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# Quadratic $\rho$-functional inequalities in complex matrix normed spaces

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**Abstract**

In this paper, we solve the following quadratic $\rho$-functional inequalities

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \|\rho(2f(\frac{x+y}{2})+2f(\frac{x-y}{2})-f(x)-f(y))\|,$$

where $\rho$ is a fixed complex number with $|\rho|<1$, and

$$\|2f(\frac{x+y}{2})+2f(\frac{x-y}{2})-f(x)-f(y)\| \le \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|,$$

where $\rho$ is a fixed complex number with $|\rho|<\frac{1}{2}$. By using the direct method, we prove the Hyers-Ulam stability of these inequalities in complex matrix normed spaces, and prove the Hyers-Ulam stability of quadratic $\rho$-functional equations associated with these inequalities in complex matrix normed spaces. 

*Keywords:* Hyers-Ulam stability, matrix normed space, quadratic $\rho$-functional equation, quadratic $\rho$-functional inequality.
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## 1. Introduction and preliminaries

The first stability problem concerning with the group homomorphisms was raised by Ulam [13] and affirmatively solved by Hyers [5]. Hyers' result was generalized by Aoki [1] for additive mappings and by

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Rassias [11] for linear mappings by considering an unbounded Cauchy difference. The paper [11] of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. In 1994, a generalization of the Rassias' theorem was obtained by Găvruţă [4] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$ in the spirit of the Rassias approach.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \tag{1.1}$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof [12] for mappings from a normed space to a Banach space. Cholewa [2] noticed that Skof's theorem remains true if the domain is replaced by an Abelian group. In 1992, Czerwik [3] gave a generalization of the Skof–Cholewa's result.

The following functional equation

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y), \tag{1.2}$$

is called a Jensen-type quadratic equation (see [6]). In [6], Jang et al. proved the Hyers-Ulam stability of the equation (1.2) in fuzzy Banach spaces. In 2014, Wang et al. [14] investigated some stability results for Jensen-type quadratic functional equation (1.2) in intuitionistic fuzzy normed spaces.

In this paper, we consider the following two quadratic $\rho$-functional inequalities

$$\|f(x+y)+f(x-y) - 2f(x) - 2f(y)\| \leq \|\rho(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y))\|, \tag{1.3}$$

where $\rho$ is a fixed complex number with $|\rho| < 1$, and

$$\|2f(\frac{x+y}{2})+2f(\frac{x-y}{2}) - f(x) - f(y)\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \tag{1.4}$$

where $\rho$ is a fixed complex number with $|\rho| < \frac{1}{2}$, in complex matrix Banach spaces. More precisely, we solve the problem of the quadratic $\rho$-functional inequalities (1.3) and (1.4), and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (1.3) and (1.4) in complex matrix Banach spaces by using the direct method. Moreover, we prove the Hyers-Ulam stability of quadratic $\rho$-functional equations associated with the quadratic $\rho$-functional inequalities (1.3) and (1.4) in complex matrix Banach spaces.

Following [7, 8, 10], we will also use the following notations. The set of all $(m \times n)$-matrices in $X$ will be denoted by $M_{m,n}(X)$. When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$. The symbol $e_j \in M_{1,n}(\mathbb{C})$ will denote the row vector whose $j$-th component is 1 and the other components are 0. Similarly, $E_{ij} \in M_n(\mathbb{C})$ will denote the $n \times n$ matrix whose $(i, j)$-component is 1 and the other components are 0. The $n \times n$ matrix whose $(i, j)$-component is $x$ and the other components are 0 will be denoted by $E_{ij} \otimes x \in M_n(X)$. For $x \in M_n(X)$, $y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer $n$ and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if $X$ is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \|\cdot\|_n)$ is called an $L^\infty$-matrix normed space if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let $E, F$ be vector spaces. For a given mapping $h : E \to F$ and a given positive integer $n$, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

**Lemma 1.1** ([7, 8, 10]). *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Then*

(1) $\|E_{kl} \otimes x\|_n = \|x\|$ *for* $x \in X$;

(2) $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^{n} \|x_{ij}\|$ *for* $[x_{ij}] \in M_n(X)$;

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space and $(Y, \{\|\cdot\|_n\})$ be a matrix Banach space.

## 2. Stability of the quadratic $\rho$-functional inequality (1.3) in complex matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (1.3) in complex matrix normed spaces. We assume that $\rho$ is a fixed complex number with $|\rho| < 1$.

