Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4780–4797
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

General convolution identities for Apostol-Bernoulli, Euler and Genocchi polynomials

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Communicated by S. H. Rim

Abstract

We perform a further investigation for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. By making use of the generating function methods and summation transform techniques, we establish some general convolution identities for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. These results are the corresponding extensions of some known formulas including the general convolution identities discovered by Dilcher and Vignat [K. Dilcher, C. Vignat, J. Math. Anal. Appl., **435** (2016), 1478–1498] on the classical Bernoulli and Euler polynomials. ©2016 All rights reserved.

Keywords: Apostol-Bernoulli polynomials and numbers, Apostol-Euler polynomials and numbers, Apostol-Genocchi polynomials and numbers, combinatorial identities.
2010 MSC: 11B68, 05A19.

1. Introduction

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are usually defined by the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi),$$

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad(|t|<\pi),$$

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Received 2016-05-18

and

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\quad(|t|<\pi).$$

The rational numbers B_n, the integers E_n and the rational numbers G_n given by

$$B_n=B_n(0),\quad E_n=2^nE_n\left(\frac{1}{2}\right)\quad\text{and}\quad G_n=G_n(0)$$

are called the classical Bernoulli numbers, the classical Euler numbers and the classical Genocchi numbers, respectively.

As is well-known, the classical Bernoulli, Euler and Genocchi polynomials and numbers play important roles in many different areas of mathematics such as number theory, combinatorics, special functions and mathematical analysis. Numerous interesting properties for them can be found in many books and papers; see, for example, [2, 3, 11, 13, 22, 23, 37]. The inspiration of the present paper stems from the general convolution identities for the classical Bernoulli and Euler polynomials recently discovered by Dilcher and Vignat [14] using identities for difference operators, techniques of symbolic computation and tools from the probability theory. We establish some general convolution identities for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials by making use of the generating function methods and summation transform techniques. These results are the corresponding extensions of some known formulas including the general convolution identities on the classical Bernoulli and Euler polynomials due to Dilcher and Vignat [14] and the convolution identities for the classical Genocchi polynomials due to Agoh [4].

We now turn to the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$, the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathcal{G}_n(x;\lambda)$, which are usually defined by means of the following generating functions (see, e.g., [25, 27–29]):

$$\frac{te^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{t^n}{n!}\quad(|t+\log\lambda|<2\pi),\tag{1.1}$$

$$\frac{2e^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(x;\lambda)\frac{t^n}{n!}\quad(|t+\log\lambda|<\pi),\tag{1.2}$$

and

$$\frac{2te^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{G}_n(x;\lambda)\frac{t^n}{n!}\quad(|t+\log\lambda|<\pi).\tag{1.3}$$

In particular, $\mathcal{B}_n(\lambda)$, $\mathcal{E}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ given by

$$\mathcal{B}_n(\lambda)=\mathcal{B}_n(0;\lambda),\quad \mathcal{E}_n(\lambda)=2^n\mathcal{E}_n\left(\frac{1}{2};\lambda\right)\quad\text{and}\quad\mathcal{G}_n(\lambda)=\mathcal{G}_n(0;\lambda)$$

are called the Apostol-Bernoulli numbers, the Apostol-Euler numbers and the Apostol-Genocchi numbers, respectively. Clearly, $\mathcal{B}_n(x;\lambda)$, $\mathcal{E}_n(x;\lambda)$ and $\mathcal{G}_n(x;\lambda)$ reduce to $B_n(x)$, $E_n(x)$ and $G_n(x)$ when $\lambda=1$. It is worth mentioning that the Apostol-Bernoulli polynomials were firstly introduced by Apostol [6] (see also Srivastava [36] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. For some nice methods and results on these polynomials and numbers, one is referred to [7, 8, 26, 32, 33].

This paper is organized as follows. In the second section, we state some notation, recall the elementary and beautiful idea stemming from Euler to discover his famous pentagonal number theorem, and give some general convolution identities for the Apostol-Bernoulli polynomials. In the third section, we present some general convolution identities for the Apostol-Euler polynomials. The fourth section is contributed to the statement of some general convolution identities for the Apostol-Genocchi polynomials.

2. Convolution identities for Apostol-Bernoulli polynomials

For convenience, in the following we adopt the common notation described in the standard books [12, 38]. The rising factorial $(a)_k$ is defined for complex number a and non-negative integer k by

$$(a)_0 = 1 \quad \text{and} \quad (a)_k = a(a+1)\cdots(a+k-1) \quad (k \geq 1).$$

The binomial coefficients $\binom{a}{k}$ is defined for complex number a and non-negative integer k by

$$\binom{a}{0} = 1 \quad \text{and} \quad \binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!} \quad (k \geq 1).$$

The multinomial coefficient $\binom{n}{r_1,\ldots,r_k}$ is defined for positive integer k and non-negative integers $n, r_1, \ldots, r_k$ by

$$\binom{n}{r_1,\ldots,r_k} = \frac{n!}{r_1!\cdots r_k!}.$$

We also write, for a subset $J \subseteq \{1,\ldots,k\}$ and complex numbers $a_1,\ldots,a_k$, $|J|$ as the cardinality of J,

$$a_J = \prod_{r\in J} a_r \quad \text{and} \quad \overline{J} = \{1,\ldots,k\} \setminus J,$$

and denote by $[t_1^{j_1}\cdots t_k^{j_k}]f(t_1,\ldots,t_k)$ the coefficients of $t_1^{j_1}\cdots t_k^{j_k}$ in $f(t_1,\ldots,t_k)$ for positive integer k and non-negative integers $j_1,\ldots,j_k$. Obviously,

$$\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]f(t_1,\ldots,t_k) = (j_1!\cdots j_k!)[t_1^{j_1}\cdots t_k^{j_k}]f(t_1,\ldots,t_k). \tag{2.1}$$

We now recall Euler's elementary and beautiful idea in the discovery of his famous pentagonal number theorem: for infinite number of complex numbers $x_1, x_2, x_3, \cdots$, (see, e.g., [5, 9, 10])

$$(1+x_1)(1+x_2)(1+x_3)\cdots = (1+x_1) + x_2(1+x_1) + x_3(1+x_1)(1+x_2) + \cdots. \tag{2.2}$$

