)} := -\frac{2c-a-b-2-\frac{2a(c-b-1)}{d}}{a-b};
$$

11. ($\Re(2c-a-b)>1$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1\,; \\ \frac{1}{2}(a+b-1), & 2c+1, & d; \end{matrix}\; 1\right] \\
&\quad = \mathcal{D}^{(1)}_{-2,1}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \\
&\qquad + \mathcal{D}^{(2)}_{-2,1}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)},
\end{aligned}
\tag{2.11}
$$

where

$$\mathcal{D}^{(1)}_{-2,1} := \frac{1}{2}(2c-a-b-1)+\frac{2abc}{d(a+b-1)},$$

and

$$
\begin{aligned}
\mathcal{D}^{(2)}_{-2,1} :=&\frac{1}{4}(4c-a-b+1)(2c-a-b-1) \\
&+\frac{2c}{d}\left\{(a+1)(2c-a+1)+(b+1)(2c-b+1)-2c-1\right\};
\end{aligned}
$$

12. ($\Re(2c-a-b)>4$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1\,; \\ \frac{1}{2}(a+b), & 2c-1, & d; \end{matrix}\; 1\right] &= \mathcal{D}^{(1)}_{-1,-1}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-2\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} \\
&\quad + \mathcal{D}^{(2)}_{-1,-1}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-2\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b\right)},
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
\mathcal{D}^{(1)}_{-1,-1} :=& \frac{(2c-a+b-2)(2c-a-b-4)}{4} \\
&+\frac{b[2c(c-1)+(a-b)(c-a-1)]}{2d},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}^{(2)}_{-1,-1} :=& \frac{(2c+a-b-2)(2c-a-b-4)}{4} \\
&+\frac{2c(c-1)-(a-b)(c-b-1)}{2d};
\end{aligned}
$$

13. ($\Re(2c-a-b)>-2$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1\,; \\ \frac{1}{2}(a+b), & 2c+2, & d; \end{matrix}\; 1\right] &= \frac{\left(c-b+1+\frac{bc}{d}\right)}{c+1} \\
&\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+\frac{3}{2}\right)} \\
&\quad+\frac{\left(c-a+1+\frac{ac}{d}\right)}{c+1} \\
&\quad\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}b+1\right)}.
\end{aligned}
\tag{2.13}
$$

Proof. The proofs of the summation formulas (2.1)-(2.13) are quite straightforward. For (2.1), first taking $i=0$ and $j=0$ in (1.4) gives the following formula:

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1\,; \\ \frac{1}{2}(a+b+1), & 2c, & d; \end{matrix}\; 1\right] &= {}_3F_2\left[\begin{matrix} a, & b, & c\,; \\ \frac{1}{2}(a+b+1), & 2c\,; & \end{matrix}\; 1\right] \\
&\quad+\frac{ab}{d(a+b+1)}\,{}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1\,; \\ \frac{1}{2}(a+b+3), & 2c+1\,; & \end{matrix}\; 1\right].
\end{aligned}
\tag{2.14}
$$

Now, the first ${}_3F_2$ on the right-hand side of (2.14) can be evaluated with the help of (1.2) and the second ${}_3F_2$ on the right-hand side of (2.14) can be evaluated with the aid of a formula obtained from (1.3) in which a, b and c are replaced by $a+1$, $b+1$ and $c+1$, respectively, and $i=1$ and $j=-1$ in the resulting identity are chosen. Then, after a little simplification, it is easy to see that the right-hand side of (2.14) is equal to that of (2.1). This completes the proof of (2.1).

The other formulas (2.2)-(2.13) can be verified by using a similar argument of the proof of (2.1). So the details of their proofs are omitted.

□

3. Terminating summation formulas

Setting $b=-2n$ ($n\in\mathbb{N}_0$) and replacing a by $a+2n$ ($n\in\mathbb{N}_0$) or setting $b=-2n-1$ ($n\in\mathbb{N}_0$) and replacing a by $a+2n+1$ ($n\in\mathbb{N}_0$) in (2.1)-(2.13), we find that one of the two terms of the right-hand sides of the resulting summation formulas will vanish. Then we obtain twenty four (presumably) new terminating summation formulas for ${}_4F_3(1)$, which are recorded in the following theorem.

