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Best proximity and coupled best proximity results for Suzuki type proximal multivalued mappings

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Abstract

We extend and generalize the best proximity results for Suzuki type α^+-ψ-proximal single valued mappings given by Hussain et al.. Some novel best proximity results and coupled best proximity results are presented for Suzuki type α^+-ψ-proximal multivalued mappings satisfying generalized conditions of existence.

Keywords: Suzuki type α^+-ψ-proximal multivalued mappings, coupled best proximity point, best proximity point.
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1. Introduction and preliminaries

Some problems of fixed points of either single-valued or multivalued mappings involving α-admissible have become a hotspot research since Samet et al. [18] introduced the notion of α-admissible in 2012, for example, following Samet's definition, Latif et al. [11] defined the concept of (α, ψ)-Meir-Keeler self mappings. Redjel et al. [17] introduced a concept of (α, ψ)-Meir-Keeler-Khan mappings, also, the class of (α, ψ)-Meir-Keeler-Khan multivalued mappings has been defined recently [23]. Hussain et al. [7] introduced the concept of proximal α^+-admissible.

Let (X, d) be a metric space and $A, B \subset X$, the following notations will be used in the sequel:

$$\operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}, \quad D(x, B) = \inf\{d(x, y) : y \in B\},$$

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$$A_0 = \{a \in A : d(a,b) = \operatorname{dist}(A,B) \text{ for some } \ b \in B\},$$
$$B_0 = \{b \in B : d(a,b) = \operatorname{dist}(A,B) \text{ for some } a \in A\}.$$

For any two nonempty sets A and B in a metric space (X,d), the point $a \in A$ is called a best proximity point of the mapping $T : A \to B$ if $d(a,Ta) = d(A,B)$. The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors since Fan [6] established a best approximation theorem in 1969. Afterward, many authors [2–5, 8, 9, 12, 13, 16, 19, 20, 22, 24] devote themselves to investigate the best proximity points of mappings in a variety of settings. For example, Hussain et al. [7] introduced new Suzuki type contractions and proved new best proximity results for these contractions in the setting of a metric space. Sintunavarat and Kumam [21] introduced the concept of coupled best proximity point and proved the coupled best proximity theorem for involving cyclic contractions. Recently, Nantadilok [14] established the coupled best proximity point theorems for multivalued mappings via the α-admissible notion and ψ function.

Inspired and motivated by Hussain et al. [7], Sintunavarat et al. [21], and Nantadilok [14], in Section 2, we introduce the new type of one-variable and two-variable multivalued mappings based on Suzuki type contractive condition. Via the admissible mappings, the notions of (α^+, ψ)-proximal multivalued mapping for one-variable and two-variable are presented. The coupled best proximity point results for two-variable (α^+, ψ)-proximal multivalued mappings with continuity or regularity and the best proximity point results for one-variable (α^+, ψ)-proximal multivalued mappings with continuity or regularity in the setting of complete metric spaces are established, respectively. These results extend and generalize the main results of Hussain et al., Nantadilok in the literatures [7, 14, 21, 22]. We also provide an example to show the generality and effectiveness of our results.

As the preliminaries, we review some definitions (see [7, 10, 11, 15] and references therein).

Definition 1.1 ([7])**.** Let T be a self-mapping on a nonempty set X and $\alpha : X \times X \to [-\infty, +\infty)$ be a mapping. The mapping T is said to be proximal α^+-admissible if the following condition holds:

$$\left.\begin{array}{r} \alpha(x,y) \geq 0, \\ d(u_1, Tx_1) = \operatorname{dist}(A,B), \\ d(u_2, Tx_2) = \operatorname{dist}(A,B), \end{array}\right\} \Rightarrow \alpha(u_1,u_2) \geq 0$$

for all $x_1, x_2, u_1, u_2 \in A$.

Let Ω be the family of (c)-comparison functions, a (c)-comparison function ψ be a nondecreasing self-mapping on $[0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n < \infty$ for each $t > 0$, ψ^n is the n-th iteration of ψ. It is clear that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ (see [10, 11]).

Definition 1.2 ([7])**.** Let (X,d) be a metric space. $T : A \to B$ is called a Suzuki type $\alpha^+\psi$-proximal mapping if there exist two functions $\psi \in \Omega$ and $\alpha : X \times X \to [-\infty, +\infty)$ such that for all $x, y \in A$, we have

$$\frac{1}{2} d^*(x,Tx) \leq d(x,Tx) \Rightarrow \alpha(x,y) + d(Tx,Ty) \leq \psi(M(x,y)), \tag{1.1}$$

where

$$d^*(x,Tx) = d(x,Tx) - \operatorname{dist}(A,B),$$

and

$$M(x,y) = \max\left\{ d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2} - \operatorname{dist}(A,B), \frac{d(x,Ty)+d(y,Tx)}{2} - \operatorname{dist}(A,B) \right\}.$$

Definition 1.3 ([15])**.** For nonempty subsets A, B of a metric space (X,d) with $A_0 \neq \emptyset$, we say the pair (A,B) satisfies

(a) the P-property if

$$\left.\begin{array}{r} d(x_1, y_1) = \operatorname{dist}(A, B), \\ d(x_2, y_2) = \operatorname{dist}(A, B), \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2)$$

for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$;

(b) the weak P-property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\left.\begin{array}{r} d(x_1, y_1) = \operatorname{dist}(A, B), \\ d(x_2, y_2) = \operatorname{dist}(A, B), \end{array}\right\} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Hussian et al. [7] established an existence theorem for the best proximity points of Suzuki type $\alpha^+\psi$-proximal mappings with continuity assumption or regularity on the mappings.