**Lemma 2.1.** *Let $V$ and $W$ be complex normed spaces. A mapping $f : V \to W$ satisfies*

$$\|f(x+y)+f(x-y) - 2f(x) - 2f(y)\| \leq \|\rho(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y))\|$$

*for all $x, y \in V$ if and only if $f : V \to W$ is quadratic.*

*Proof.* The proof is similar to the proof of [9, Lemma 2.2]. □

**Corollary 2.2.** *A mapping $f : V \to W$ satisfies*

$$\|f(x+y)+f(x-y) - 2f(x) - 2f(y)\| = \|\rho(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y))\|$$

*for all $x, y \in V$ if and only if $f : V \to W$ is quadratic.*

**Theorem 2.3.** *Let $r, \theta$ be positive real numbers with $r < 2$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\quad \leq \|\rho(2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n \\ &\quad\quad + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned} \tag{2.1}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r} \|x_{ij}\|^r \tag{2.2}$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* When $n = 1$, (2.1) is equivalent to

$$\begin{aligned}\|f(a+b) + f(a-b) - 2f(a) - 2f(b)\| &\leq \|\rho(2f(\frac{a+b}{2}) + 2f(\frac{a-b}{2}) - f(a) - f(b))\| \\ &\quad + \theta(\|a\|^r + \|b\|^r)\end{aligned} \tag{2.3}$$

for all $a, b \in X$. By letting $a = b = 0$ in (2.3), we get $\|2f(0)\| \le |\rho|\|2f(0)\|$, implying that $f(0) = 0$. Next, by letting $b = a$ in (2.3), we obtain

$$\|f(2a) - 4f(a)\| \le 2\theta\|a\|^r \tag{2.4}$$

for all $a \in X$. It follows from (2.4) that

$$\|f(a) - \frac{1}{4}f(2a)\| \le \frac{1}{2}\theta\|a\|^r$$

for all $a \in X$. Hence

$$\begin{aligned} \|\frac{1}{4^l}f(2^l a) - \frac{1}{4^m}f(2^m a)\| &\le \sum_{j=l}^{m-1} \|\frac{1}{4^j}f(2^j a) - \frac{1}{4^{j+1}}f(2^{j+1}a)\| \\ &\le \frac{1}{2}\sum_{j=l}^{m-1} \frac{2^{rj}}{4^j}\theta\|a\|^r \end{aligned} \tag{2.5}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $a \in X$. It follows from (2.5) that the sequence $\{\frac{f(2^n a)}{4^n}\}$ is a Cauchy sequence in $Y$ for all $a \in X$. Since $Y$ is complete, the sequence $\{\frac{f(2^n a)}{4^n}\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(a) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n a) \tag{2.6}$$

for all $a \in X$. Moreover, by letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get

$$\|f(a) - Q(a)\| \le \frac{2\theta}{4 - 2^r}\|a\|^r \tag{2.7}$$

for all $a \in X$.

Now, we show that the mapping $Q$ is quadratic. It follows from (2.3) and (2.6) that

$$\begin{aligned} \|Q(a+b) + Q(a-b) - 2Q(a) - 2Q(b)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|f(2^n(a+b)) + f(2^n(a-b)) - 2f(2^n a) - 2f(2^n b)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n}\|\rho(2f(\frac{2^n(a+b)}{2}) + 2f(\frac{2^n(a-b)}{2}) - f(2^n a) - f(2^n b))\| \\ &\quad + \lim_{n\to\infty} \frac{2^{rn}}{4^n}\theta(\|a\|^r + \|b\|^r) \\ &= \|\rho(2Q(\frac{a+b}{2}) + 2Q(\frac{a-b}{2}) - Q(a) - Q(b))\| \end{aligned}$$

for all $a, b \in X$. Thus, by Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

To prove the uniqueness of $Q$, let $Q' : X \to Y$ be another quadratic mapping satisfying (2.2). Let $n = 1$. Then we get

$$\begin{aligned} \|Q(a) - Q'(a)\| &= \|\frac{1}{4^n}Q(2^n a) - \frac{1}{4^n}Q'(2^n a)\| \\ &\le \|\frac{1}{4^n}Q(2^n a) - \frac{1}{4^n}f(2^n a)\| + \|\frac{1}{4^n}Q'(2^n a) - \frac{1}{4^n}f(2^n a)\| \\ &\le \frac{4\theta}{4 - 2^r}\frac{2^{rn}}{4^n}\|a\|^r \end{aligned}$$

for all $a \in X$. By letting $n \to \infty$ in the above inequality, we get $Q(a) = Q'(a)$ for all $a \in X$, which gives the conclusion.