Theorem 3.1. *Each of the following summation formulas holds true:*

1. ($n\in\mathbb{N}_0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ 2c,\ d; \end{matrix}\ 1\right] = \left(1-\frac{2n(a+2n)}{d(2c-a-1)}\right) \times \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}; \tag{3.1}$$

2. ($n\in\mathbb{N}_0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ 2c,\ d; \end{matrix}\ 1\right] = -\frac{1}{d}\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}; \tag{3.2}$$

3. ($n\in\mathbb{N}_0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ 2c+1,\ d; \end{matrix}\ 1\right] = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}; \tag{3.3}$$

4. ($n\in\mathbb{N}_0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ 2c+1,\ d; \end{matrix}\ 1\right] = \frac{(d-2c)}{d(2c+1)}\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{3}{2}\right)_n}; \tag{3.4}$$

5. ($n\in\mathbb{N}_0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ 2c+2,\ d; \end{matrix}\ 1\right] = \frac{\alpha}{(c+1)(2c-a+1)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c-\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{3}{2}\right)_n}, \tag{3.5}$$

where

$$\alpha := (c-a-2n+1)(c+2n+1)+c(c+1)+\frac{2nc(a+2n)}{d};$$

6. ($n\in\mathbb{N}_0$ *and* $d\in\mathbb{C}\setminus\mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{matrix} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ 2c+2,\ d; \end{matrix}\ 1\right] = \frac{d-c}{d(c+1)}\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{3}{2}\right)_n}; \tag{3.6}$$

7. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}a,\ 2c,\ d; \end{array} 1\right] = \left(1 - \frac{2n(2c+a+4n)}{d(2c-a-2)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.7} $$

8. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ \begin{aligned} {}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}a,\ 2c,\ d; \end{array} 1\right] &= -\frac{1}{a+2n}\left(1 + \frac{(a+2n+1)(2c-a-4n-2)}{d(2c-a-2)}\right) \\ &\quad \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}; \end{aligned} \tag{3.8} $$

9. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}a,\ 2c+1,\ d; \end{array} 1\right] = \frac{\left(2c-a-\frac{4nc}{d}\right)}{(2c-a-2n)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.9} $$

10. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}a,\ 2c+1,\ d; \end{array} 1\right] = \frac{a-2c+2n+1+\frac{4nc}{d}}{(2c-a-4n-1)(a+2n)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.10} $$

11. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ 2c,\ d; \end{array} 1\right] = \frac{a(d+2n)}{d(a+4n)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.11} $$

12. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ 2c,\ d; \end{array} 1\right] = \frac{(d-a-2n-1)}{d(a+4n+2)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.12} $$

13. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ 2c+1,\ d; \end{array} 1\right] = \frac{a\left(2c-a-4n+\frac{4nc}{d}\right)}{(2c-a-2n)(a+4n)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.13} $$

14. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ \begin{aligned} {}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ 2c+1,\ d; \end{array} 1\right] &= \frac{\left(2c+a+4n+2-\frac{2c}{d}(a+2n+1)\right)}{(2c+2n+1)(a+4n+2)} \\ &\quad \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c+\frac{1}{2}\right)_n}; \end{aligned} \tag{3.14} $$

15. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ 2c+2,\ d; \end{array} 1\right] = \frac{\beta}{(2c-a)(c+1)(a+4n)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{3}{2}\right)_n}, \tag{3.15} $$

where

$$ \beta := a\left(2c(c+1) - (a+4n)(c+2n+1) + \frac{2nc}{d}(2c+a+4n+2)\right); $$

16. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ \ 2c+2,\ \ d; \end{array} 1\right] = \frac{\gamma}{(a-2c)(c+1)(a+4n+2)} \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c+\frac{3}{2}\right)_n}, \tag{3.16}
$$

where

$$
\gamma := \frac{c}{d}(a+2n+1)(2c-4n-a) - 2c(c+1) - (a+4n+2)(c-a-2n);
$$

17. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ \ 2c-1,\ \ d; \end{array} 1\right] = \left(1 - \frac{4n(a+2n)}{d(2c-a-3)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c-\frac{1}{2}\right)_n}; \tag{3.17}
$$

18. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+1),\ \ 2c-1,\ \ d; \end{array} 1\right] = \frac{\delta}{d(2c-1)(2c-a-3)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{3.18}
$$

where

$$
\delta := d(a-2c+3) - 2\{c(c-1) + (c+2n)(c-a-2n-2)\};
$$

19. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ \ 2c-1,\ \ d; \end{array} 1\right] = \frac{a}{a+4n}\left(1 + \frac{4n(c-a-2n-1)}{d(2c-a-2)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c-\frac{1}{2}\right)_n}; \tag{3.19}
$$

20. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}(a+2),\ \ 2c-1,\ \ d; \end{array} 1\right] = \frac{2c-a-2-\frac{2}{d}(c+2n)(a+2n+1)}{(2c-1)(a+4n+2)} \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}; \tag{3.20}
$$

21. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}a,\ \ 2c-1,\ \ d; \end{array} 1\right] = \epsilon\, \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{3.21}
$$

where

$$
\epsilon := 1 - \frac{2n\left\{d(2c-a+4) + 4c(c-1) + 2(a+4n)(c-a-2n-1)\right\}}{d(2c-a-4)(2c-a-4n-2)};
$$

22. $(n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-)$

$$
{}_4F_3\left[\begin{array}{r} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}a,\ \ 2c-1,\ \ d; \end{array} 1\right] = -\frac{\varepsilon}{ad(2c-1+2n)(2c-a-4)} \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{3.22}
$$

where

$$
\begin{aligned}
\varepsilon := {} & d(2c+a+4n)(2c-a-4) \\
& + 2(a+2n+1)\left\{2c(c-1) - (a+4n+2)(c+2n)\right\};
\end{aligned}
$$

23. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ {}_4F_3 \left[\begin{array}{c} -2n,\ a+2n,\ c,\ d+1\,; \\ \frac{1}{2}a,\ 2c+2,\ d; \end{array} 1 \right] = \frac{\left(c+2n+1-\frac{2nc}{d}\right)}{(c+1)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{3}{2}\right)_n}; \tag{3.23} $$

24. ($n \in \mathbb{N}_0$ *and* $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

$$ \begin{aligned} {}_4F_3 \left[\begin{array}{c} -2n-1,\ a+2n+1,\ c,\ d+1\,; \\ \frac{1}{2}a,\ 2c+2,\ d; \end{array} 1 \right] &= -\frac{\left(c-a-2n+\frac{c}{d}(a+2n+1)\right)}{a(c+1)} \\ &\quad \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c+\frac{3}{2}\right)_n}. \end{aligned} \tag{3.24} $$

4. Special cases

Here we consider further special cases of the results in Section 3, which are known identities and, for completeness, recorded in the following theorem.

Theorem 4.1. *Each of the following summation formulas holds true:*

1. ($d = 2c-1$ *in* (3.1) *and* (3.2)*;* $n \in \mathbb{N}_0$)

$$ {}_3F_2 \left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+1),\ 2c-1\,; \end{array} 1 \right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{4.1} $$

and

$$ {}_3F_2 \left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+1),\ 2c-1\,; \end{array} 1 \right] = -\frac{1}{2c-1} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}. \tag{4.2} $$

2. ($d = \frac{1}{2}(a-1)$ *in* (3.1) *and* (3.2)*;* $n \in \mathbb{N}_0$)

$$ {}_3F_2 \left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a-1),\ 2c\,; \end{array} 1 \right] = \left(1-\frac{4n(a+2n)}{(a-1)(2c-a-1)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{4.3} $$

and

$$ {}_3F_2 \left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a-1),\ 2c\,; \end{array} 1 \right] = -\frac{2}{(a-1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}. \tag{4.4} $$