We shall make use of the finite form of (2.2) to establish some general convolution identities for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Clearly, the finite form of (2.2) can be written as

$$\begin{aligned}(1+x_1)(1+x_2)(1+x_3)\cdots(1+x_n) &= (1+x_1) + x_2(1+x_1) + x_3(1+x_1)(1+x_2) \\ &\quad + \cdots + x_n(1+x_1)(1+x_2)\cdots(1+x_{n-1}), \quad (n \geq 1).\end{aligned} \tag{2.3}$$

We replace x_r by $x_r - 1$ for $1 \leq r \leq n$ in (2.3) to obtain

$$x_1\cdots x_n - 1 = \sum_{r=1}^{n}(x_r - 1)x_1\cdots x_{r-1}, \tag{2.4}$$

where the product $x_1\cdots x_{r-1}$ is considered to be equal to 1 when $r = 1$. If we take $x_r = \lambda_r e^{t_r}$ for $1 \leq r \leq n$ and substitute k for n in (2.4), then for positive integer k,

$$\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1 = \sum_{r=1}^{k}(\lambda_r e^{t_r} - 1)\prod_{i=1}^{r-1}\lambda_i e^{t_i}. \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}\prod_{i=1}^{k}\frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} &= \sum_{r=1}^{k}\frac{\lambda_r e^{t_r} - 1}{\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1}\prod_{i=1}^{r-1}\lambda_i e^{t_i}\prod_{i=1}^{k}\frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} \\ &= \sum_{r=1}^{k}\frac{t_r e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1}\prod_{i=1}^{r-1}\lambda_i\frac{t_i e^{(x_i - x_r + 1)t_i}}{\lambda_i e^{t_i} - 1}\prod_{i=r+1}^{k}\frac{t_i e^{(x_i - x_r)t_i}}{\lambda_i e^{t_i} - 1},\end{aligned}$$

which means for non-negative integer n and complex numbers $a_1,\ldots,a_k$,

$$\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{t_ie^{x_it_i}}{\lambda_ie^{t_i}-1}\right)$$
$$=\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left(\sum_{r=1}^{k}t_r\frac{(t_1+\cdots+t_k)e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}-1}\right. \tag{2.6}$$
$$\left.\times\prod_{i=1}^{r-1}\lambda_i\frac{t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}-1}\prod_{i=r+1}^{k}\frac{t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}-1}\right).$$

It is easily seen that for complex number a and non-negative integer k,

$$(k+1)\binom{-a}{k+1}=(-a-k)\binom{-a}{k}\quad\text{and}\quad(a)_k=(-1)^kk!\cdot\binom{-a}{k}. \tag{2.7}$$

It follows from (1.1) and (2.7) that

$$\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{t_ie^{x_it_i}}{\lambda_ie^{t_i}-1}\right)$$
$$=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\sum_{r=1}^{k}(j_r+1)\binom{-a_r}{j_r+1}\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{j_i}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\prod_{i=1}^{k}\frac{t_ie^{x_it_i}}{\lambda_ie^{t_i}-1}\right) \tag{2.8}$$
$$=\frac{(-1)^{n+1}(n+a_1+\cdots+a_k)}{n!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}(a_1)_{j_1}\cdots(a_k)_{j_k}$$
$$\times\mathcal{B}_{j_1}(x_1;\lambda_1)\cdots\mathcal{B}_{j_k}(x_k;\lambda_k).$$

On the other hand, since for positive integer k and non-negative integer N (see, e.g., [12, 38]),

$$(t_1+\cdots+t_k)^N=\sum_{\substack{l_1+\cdots+l_k=N\\ l_1,\ldots,l_k\geq 0}}\binom{N}{l_1,\ldots,l_k}t_1^{l_1}\cdots t_k^{l_k}, \tag{2.9}$$

so by (1.1) and (2.9) we have

$$\frac{(t_1+\cdots+t_k)e^{x(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}-1}=\sum_{N=0}^{\infty}\mathcal{B}_N(x;\lambda_1\cdots\lambda_k)\sum_{\substack{l_1+\cdots+l_k=N\\ l_1,\ldots,l_k\geq 0}}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_k^{l_k}}{l_k!}. \tag{2.10}$$

It follows from (1.1), (2.1), (2.6) and (2.10) that

$$\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{t_ie^{x_it_i}}{\lambda_ie^{t_i}-1}\right)$$
$$=\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}(j_1!\cdots j_k!)$$
$$\times\sum_{r=1}^{k}\sum_{\substack{l_1,\ldots,l_{r-1},\\ l_{r+1},\ldots,l_k\geq 0}}\frac{\mathcal{B}_{l_1+\cdots+l_{r-1}+(j_r-1)+l_{r+1}+\cdots+l_k}(x_r;\lambda_1\cdots\lambda_k)}{l_1!\cdots l_{r-1}!\cdot(j_r-1)!\cdot l_{r+1}!\cdots l_k!} \tag{2.11}$$

$$
\begin{aligned}
&\times \prod_{i=1}^{r-1} \lambda_i \frac{\mathcal{B}_{j_i-l_i}(x_i-x_r+1;\lambda_i)}{(j_i-l_i)!} \prod_{i=r+1}^{k} \frac{\mathcal{B}_{j_i-l_i}(x_i-x_r;\lambda_i)}{(j_i-l_i)!} \\
&= \sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}} \binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k} \sum_{r=1}^{k} \sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}} j_r \mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k) \\
&\times \prod_{i=1}^{r-1} \binom{j_i}{l_i} \lambda_i \mathcal{B}_{l_i}(x_i-x_r+1;\lambda_i) \prod_{i=r+1}^{k} \binom{j_i}{l_i} \mathcal{B}_{l_i}(x_i-x_r;\lambda_i).
\end{aligned}
$$