By Lemma 1.1 and (2.7), we get

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{4-2^r}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $Q : X \to Y$ is a unique quadratic mapping satisfying (2.2), as desired. This completes the proof of the theorem. $\square$

**Theorem 2.4.** *Let $r, \theta$ be positive real numbers with $r > 2$, and let $f : X \to Y$ be a mapping satisfying* (2.1) *for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2^r-4}\|x_{ij}\|^r \tag{2.8}$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* It follows from (2.4) that

$$\|f(a) - 4f(\frac{a}{2})\| \leq \frac{2}{2^r}\theta\|a\|^r$$

for all $a \in X$. Hence

$$\begin{aligned}\|4^l f(\frac{a}{2^l}) - 4^m f(\frac{a}{2^m})\| &\leq \sum_{j=l}^{m-1} \|4^j f(\frac{a}{2^j}) - 4^{j+1} f(\frac{a}{2^{j+1}})\| \\ &\leq \frac{2}{2^r}\sum_{j=l}^{m-1} \frac{4^j}{2^{rj}}\theta\|a\|^r \end{aligned} \tag{2.9}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $a \in X$. It follows from (2.9) that the sequence $\{4^n f(\frac{a}{2^n})\}$ is a Cauchy sequence in $Y$ for all $a \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{a}{2^n})\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(a) = \lim_{n\to\infty} 4^n f(\frac{a}{2^n})$$

for all $a \in X$. Moreover, by letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get

$$\|f(a) - Q(a)\| \leq \frac{2\theta}{2^r-4}\|a\|^r$$

for all $a \in X$. The rest of the proof is similar to that of Theorem 2.3 and thus it is omitted. $\square$

By the triangle inequality, we obtain

$$\begin{aligned}&\|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])\|_n \\ &\qquad - \|\rho(2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n \\ &\leq \|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]) \\ &\qquad - \rho(2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n.\end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the quadratic $\rho$-functional equation associated with the quadratic $\rho$-functional inequality (1.3) in complex matrix Banach spaces.

**Corollary 2.5.** *Let $r, \theta$ be positive real numbers with $r < 2$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]) \\ &\quad - \rho(2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}]))\|_n \le \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned} \tag{2.10}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.2) *for all $x = [x_{ij}] \in M_n(X)$.*

**Corollary 2.6.** *Let $r, \theta$ be positive real numbers with $r > 2$, and let $f : X \to Y$ be a mapping satisfying* (2.10) *for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.8) *for all $x = [x_{ij}] \in M_n(X)$.*

*Remark* 2.7. If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

## 3. Stability of the quadratic $\rho$-functional inequality (1.4) in complex matrix normed spaces

In this section, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (1.4) in complex matrix normed spaces. We assume that $\rho$ is a fixed complex number with $|\rho| < \frac{1}{2}$.

**Lemma 3.1.** *Let $V$ and $W$ be complex normed spaces. A mapping $f : V \to W$ satisfies*

$$\|2f(\frac{x+y}{2})+2f(\frac{x-y}{2}) - f(x) - f(y)\| \le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|$$

*for all $x, y \in V$ if and only if $f : V \to W$ is quadratic.*

*Proof.* The proof is similar to the proof of [9, Lemma 3.1]. □

**Corollary 3.2.** *A mapping $f : V \to W$ satisfies*

$$\|2f(\frac{x+y}{2})+2f(\frac{x-y}{2}) - f(x) - f(y)\| = \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|$$

*for all $x, y \in V$ if and only if $f : V \to W$ is quadratic.*

**Theorem 3.3.** *Let $r, \theta$ be positive real numbers with $r < 2$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n \\ &\qquad \le \|\rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]))\|_n \\ &\qquad\quad + \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \end{aligned} \tag{3.1}$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2^r \theta}{4 - 2^r} \|x_{ij}\|^r \tag{3.2}$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* When $n = 1$, (3.1) is equivalent to

$$\begin{aligned} &\|2f(\frac{a+b}{2}) + 2f(\frac{a-b}{2}) - f(a) - f(b)\| \\ &\quad \le \|\rho(f(a+b) + f(a-b) - 2f(a) - 2f(b))\| + \theta(\|a\|^r + \|b\|^r) \end{aligned} \tag{3.3}$$

for all $a, b \in X$. By letting $a = b = 0$ in (3.3), we get $\|2f(0)\| \le |\rho|\|2f(0)\|$, implying that $f(0) = 0$. Next, by letting $b = 0$ in (3.3), we obtain