3. ($d = 2c$ *in* (3.3) *and* (3.4)*;* $n \in \mathbb{N}_0$)

$$ {}_3F_2 \left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+1),\ 2c\,; \end{array} 1 \right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{4.5} $$

and

$$ {}_3F_2 \left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+1),\ 2c\,; \end{array} 1 \right] = 0. \tag{4.6} $$

4. ($d = \frac{1}{2}(a-1)$ *in* (3.3) *and* (3.4)*;* $n \in \mathbb{N}_0$)

$$ {}_3F_2 \left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a-1),\ 2c+1\,; \end{array} 1 \right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{4.7} $$

and

$$ {}_3F_2 \left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a-1),\ 2c+1\,; \end{array} 1 \right] = \frac{(a-4c-1)}{(a-1)(2c+1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{3}{2}\right)_n}. \tag{4.8} $$

5. ($d = 2c + 1$ *in* (3.5) *and* (3.6); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+1),\ 2c+1\,; \end{array} 1\right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c - \frac{1}{2}\right)_n}{\left(\frac{1}{2}a + \frac{1}{2}\right)_n \left(c + \frac{1}{2}\right)_n}, \tag{4.9} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+1),\ 2c+1\,; \end{array} 1\right] = \frac{1}{(2c+1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + \frac{1}{2}\right)_n}{\left(\frac{1}{2}a + \frac{1}{2}\right)_n \left(c + \frac{3}{2}\right)_n}. \tag{4.10} $$

6. ($d = \frac{1}{2}(a-1)$ *in* (3.5) *and* (3.6); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a-1),\ 2c+2\,; \end{array} 1\right] = \left(1 + \frac{2n(a+2n)}{(c+1)(a-1)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c - \frac{1}{2}\right)_n}{\left(\frac{1}{2}a + \frac{1}{2}\right)_n \left(c + \frac{3}{2}\right)_n}, \tag{4.11} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a-1),\ 2c+2\,; \end{array} 1\right] = \frac{(a-2c-1)}{(a-1)(c+1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + \frac{1}{2}\right)_n}{\left(\frac{1}{2}a + \frac{1}{2}\right)_n \left(c + \frac{3}{2}\right)_n}. \tag{4.12} $$

7. ($d = 2c - 1$ *in* (3.7) *and* (3.8); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}a,\ 2c-1\,; \end{array} 1\right] = \left(1 + \frac{4n}{a-2c+2}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + 1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c - \frac{1}{2}\right)_n}, \tag{4.13} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}a,\ 2c-1\,; \end{array} 1\right] = -\frac{a+2c+4n}{a(2c-1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + 2\right)_n}{\left(\frac{1}{2}a + 1\right)_n \left(c + \frac{1}{2}\right)_n}. \tag{4.14} $$

8. ($d = 2c$ *in* (3.9) *and* (3.10); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}a,\ 2c\,; \end{array} 1\right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + 1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c + \frac{1}{2}\right)_n}, \tag{4.15} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}a,\ 2c\,; \end{array} 1\right] = -\frac{1}{a} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + 1\right)_n}{\left(\frac{1}{2}a + 1\right)_n \left(c + \frac{1}{2}\right)_n}. \tag{4.16} $$

9. ($d = 2c - 1$ *in* (3.11) *and* (3.12); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+2),\ 2c-1\,; \end{array} 1\right] = \frac{a}{a+4n} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + 1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c - \frac{1}{2}\right)_n}, \tag{4.17} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+2),\ 2c-1\,; \end{array} 1\right] = \frac{(2c-a-2)}{(a+4n+2)(2c-1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + 2\right)_n}{\left(\frac{1}{2}a + 1\right)_n \left(c + \frac{1}{2}\right)_n}. \tag{4.18} $$