Notice that for non-negative integers n, k and complex number a,

$$
k\binom{-a}{k} = -a\binom{-a-1}{k-1} \quad \text{and} \quad \binom{n}{k}\binom{a}{n} = \binom{a}{k}\binom{a-k}{n-k}, \tag{2.12}
$$

which together with the famous Chu-Vandermonde convolution showed in [17] yields for non-negative integers $l_1, \ldots, l_k$ with $l_1 + \cdots + l_k = n$,

$$
\begin{aligned}
\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}} j_r \binom{-a_r}{j_r} \prod_{\substack{i=1\\ i\neq r}}^{k} \binom{-a_i}{j_i}\binom{j_i}{l_i} &= -a_r \prod_{\substack{i=1\\ i\neq r}}^{k} \binom{-a_i}{l_i} \sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}} \binom{-a_r-1}{j_r-1} \prod_{\substack{i=1\\ i\neq r}}^{k} \binom{-a_i-l_i}{j_i-l_i} \\
&= -a_r \prod_{\substack{i=1\\ i\neq r}}^{k} \binom{-a_i}{l_i} \binom{-(a_1+\cdots+a_k)-1-(n-l_r)}{n-(n-l_r)}.
\end{aligned} \tag{2.13}
$$

If we apply (2.7) and (2.13) to (2.11) then

$$
\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}} \binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k} \left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right] \left(\sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i}-1}\right) \\
&= \frac{(-1)^{n+1}}{n!} \sum_{r=1}^{k} a_r \sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}} \binom{n}{l_1,\ldots,l_k} (a_1+\cdots+a_k+n+1-l_r)_{l_r} \mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k) \\
&\times \prod_{i=1}^{r-1} \lambda_i (a_i)_{l_i} \mathcal{B}_{l_i}(x_i-x_r+1;\lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} \mathcal{B}_{l_i}(x_i-x_r;\lambda_i).
\end{aligned} \tag{2.14}
$$

Thus, by equations (2.8) and (2.14), we get the following result.

Theorem 2.1. *Let $a_1, \ldots, a_k$ be complex numbers with k being a positive integer. Then, for non-negative integer n,*

$$
\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} \mathcal{B}_{j_1}(x_1;\lambda_1)\cdots\mathcal{B}_{j_k}(x_k;\lambda_k) \\
&= \sum_{r=1}^{k} \sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}} \binom{n}{l_1,\ldots,l_k} \frac{a_r}{(a_1+\cdots+a_k)_{n+1-l_r}} \mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k) \\
&\times \prod_{i=1}^{r-1} (a_i)_{l_i} \lambda_i \mathcal{B}_{l_i}(x_i-x_r+1;\lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} \mathcal{B}_{l_i}(x_i-x_r;\lambda_i).
\end{aligned}
$$

It follows that we show some special cases of Theorem 2.1. Since the Apostol-Bernoulli polynomials satisfy the difference equation (see, e.g., [28]):

$$\lambda\mathcal{B}_n(x+1;\lambda)-\mathcal{B}_n(x;\lambda)=nx^{n-1}\quad(n\geq 0),\tag{2.15}$$

so from (2.15) we have

$$\prod_{i=1}^{r-1}(a_i)_{l_i}\lambda_i\mathcal{B}_{l_i}(x_i-x_r+1;\lambda_i)=\sum_{J\subseteq\{1,\ldots,r-1\}}\prod_{i\in J}(a_i)_{l_i}\mathcal{B}_{l_i}(x_i-x_r;\lambda_i)\prod_{i\in\overline{J}}(a_i)_{l_i}l_i(x_i-x_r)^{l_i-1}.\tag{2.16}$$

Thus, by applying (2.16) to Theorem 2.1 and then taking $x_1=\cdots=x_k=x$, we get the general convolution identity for the Apostol-Bernoulli polynomials as follows.

Corollary 2.2. *Let $a_1,\ldots,a_k$ be complex numbers with k being a positive integer. Then, for non-negative integer n,*

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}\mathcal{B}_{j_1}(x;\lambda_1)\cdots\mathcal{B}_{j_k}(x;\lambda_k)\\&\quad=\sum_{r=1}^{k}\sum_{|J|=r}\frac{n!\cdot a_J}{(n+1-r)!}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+1-r}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}\\&\qquad\times\mathcal{B}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{B}_{l_1}(\lambda_{i_{r+1}})\cdots\mathcal{B}_{l_{k-r}}(\lambda_{i_k}),\end{aligned}$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

In particular, if we take $\lambda_1=\cdots=\lambda_k=1$ in Corollary 2.2, we get the following general convolution identity for the classical Bernoulli polynomials.

Corollary 2.3. *Let $a_1,\ldots,a_k$ be complex numbers with k being a positive integer. Then, for non-negative integer n,*

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}B_{j_1}(x)\cdots B_{j_k}(x)\\&\quad=\sum_{r=1}^{k}\sum_{|J|=r}\frac{n!\cdot a_J}{(n+1-r)!}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+1-r}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}\\&\qquad\times B_{l_0}(x)B_{l_1}\cdots B_{l_{k-r}},\end{aligned}$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

The case $a_1,\ldots,a_k$ being positive real numbers in Corollary 2.3 is due to Dilcher and Vignat [14, Theorem 2], and leads to the corresponding higher-order convolution identity for the classical Bernoulli polynomials due to Agoh and Dilcher [4, Theorem 1] when $a_1=\cdots=a_k=1$. In the same way, if we take $a_1=\cdots=a_k=1$ in Corollary 2.2, we obtain the higher-order convolution identity for the Apostol-Bernoulli polynomials as follows.

Corollary 2.4. *Let k be a positive integer. Then, for non-negative integer n,*

$$(n+k)\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\mathcal{B}_{j_1}(x;\lambda_1)\cdots\mathcal{B}_{j_k}(x;\lambda_k)$$

$$= \sum_{r=1}^{k} \sum_{|J|=r} \sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r \\ l_0,l_1,\ldots,l_{k-r}\geq 0}} \binom{n+k}{l_0} \mathcal{B}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{B}_{l_1}(\lambda_{i_{r+1}})\cdots\mathcal{B}_{l_{k-r}}(\lambda_{i_k}),$$

where $i_{r+1},\ldots,i_k \in \overline{J}$.