$$\|f(2a) - 4f(a)\| \le 2^r\theta\|a\|^r \tag{3.4}$$

for all $a \in X$. It follows from (2.4) that

$$\|f(a) - \frac{1}{4}f(2a)\| \le \frac{2^r}{4}\theta\|a\|^r$$

for all $a \in X$. Hence

$$\begin{aligned} \|\frac{1}{4^l}f(2^l a) - \frac{1}{4^m}f(2^m a)\| &\le \sum_{j=l}^{m-1} \|\frac{1}{4^j}f(2^j a) - \frac{1}{4^{j+1}}f(2^{j+1}a)\| \\ &\le \frac{2^r}{4}\sum_{j=l}^{m-1}\frac{2^{rj}}{4^j}\theta\|a\|^r \end{aligned} \tag{3.5}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $a \in X$. It follows from (3.5) that the sequence $\{\frac{f(2^n a)}{4^n}\}$ is a Cauchy sequence in $Y$ for all $a \in X$. Since $Y$ is complete, the sequence $\{\frac{f(2^n a)}{4^n}\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(a) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n a) \tag{3.6}$$

for all $a \in X$. Moreover, by letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get

$$\|f(a) - Q(a)\| \le \frac{2^r\theta}{4 - 2^r}\|a\|^r$$

for all $a \in X$.

Now, we show that the mapping $Q$ is quadratic. It follows from (3.3) and (3.6) that

$$\begin{aligned} \|2Q(\frac{a+b}{2}) + 2Q(\frac{a-b}{2}) - Q(a) - Q(b)\| &= \lim_{n\to\infty}\frac{1}{4^n}\|2f(\frac{2^n(a+b)}{2}) + 2f(\frac{2^n(a-b)}{2}) - f(2^n a) - f(2^n b)\| \\ &\le \lim_{n\to\infty}\frac{1}{4^n}\|\rho(f(2^n(a+b)) + f(2^n(a-b)) - 2f(2^n a) - 2f(2^n b))\| \\ &\quad + \lim_{n\to\infty}\frac{2^{rn}}{4^n}\theta(\|a\|^r + \|b\|^r) \\ &= \|\rho(Q(a+b) + Q(a-b) - 2Q(a) - 2Q(b))\| \end{aligned}$$

for all $a, b \in X$. Thus, by Lemma 3.1, the mapping $Q : X \to Y$ is quadratic. The rest of the proof is similar to that of Theorem 2.3 and thus it is omitted. □

**Theorem 3.4.** *Let $r, \theta$ be positive real numbers with $r > 2$, and let $f : X \to Y$ be a mapping satisfying (3.1) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2^r\theta}{2^r - 4}\|x_{ij}\|^r \tag{3.7}$$

*for all $x = [x_{ij}] \in M_n(X)$.*

*Proof.* It follows from (3.4) that

$$\|f(a) - 4f(\frac{a}{2})\| \le \theta\|a\|^r$$

for all $a \in X$. Hence

$$\begin{aligned}\|4^l f(\frac{a}{2^l}) - 4^m f(\frac{a}{2^m})\| &\le \sum_{j=l}^{m-1} \|4^j f(\frac{a}{2^j}) - 4^{j+1} f(\frac{a}{2^{j+1}})\| \\ &\le \sum_{j=l}^{m-1} \frac{4^j}{2^{rj}}\theta\|a\|^r \qquad (3.8)\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $a \in X$. It follows from (3.8) that the sequence $\{4^n f(\frac{a}{2^n})\}$ is a Cauchy sequence in $Y$ for all $a \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{a}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(a) = \lim_{n\to\infty} 4^n f(\frac{a}{2^n})$$

for all $a \in X$. Moreover, by letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get

$$\|f(a) - Q(a)\| \le \frac{2^r\theta}{2^r - 4}\|a\|^r$$

for all $a \in X$. The rest of the proof is similar to that of Theorem 3.3 and thus it is omitted. □

By the triangle inequality, we obtain

$$\begin{aligned}&\|2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}])\|_n \\ &\quad - \|\rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]))\|_n \\ &\le \|2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}]) \\ &\quad - \rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]))\|_n.\end{aligned}$$

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the quadratic $\rho$-functional equation associated with the quadratic $\rho$-functional inequality (1.4) in complex matrix Banach spaces.

**Corollary 3.5.** *Let $r, \theta$ be positive real numbers with $r < 2$, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|2f_n(\frac{[x_{ij}] + [y_{ij}]}{2}) + 2f_n(\frac{[x_{ij}] - [y_{ij}]}{2}) - f_n([x_{ij}]) - f_n([y_{ij}]) \\ &\quad - \rho(f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}]))\|_n \le \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)\end{aligned} \qquad (3.9)$$

*for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (3.2) *for all $x = [x_{ij}] \in M_n(X)$.*

**Corollary 3.6.** *Let $r, \theta$ be positive real numbers with $r > 2$, and let $f : X \to Y$ be a mapping satisfying* (3.9) *for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (3.7) *for all $x = [x_{ij}] \in M_n(X)$.*

*Remark* 3.7. If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

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