10. ($d = 2c$ *in* (3.13) *and* (3.14); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+2),\ 2c\,; \end{array} 1\right] = \frac{a}{a+4n} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + 1\right)_n}{\left(\frac{1}{2}a\right)_n \left(c + \frac{1}{2}\right)_n}, \tag{4.19} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+2),\ 2c\,; \end{array} 1\right] = \frac{1}{(a+4n+2)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + 1\right)_n}{\left(\frac{1}{2}a + 1\right)_n \left(c + \frac{1}{2}\right)_n}. \tag{4.20} $$

11. ($d = \frac{1}{2}a$ *in* (3.13) *and* (3.14); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}a,\ \ 2c+1\,; \end{array} 1\right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{4.21} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}a,\ \ 2c+1\,; \end{array} 1\right] = \frac{a-2c}{a(2c+1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c+\frac{3}{2}\right)_n}. \tag{4.22} $$

12. ($d = 2c+1$ *in* (3.15) *and* (3.16); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+2),\ \ 2c+1\,; \end{array} 1\right] = \frac{a(2c-a-4n)}{(2c-a)(a+4n)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c\right)_n}{\left(\frac{1}{2}a\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{4.23} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+2),\ \ 2c+1\,; \end{array} 1\right] = \frac{(a+2c+4n+2)}{(2c+1)(a+4n+2)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+1\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c+\frac{3}{2}\right)_n}. \tag{4.24} $$

13. ($d = 2c-2$ *in* (3.17) *and* (3.18); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+1),\ \ 2c-2\,; \end{array} 1\right] = \left(1+\frac{2n(a+2n)}{(c-1)(a-2c+3)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{4.25} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+1),\ \ 2c-2\,; \end{array} 1\right] = -\frac{1}{c-1} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+\frac{5}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c-\frac{1}{2}\right)_n}. \tag{4.26} $$

14. ($d = \frac{1}{2}(a-1)$ *in* (3.17) *and* (3.18); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a-1),\ \ 2c-1\,; \end{array} 1\right] = \left(1+\frac{8n(a+2n)}{(a-1)(a-2c+3)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{4.27} $$

and

$$ \begin{aligned} {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a-1),\ \ 2c-1\,; \end{array} 1\right] &= \frac{(a+4c-1)(2c-a-3)-8n(a+2n+2)}{(a-1)(2c-1)(a-2c+3)} \\ &\quad \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}. \end{aligned} \tag{4.28} $$

15. ($d = 2c-2$ *in* (3.19) *and* (3.20); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}(a+2),\ \ 2c-2\,; \end{array} 1\right] = \frac{a(c+2n-1)}{(c-1)(a+4n)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{4.29} $$

and

$$ {}_3F_2\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c\,; \\ \frac{1}{2}(a+2),\ \ 2c-2\,; \end{array} 1\right] = \frac{(c-a-2n-2)}{(c-1)(a+4n+2)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c-\frac{1}{2}\right)_n}. \tag{4.30} $$

16. ($d = 2c-2$ *in* (3.21) *and* (3.22); $n \in \mathbb{N}_0$)

$$ {}_3F_2\left[\begin{array}{c} -2n,\ a+2n,\ c\,; \\ \frac{1}{2}a,\ \ 2c-2\,; \end{array} 1\right] = \left(1+\frac{2n(a+2c+4n-2)}{(c-1)(a-2c+4)}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{4.31} $$

and

$$ {}_3F_2\left[\begin{matrix} -2n-1, \ a+2n+1, \ c\,; \\ \frac{1}{2}a, \ \ 2c-2\,; \end{matrix} \ 1\right] = \eta\,\frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a-c+2\right)_n}{\left(\frac{1}{2}a+1\right)_n \left(c-\frac{1}{2}\right)_n}, \tag{4.32} $$

where

$$ \eta := \frac{(a+c)(2c-a-4)-2n(3a-2c+4n+6)}{a(c-1)(a-2c+4)}. $$

Remark 4.2. The results (4.1)-(4.32) are recorded in [4], which are derived here in a different manner. The result (2.2) is also given in [2]. The identities (3.3) and (3.4) are recorded in [2].

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