Remark 2.5. Note that the corresponding higher-order convolution identity for the Apostol-Bernoulli polynomials stated in [18, Theorem 3.1] is only complete on condition that $\lambda_1 = \cdots = \lambda_k$. The case $k = 2$ in Corollary 2.3 can be easily used to give for positive integer n (see, e.g., [14]),

$$(n+2)\sum_{k=0}^{n} B_k(x)B_{n-k}(x) = 2\sum_{k=0}^{n}\binom{n+2}{k+2}B_kB_{n-k}(x) + \binom{n+2}{3}B_{n-1}(x), \tag{2.17}$$

and for positive integer $n \geq 2$,

$$\frac{n}{2}\sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(x)}{k(n-k)} = \sum_{k=1}^{n}\binom{n}{k}\frac{B_kB_{n-k}(x)}{k} + \frac{n}{2}B_{n-1}(x) + H_{n-1}B_n(x), \tag{2.18}$$

where H_n is the n-th Harmonic numbers. And the case $x = 0$ in (2.17) and (2.18) gives the famous identities of Matiyasevich [30] and Miki [31] for the classical Bernoulli numbers, respectively. For some equivalent versions and different proofs of (2.17) and (2.18), one is referred to [1, 15, 16, 21, 24, 34]. For more applications of Corollary 2.3, see [14] for details.

3. Convolution identities for Apostol-Euler polynomials

We next apply (2.4) to establish some general convolution identities for the Apostol-Euler polynomials. By taking $x_r = -\lambda_r e^{t_r}$ for $1 \leq r \leq n$ and substituting k for n in (2.4), we get for positive integer k,

$$(-1)^k\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1 = \sum_{r=1}^{k}(-1)^r(\lambda_r e^{t_r}+1)\prod_{i=1}^{r-1}\lambda_i e^{t_i}. \tag{3.1}$$

It follows from (3.1) that

$$\begin{aligned}\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1} &= \sum_{r=1}^{k}(-1)^r\frac{\lambda_re^{t_r}+1}{(-1)^k\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}-1}\prod_{i=1}^{r-1}\lambda_ie^{t_i}\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\\ &= \sum_{r=1}^{k}(-1)^r\frac{2e^{x_r(t_1+\cdots+t_k)}}{(-1)^k\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}-1}\prod_{i=1}^{r-1}\lambda_i\frac{2e^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{k}\frac{2e^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}.\end{aligned} \tag{3.2}$$

We discuss (3.2) on two cases. We firstly consider the case k being an even integer. In this case, we get for non-negative integer n and complex numbers $a_1,\ldots,a_k$,

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\ &\quad= 2\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left(\sum_{r=1}^{k}(-1)^r\frac{(t_1+\cdots+t_k)e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}-1}\right.\\ &\quad\quad\left.\times\prod_{i=1}^{r-1}\lambda_i\frac{2e^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{k}\frac{2e^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}\right).\end{aligned} \tag{3.3}$$

If we make the operation $[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}]$ in both sides of the above identity, in view of (1.2) and (2.7), the left hand side of (3.3) can be written as

$$\begin{aligned}\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}&\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\ &=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\sum_{r=1}^{k}(j_r+1)\binom{-a_r}{j_r+1}\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{j_i}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\ &=\frac{(-1)^{n+1}(n+a_1+\cdots+a_k)}{n!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}(a_1)_{j_1}\cdots(a_k)_{j_k}\\ &\quad\times\mathcal{E}_{j_1}(x_1;\lambda_1)\cdots\mathcal{E}_{j_k}(x_k;\lambda_k),\end{aligned}\tag{3.4}$$

and in light of (1.2) and (2.10), the right hand side of (3.3) can be written as

$$\begin{aligned}\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}&\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\ &=2\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\sum_{r=1}^{k}(-1)^r\sum_{\substack{l_1+\cdots+l_k=n+1\\ l_1,\ldots,l_k\geq 0}}\mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k)\\ &\quad\times\prod_{i=1}^{r-1}\binom{j_i}{l_i}\lambda_i\mathcal{E}_{l_i}(x_i-x_r+1;\lambda_i)\prod_{i=r+1}^{k}\binom{j_i}{l_i}\mathcal{E}_{l_i}(x_i-x_r;\lambda_i).\end{aligned}\tag{3.5}$$

It is easy to see from (2.12) and the famous Chu-Vandermonde convolution stated in [17] that for non-negative integers $l_1,\ldots,l_k$ with $l_1+\cdots+l_k=n+1$,

$$\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}\binom{-a_r}{j_r}\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{j_i}\binom{j_i}{l_i}=\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{l_i}\binom{-(a_1+\cdots+a_k)-(n+1-l_r)}{n+1-(n+1-l_r)}.\tag{3.6}$$

Hence, applying (2.7) and (3.6) to (3.5) gives

$$\begin{aligned}\sum_{\substack{j_1+\cdots+j_k=n+1\\ j_1,\ldots,j_k\geq 0}}&\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\sum_{r=1}^{k}t_r\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\ &=2\frac{(-1)^{n+1}}{(n+1)!}\sum_{r=1}^{k}(-1)^r\sum_{\substack{l_1+\cdots+l_k=n+1\\ l_1,\ldots,l_k\geq 0}}\binom{n+1}{l_1,\ldots,l_k}(a_1+\cdots+a_k+n+1-l_r)_{l_r}\\ &\quad\times\mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k)\prod_{i=1}^{r-1}\lambda_i(a_i)_{l_i}\mathcal{E}_{l_i}(x_i-x_r+1;\lambda_i)\prod_{i=r+1}^{k}(a_i)_{l_i}\mathcal{E}_{l_i}(x_i-x_r;\lambda_i).\end{aligned}\tag{3.7}$$

Thus, by equations (3.4) and (3.7), we obtain the following result.

Theorem 3.1. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}\mathcal{E}_{j_1}(x_1;\lambda_1)\cdots\mathcal{E}_{j_k}(x_k;\lambda_k)$$

$$= \frac{2}{n+1}\sum_{r=1}^{k}(-1)^r \sum_{\substack{l_1+\cdots+l_k=n+1\\ l_1,\ldots,l_k\geq 0}} \binom{n+1}{l_1,\ldots,l_k} \frac{\mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k)}{(a_1+\cdots+a_k)_{n+1-l_r}}$$
$$\times \prod_{i=1}^{r-1}(a_i)_{l_i}\lambda_i\mathcal{E}_{l_i}(x_i-x_r+1;\lambda_i)\prod_{i=r+1}^{k}(a_i)_{l_i}\mathcal{E}_{l_i}(x_i-x_r;\lambda_i).$$

It follows that we give some special cases of Theorem 3.1. Since the Apostol-Euler polynomials satisfy the difference equation (see, e.g., [25])

$$\lambda\mathcal{E}_n(x+1;\lambda)+\mathcal{E}_n(x;\lambda)=2x^n \quad (n\geq 0), \tag{3.8}$$

so from (3.8) we have

$$\prod_{i=1}^{r-1}\{-(a_i)_{l_i}\lambda_i\mathcal{E}_{l_i}(x_i-x_r+1;\lambda_i)\} = \sum_{J\subseteq\{1,\ldots,r-1\}}\prod_{i\in J}(a_i)_{l_i}\mathcal{E}_{l_i}(x_i-x_r;\lambda_i)\prod_{i\in\overline{J}}\{-2(a_i)_{l_i}(x_i-x_r)^{l_i}\}. \tag{3.9}$$

Thus, applying (3.9) to Theorem 3.1 and then taking $x_1=\cdots=x_k=x$ gives the general convolution identity for the Apostol-Euler polynomials as follows.

Corollary 3.2. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}\mathcal{E}_{j_1}(x;\lambda_1)\cdots\mathcal{E}_{j_k}(x;\lambda_k)$$
$$= \frac{1}{n+1}\sum_{r=1}^{k}\sum_{|J|=r}(-2)^r\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+1}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}$$
$$\times\mathcal{B}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{E}_{l_1}(0;\lambda_{i_{r+1}})\cdots\mathcal{E}_{l_{k-r}}(0;\lambda_{i_k}),$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

If we take $\lambda_1=\cdots=\lambda_k=1$ in Corollary 3.2, we get the following general convolution identity for the classical Euler polynomials.

Corollary 3.3. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}E_{j_1}(x)\cdots E_{j_k}(x)$$
$$= \frac{1}{n+1}\sum_{r=1}^{k}\sum_{|J|=r}(-2)^r\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+1}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}$$
$$\times B_{l_0}(x)E_{l_1}(0)\cdots E_{l_{k-r}}(0),$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

If we take $a_1=\cdots=a_k=1$ in Corollary 3.2, we get the following higher-order convolution identity for the Apostol-Euler polynomials.

Corollary 3.4. *Let k be an even positive integer. Then, for non-negative integer n,*

$$\begin{aligned}(n+k)&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\mathcal{E}_{j_1}(x;\lambda_1)\cdots\mathcal{E}_{j_k}(x;\lambda_k)\\ &=\sum_{r=1}^{k}\sum_{|J|=r}(-2)^r\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k}{l_0}\mathcal{B}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{E}_{l_1}(0;\lambda_{i_{r+1}})\cdots\mathcal{E}_{l_{k-r}}(0;\lambda_{i_k}),\end{aligned}$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

We next consider the case k being an odd positive integer in (3.2). In this case, it is easily seen that for non-negative integer n and complex numbers $a_1,\ldots,a_k$,

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left(\prod_{i=1}^{k}\frac{2e^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\ &\quad=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left(\sum_{r=1}^{k}(-1)^{r-1}\frac{2e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}+1}\right.\\ &\quad\quad\left.\times\prod_{i=1}^{r-1}\lambda_i\frac{2e^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{k}\frac{2e^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}\right).\end{aligned}\tag{3.10}$$

Notice that from (1.2) and (2.9) we have

$$\frac{2e^{x(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_ke^{t_1+\cdots+t_k}+1}=\sum_{N=0}^{\infty}\mathcal{E}_N(x;\lambda_1\cdots\lambda_k)\sum_{\substack{l_1+\cdots+l_k=N\\ l_1,\ldots,l_k\geq 0}}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_k^{l_k}}{l_k!}.\tag{3.11}$$

By making the operation $[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}]$ in both sides of (3.10), in view of (1.2) and (3.11), we obtain

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\mathcal{E}_{j_1}(x_1;\lambda_1)\cdots\mathcal{E}_{j_k}(x_k;\lambda_k)\\ &\quad=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\sum_{r=1}^{k}(-1)^{r-1}\sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}}\mathcal{E}_{l_r}(x_r;\lambda_1\cdots\lambda_k)\\ &\quad\quad\times\prod_{i=1}^{r-1}\binom{j_i}{l_i}\lambda_i\mathcal{E}_{l_i}(x_i-x_r+1;\lambda_i)\prod_{i=r+1}^{k}\binom{j_i}{l_i}\mathcal{E}_{l_i}(x_i-x_r;\lambda_i),\end{aligned}$$

which together with (2.7) and (3.6) yields the following result.

Theorem 3.5. *Let $a_1,\ldots,a_k$ be complex numbers with k being an odd positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}\mathcal{E}_{j_1}(x_1;\lambda_1)\cdots\mathcal{E}_{j_k}(x_k;\lambda_k)$$

$$= \sum_{r=1}^{k} (-1)^{r-1} \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k \geq 0}} \binom{n}{l_1,\ldots,l_k} \frac{\mathcal{E}_{l_r}(x_r;\lambda_1\cdots\lambda_k)}{(a_1+\cdots+a_k)_{n-l_r}}$$
$$\times \prod_{i=1}^{r-1} (a_i)_{l_i} \lambda_i \mathcal{E}_{l_i}(x_i - x_r + 1;\lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} \mathcal{E}_{l_i}(x_i - x_r;\lambda_i).$$

If we apply (3.9) to Theorem 3.5 and then take $x_1 = \cdots = x_k = x$, we get another general convolution identity for the Apostol-Euler polynomials as follows.

Corollary 3.6. *Let $a_1,\ldots,a_k$ be complex numbers with k being an odd positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} \mathcal{E}_{j_1}(x;\lambda_1)\cdots\mathcal{E}_{j_k}(x;\lambda_k)$$
$$= \sum_{r=1}^{k} \sum_{|J|=r} (-2)^{r-1} \sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n \\ l_0,l_1,\ldots,l_{k-r} \geq 0}} \binom{n}{l_0,l_1,\ldots,l_{k-r}} \frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n-l_0}}$$
$$\times \mathcal{E}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{E}_{l_1}(0;\lambda_{i_{r+1}})\cdots\mathcal{E}_{l_{k-r}}(0;\lambda_{i_k}),$$

where $i_{r+1},\ldots,i_k \in \overline{J}$.

The case $\lambda_1 = \cdots = \lambda_k = 1$ in Corollary 3.6 gives the following general convolution identity for the classical Euler polynomials.

Corollary 3.7. *Let $a_1,\ldots,a_k$ be complex numbers with k being an odd positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} E_{j_1}(x)\cdots E_{j_k}(x)$$
$$= \sum_{r=1}^{k} \sum_{|J|=r} (-2)^{r-1} \sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n \\ l_0,l_1,\ldots,l_{k-r} \geq 0}} \binom{n}{l_0,l_1,\ldots,l_{k-r}} \frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n-l_0}}$$
$$\times E_{l_0}(x)E_{l_1}(0)\cdots E_{l_{k-r}}(0),$$

where $i_{r+1},\ldots,i_k \in \overline{J}$.

The case $a_1,\ldots,a_k$ being positive real numbers in the above Corollaries 3.3 and 3.7 are due to Dilcher and Vignat [14, Theorem 4], which gives the corresponding higher-order convolution identity for the classical Euler polynomials due to Agoh and Dilcher [4, Theorems 2 and 3] when $a_1 = \cdots = a_k = 1$, respectively. In particular, the case $k = 2$ in Corollaries 3.3 and 3.7 will give some similar convolution identities for the classical Euler polynomials to (2.17) and (2.18) (see [14] for details). If we take $a_1 = \cdots = a_k = 1$ in Corollary 3.6, we obtain the following higher-order convolution identity for the Apostol-Euler polynomials.

Corollary 3.8. *Let k be an odd positive integer. Then, for non-negative integer n,*

$$\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \mathcal{E}_{j_1}(x;\lambda_1)\cdots\mathcal{E}_{j_k}(x;\lambda_k)$$

$$
= \sum_{r=1}^{k} \sum_{|J|=r} (-2)^{r-1} \sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n \\ l_0,l_1,\ldots,l_{k-r}\geq 0}} \binom{n+k-1}{l_0} \mathcal{E}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{E}_{l_1}(0;\lambda_{i_{r+1}})\cdots\mathcal{E}_{l_{k-r}}(0;\lambda_{i_k}),
$$

where $i_{r+1},\ldots,i_k \in \overline{J}$.

Remark 3.9. Note that the corresponding two higher-order convolution identities for the Apostol-Euler polynomials stated in [18, Theorem 3.2] are only complete on condition that $\lambda_1 = \cdots = \lambda_k$.

4. Convolution identities for Apostol-Genocchi polynomials

We finally apply (3.2) to establish some general convolution identities for the Apostol-Genocchi polynomials. By substituting $2t_i$ for 2 in the left hand side of (3.2), we discover

$$
\prod_{i=1}^{k} \frac{2t_i e^{x_i t_i}}{\lambda_i e^{t_i}+1} = \sum_{r=1}^{k} (-1)^r \frac{2t_r e^{x_r(t_1+\cdots+t_k)}}{(-1)^k \lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k}-1} \prod_{i=1}^{r-1} \lambda_i \frac{2t_i e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i}+1} \prod_{i=r+1}^{k} \frac{2t_i e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i}+1}, \tag{4.1}
$$

which means for even positive integer k, non-negative integer n and complex numbers $a_1,\ldots,a_k$,

$$
\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n+1 \\ j_1,\ldots,j_k\geq 0}} \binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k} \left(\sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{2t_i e^{x_i t_i}}{\lambda_i e^{t_i}+1} \right) \\
&\quad = 2 \sum_{\substack{j_1+\cdots+j_k=n+1 \\ j_1,\ldots,j_k\geq 0}} \binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k} \left(\sum_{r=1}^{k} (-1)^r t_r \frac{(t_1+\cdots+t_k)e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k}-1} \right. \\
&\qquad \left. \times \prod_{i=1}^{r-1} \lambda_i \frac{2t_i e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i}+1} \prod_{i=r+1}^{k} \frac{2t_i e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i}+1} \right).
\end{aligned} \tag{4.2}
$$

By making the operation $[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}]$ in both sides of (4.2), in similar considerations to (2.8) and (2.14), we discover the following result.

Theorem 4.1. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$
\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k\geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} \mathcal{G}_{j_1}(x_1;\lambda_1)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k) \\
&\quad = 2\sum_{r=1}^{k} (-1)^r \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k\geq 0}} \binom{n}{l_1,\ldots,l_k} \frac{a_r}{(a_1+\cdots+a_k)_{n+1-l_r}} \mathcal{B}_{l_r}(x_r;\lambda_1\cdots\lambda_k) \\
&\qquad \times \prod_{i=1}^{r-1} (a_i)_{l_i} \lambda_i \mathcal{G}_{l_i}(x_i-x_r+1;\lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} \mathcal{G}_{l_i}(x_i-x_r;\lambda_i).
\end{aligned}
$$

It follows that we discuss some special cases of Theorem 4.1. Since the Apostol-Genocchi polynomials satisfy the difference equation (see, e.g., [26])

$$
\lambda\mathcal{G}_n(x+1;\lambda) + \mathcal{G}_n(x;\lambda) = 2nx^{n-1} \quad (n\geq 0), \tag{4.3}
$$

so from (4.3) we have

$$\prod_{i=1}^{r-1}\{-(a_i)_{l_i}\lambda_i\mathcal{G}_{l_i}(x_i-x_r+1;\lambda_i)\}=\sum_{J\subseteq\{1,\ldots,r-1\}}\prod_{i\in J}(a_i)_{l_i}\mathcal{G}_{l_i}(x_i-x_r;\lambda_i)\prod_{i\in\overline{J}}\{-2(a_i)_{l_i}l_i(x_i-x_r)^{l_i-1}\}. \quad (4.4)$$

Thus, applying (4.4) to Theorem 4.1 and then taking $x_1=\cdots=x_k=x$ gives the general convolution identity of the Apostol-Genocchi polynomials, as follows.

Corollary 4.2. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$\begin{aligned}\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}&\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}\mathcal{G}_{j_1}(x;\lambda_1)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\ &=\sum_{r=1}^{k}\sum_{|J|=r}(-2)^r\frac{n!\cdot a_J}{(n+1-r)!}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+1-r}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}\\ &\quad\times\mathcal{B}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{G}_{l_1}(\lambda_{i_{r+1}})\cdots\mathcal{G}_{l_{k-r}}(\lambda_{i_k}),\end{aligned}$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

It is obvious that the case $\lambda_1=\cdots=\lambda_k=1$ in Corollary 4.2 gives the following general convolution identity for the classical Genocchi polynomials.

Corollary 4.3. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$\begin{aligned}\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}&\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}G_{j_1}(x)\cdots G_{j_k}(x)\\ &=\sum_{r=1}^{k}\sum_{|J|=r}(-2)^r\frac{n!\cdot a_J}{(n+1-r)!}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+1-r}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}\\ &\quad\times B_{l_0}(x)G_{l_1}\cdots G_{l_{k-r}},\end{aligned}$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

If we take $a_1=\cdots=a_k=1$ in Corollary 4.2, we get the following higher-order convolution identity for the Apostol-Genocchi polynomials.

Corollary 4.4. *Let $a_1,\ldots,a_k$ be complex numbers with k being an even positive integer. Then, for non-negative integer n,*

$$\begin{aligned}(n+k)&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\mathcal{G}_{j_1}(x;\lambda_1)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\ &=\sum_{r=1}^{k}\sum_{|J|=r}(-2)^r\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k}{l_0}\mathcal{B}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{G}_{l_1}(\lambda_{i_{r+1}})\cdots\mathcal{G}_{l_{k-r}}(\lambda_{i_k}),\end{aligned}$$

where $i_{r+1},\ldots,i_k\in\overline{J}$.

In particular, the case $\lambda_1 = \cdots = \lambda_k = 1$ in Corollary 4.4 gives the higher-order convolution identity for the classical Genocchi polynomials as follows,

$$(n+k) \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} G_{j_1}(x) \cdots G_{j_k}(x) = \sum_{r=1}^{k} \binom{k}{r} (-2)^r \sum_{\substack{l_0+l_1\cdots+l_{k-r}=n+1-r \\ l_0,l_1,\ldots,l_{k-r} \geq 0}} \binom{n+k}{l_0} B_{l_0}(x) G_{l_1} \cdots G_{l_{k-r}}, \quad (4.5)$$

where n, k are positive integers with k being an even integer. If we take $k = 2$ in (4.5), in view of $G_0 = 0$ and $G_1 = 1$ (see, e.g., [27]), we get the convolution identity for the classical Genocchi polynomials due to Agoh [1, 20], namely

$$\sum_{k=1}^{n-1} G_k(x) G_{n-k}(x) + \frac{4}{n+2} \sum_{k=0}^{n-2} \binom{n+2}{k} B_k(x) G_{n-k} = 0 \quad (n \geq 2).$$

If we take $k = 2$ in Corollary 4.3, by applying the methods described in [14] to yield (2.18), we obtain another convolution identity for the classical Genocchi polynomials due to Agoh [1, 20], namely

$$\sum_{k=1}^{n-1} \frac{G_k(x) G_{n-k}(x)}{k(n-k)} + \frac{4}{n} \sum_{k=0}^{n-2} \binom{n}{k} \frac{B_k(x) G_{n-k}}{n-k} = 0 \quad (n \geq 2).$$

We next consider the case k being an odd positive integer in (4.1). Obviously, in this case, for non-negative integer n and complex numbers $a_1, \ldots, a_k$,

$$\begin{aligned} &\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{-a_1}{j_1} \cdots \binom{-a_k}{j_k} \left(\prod_{i=1}^{k} \frac{2t_i e^{x_i t_i}}{\lambda_i e^{t_i} + 1} \right) \\ &\quad = \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{-a_1}{j_1} \cdots \binom{-a_k}{j_k} \left(\sum_{r=1}^{k} (-1)^{r-1} \frac{2t_r e^{x_r(t_1+\cdots+t_k)}}{\lambda_1 \cdots \lambda_k e^{t_1+\cdots+t_k} + 1} \right. \qquad (4.6) \\ &\qquad \left. \times \prod_{i=1}^{r-1} \lambda_i \frac{2t_i e^{(x_i - x_r + 1)t_i}}{\lambda_i e^{t_i} + 1} \prod_{i=r+1}^{k} \frac{2t_i e^{(x_i - x_r)t_i}}{\lambda_i e^{t_i} + 1} \right). \end{aligned}$$

Since $\mathcal{G}_0(x;\lambda) = 0$ (see, e.g., [27]), so by (2.9) we have

$$\frac{2e^{x(t_1+\cdots+t_k)}}{\lambda_1 \cdots \lambda_k e^{t_1+\cdots+t_k} + 1} = \sum_{N=0}^{\infty} \frac{\mathcal{G}_{N+1}(x; \lambda_1 \cdots \lambda_k)}{N+1} \sum_{\substack{l_1+\cdots+l_k=N \\ l_1,\ldots,l_k \geq 0}} \frac{t_1^{l_1}}{l_1!} \cdots \frac{t_k^{l_k}}{l_k!}. \quad (4.7)$$

By making the operation $[\frac{t_1^{j_1}}{j_1!} \cdots \frac{t_k^{j_k}}{j_k!}]$ in both sides of (4.6), in view of (1.3) and (4.7), we get

$$\begin{aligned} &\sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{-a_1}{j_1} \cdots \binom{-a_k}{j_k} \mathcal{G}_{j_1}(x_1;\lambda_1) \cdots \mathcal{G}_{j_k}(x_k;\lambda_k) \\ &\quad = \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k \geq 0}} \binom{-a_1}{j_1} \cdots \binom{-a_k}{j_k} \sum_{r=1}^{k} (-1)^{r-1} j_r \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k \geq 0}} \frac{\mathcal{G}_{l_r}(x_r; \lambda_1 \cdots \lambda_k)}{l_r} \qquad (4.8) \\ &\qquad \times \prod_{i=1}^{r-1} \binom{j_i}{l_i} \lambda_i \mathcal{G}_{l_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} \binom{j_i}{l_i} \mathcal{G}_{l_i}(x_i - x_r; \lambda_i). \end{aligned}$$

It is easily seen from (2.13) that for non-negative integers $l_1, \ldots, l_k$ with $l_1 + \cdots + l_k = n$,

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} j_r \binom{-a_r}{j_r} \prod_{\substack{i=1\\ i\neq r}}^{k} \binom{-a_i}{j_i}\binom{j_i}{l_i} = -a_r \prod_{\substack{i=1\\ i\neq r}}^{k} \binom{-a_i}{l_i}\binom{-(a_1+\cdots+a_k)-1-(n-l_r)}{n-1-(n-l_r)}. \tag{4.9}$$

Thus, by applying (4.9) to (4.8), in light of (2.7), we obtain the following result.

Theorem 4.5. *Let $a_1, \ldots, a_k$ be complex numbers with k being an odd positive integer. Then, for non-negative integer n,*

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} \mathcal{G}_{j_1}(x_1;\lambda_1)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k)\\
&= \sum_{r=1}^{k}(-1)^{r-1} \sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}} \binom{n}{l_1,\ldots,l_k} \frac{a_r}{(a_1+\cdots+a_k)_{n+1-l_r}} \mathcal{G}_{l_r}(x_r;\lambda_1\cdots\lambda_k)\\
&\quad\times \prod_{i=1}^{r-1}(a_i)_{l_i}\lambda_i\mathcal{G}_{l_i}(x_i-x_r+1;\lambda_i) \prod_{i=r+1}^{k}(a_i)_{l_i}\mathcal{G}_{l_i}(x_i-x_r;\lambda_i).
\end{aligned}$$

If we apply (4.4) to Theorem 4.5 and then take $x_1 = \cdots = x_k = x$, we get the general convolution identity for the Apostol-Genocchi polynomials, as follows.

Corollary 4.6. *Let $a_1, \ldots, a_k$ be complex numbers with k being an odd positive integer. Then, for non-negative integer n,*

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} \mathcal{G}_{j_1}(x;\lambda_1)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\
&= \sum_{r=1}^{k}\sum_{|J|=r}(-2)^{r-1}\frac{n!\cdot a_J}{(n+1-r)!} \sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}} \binom{n+1-r}{l_0,l_1,\ldots,l_{k-r}} \frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}\\
&\quad\times \mathcal{G}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{G}_{l_1}(\lambda_{i_{r+1}})\cdots\mathcal{G}_{l_{k-r}}(\lambda_{i_k}),
\end{aligned}$$

where $i_{r+1}, \ldots, i_k \in \overline{J}$.

It is clear that the case $\lambda_1 = \cdots = \lambda_k = 1$ in Corollary 4.6 gives the general convolution identity for the classical Genocchi polynomials, as follows.

Corollary 4.7. *Let $a_1, \ldots, a_k$ be complex numbers with k being an odd positive integer. Then, for non-negative integer n,*

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} \binom{n}{j_1,\ldots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} G_{j_1}(x)\cdots G_{j_k}(x)\\
&= \sum_{r=1}^{k}\sum_{|J|=r}(-2)^{r-1}\frac{n!\cdot a_J}{(n+1-r)!} \sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r\\ l_0,l_1,\ldots,l_{k-r}\geq 0}} \binom{n+1-r}{l_0,l_1,\ldots,l_{k-r}} \frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n+1-l_0}}\\
&\quad\times G_{l_0}(x)G_{l_1}\cdots G_{l_{k-r}},
\end{aligned}$$

where $i_{r+1}, \ldots, i_k \in \overline{J}$.

If we take $a_1 = \cdots = a_k = 1$ in Corollary 4.6, we get the following higher-order convolution identity for the Apostol-Genocchi polynomials.

Corollary 4.8. *Let k be an odd positive integer. Then, for non-negative integer n,*

$$\begin{aligned}(n+k) & \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k\geq 0}} \mathcal{G}_{j_1}(x;\lambda_1)\cdots\mathcal{G}_{j_k}(x;\lambda_k) \\ & = \sum_{r=1}^{k}\sum_{|J|=r}(-2)^{r-1}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r \\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k}{l_0}\mathcal{G}_{l_0}(x;\lambda_1\cdots\lambda_k)\mathcal{G}_{l_1}(\lambda_{i_{r+1}})\cdots\mathcal{G}_{l_{k-r}}(\lambda_{i_k}),\end{aligned}$$

where $i_{r+1},\ldots,i_k \in \overline{J}$.

The above Corollaries 4.4 and 4.8 imply the corresponding higher-order convolution identities for the Apostol-Genocchi polynomials described in [19, Theorem 2.3] are only complete on condition that $\lambda_1 = \cdots = \lambda_k$. In particular, if we take $\lambda_1 = \cdots = \lambda_k = 1$ in Corollary 4.8, we get the following higher-order convolution identity for the classical Genocchi polynomials:

$$\begin{aligned}(n+k) & \sum_{\substack{j_1+\cdots+j_k=n \\ j_1,\ldots,j_k\geq 0}} G_{j_1}(x)\cdots G_{j_k}(x) \\ & = \sum_{r=1}^{k}\binom{k}{r}(-2)^{r-1}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n+1-r \\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k}{l_0}G_{l_0}(x)G_{l_1}\cdots G_{l_{k-r}},\end{aligned} \tag{4.10}$$

where n, k are positive integers with k being an odd integer. And the case $k = 3$ in (4.10) gives for positive integer $n \geq 2$,

$$\begin{aligned}\sum_{j_1+j_2+j_3=n} G_{j_1}(x)G_{j_2}(x)G_{j_3}(x) & - \frac{3}{n+3}\sum_{j_1+j_2+j_3=n}\binom{n+3}{j_3}G_{j_1}G_{j_2}G_{j_3}(x) \\ & + \frac{6}{n+3}\sum_{k=0}^{n-1}\binom{n+3}{k}G_k(x)G_{n-1-k} = \frac{4}{n+3}\binom{n+3}{5}G_{n-2}(x),\end{aligned}$$

which is very analogous to the convolution identity on the classical Bernoulli and Euler polynomials presented in [4, Corollaries 1 and 3]. In fact, by using the methods showed in [14], one can derive the similar convolution identity for the classical Genocchi polynomials to Corollary 9 stated in [14]. The details are left as an exercise for the interested readers. It is also informed that the results presented in [35, Corollary 2.3, (2.19)] are only complete on condition that $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1}$, which implies the result stated in [35, (2.20)] is only complete when $\alpha_0 = \alpha_1$.

Acknowledgment

This work was partly done when the first author visited the Department of Mathematics, Tianjin Polytechnic University in February 2016 during the second author appointed as a Chair Professor at Tianjin Polytechnic University by Tianjin city in China from August 2015 to August 2019. The first author is very grateful for Professor Feng Qi's kind hospitality. This work is supported by the Foundation for Fostering Talents in Kunming University of Science and Technology (Grant No. KKSY201307047) and the National Natural Science Foundation of China (Grant No. 11326050).

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