Theorem 1.4 ([7])**.** *Suppose A and B are nonempty closed subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Let $T : A \to B$ satisfies* (1.1) *together with the following assertions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;*

(ii) *T is proximal α^+-admissible;*

(iii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x, Tx) = \operatorname{dist}(A, B) \text{ and } \alpha(x_0, x_1) \geq 0;$$

(iv) *T is continuous; or*

(v) *A is α-regular, that is, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 0$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.*

Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = \operatorname{dist}(A, B)$.

2. Coupled best proximity points and best proximity points for Suzuki type α^+-ψ-proximal mappings

In the sequel, $\mathbb{N}$ denotes the set of all nonnegative integers, $\operatorname{Bpp}(T)$ denotes the set of best proximity points of T, $\operatorname{CBpp}(T)$ denotes the set of coupled best proximity points of T, and $\mathcal{CL}(X)$ denotes the family of nonempty closed subsets of X.

For any $A, B \in \mathcal{CL}(X)$, let the mapping $H(\cdot, \cdot)$ be the generalized Hausdorff distance with respect to d defined by

$$H(A, B) = \begin{cases} \max\left\{\sup\limits_{x \in A} \operatorname{dist}(x, B), \sup\limits_{y \in B} \operatorname{dist}(y, A)\right\}, & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Before stating the results, we need to present some definitions.

Definition 2.1 ([1, 14])**.** An element $x^* \in A$ is said to be the best proximity point of a multivalued non-self mapping $T : A \to 2^B\backslash\{\emptyset\}$ if $D(x^*, Tx^*) = \operatorname{dist}(A, B)$.

Definition 2.2. Let (X, d) be a metric space and $A, B \in \mathcal{CL}(X)$. A multivalued mapping $T : A \to 2^B\backslash\{\emptyset\}$ is called *proximal α^+-admissible* if there exists a mapping $\alpha : A \times A \to [-\infty, +\infty)$ such that for any $x_1, x_2, u_1, u_2 \in A$ and $y_1 \in Tx_1, y_2 \in Tx_2$,

$$\left.\begin{array}{r} \alpha(x_1, x_2) \geq 0, \\ d(u_1, y_1) = \operatorname{dist}(A, B), \\ d(u_2, y_2) = \operatorname{dist}(A, B), \end{array}\right\} \Rightarrow \alpha(u_1, u_2) \geq 0.$$

Definition 2.3. Let (X,d) be a metric space and $A,B \in \mathcal{CL}(X)$. A multivalued mapping $T : A \to \mathcal{CL}(B)$ is called a *one-variable Suzuki type α^+-ψ-proximal multivalued mapping* if there exist two functions $\psi \in \Omega$ and $\alpha : X \times X \to [-\infty, +\infty)$ such that for all $x, y \in A$,

$$\frac{1}{2}D^*(x,Tx) \le d(x,y) \Rightarrow \alpha(x,y) + H(Tx,Ty) \le \psi(M(x,y)),$$

where

$$D^*(x,Tx) = D(x,Tx) - \operatorname{dist}(A,B),$$

and

$$M(x,y) = \max\left\{d(x,y), \frac{D(x,Tx)+D(y,Ty)}{2} - \operatorname{dist}(A,B), \frac{D(x,Ty)+D(y,Tx)}{2} - \operatorname{dist}(A,B)\right\}.$$

Definition 2.4. Let (X,d) be a metric space and $A,B \in \mathcal{CL}(X)$. A mapping $T : A \times A \to 2^B \backslash \{\emptyset\}$ is called *proximal α^+-admissible* if there exists a mapping $\alpha : A \times A \to [-\infty, +\infty)$ such that for any $x_1, x_2, w_1, w_2, w_1', w_2', y_1, y_2 \in A$ and $u_1 \in T(x_1,y_1), u_2 \in T(x_2,y_2)$, $v_1 \in T(y_1,x_1), v_2 \in T(y_2,x_2)$,

$$\left.\begin{aligned} \alpha(x_1,x_2) &\ge 0, \\ d(w_1,u_1) &= \operatorname{dist}(A,B), \\ d(w_2,u_2) &= \operatorname{dist}(A,B), \end{aligned}\right\} \Rightarrow \alpha(w_1,w_2) \ge 0,$$

and

$$\left.\begin{aligned} \alpha(y_1,y_2) &\ge 0, \\ d(w_1',v_1) &= \operatorname{dist}(A,B), \\ d(w_2',v_2) &= \operatorname{dist}(A,B), \end{aligned}\right\} \Rightarrow \alpha(w_1',w_2') \ge 0.$$

Definition 2.5 ([14])**.** Let (X,d) be a complete metric space and $A,B \in \mathcal{CL}(X)$. An element $(x^*,y^*) \in (A \times A)$ is said to be the *coupled best proximity point* of a multivalued mapping $T : A \times A \to \mathcal{CL}(B)$ if $D(x^*, T(x^*,y^*)) = \operatorname{dist}(A,B)$ and $D(y^*, T(y^*,x^*)) = \operatorname{dist}(A,B)$.

Next, we introduce the class of Suzuki type α^+-ψ-proximal multivalued mappings and then study the existence of coupled best proximity points for such mappings via the α^+-admissibility.

Definition 2.6. Let (X,d) be a metric space and $A,B \in \mathcal{CL}(X)$. A mapping $T : A \times A \to \mathcal{CL}(B)$ is called a *two-variable Suzuki type α^+-ψ-proximal multivalued mapping* if there exist two functions $\psi \in \Omega$ and $\alpha : X \times X \to [-\infty, +\infty)$ such that for all $x, y, x', y' \in A$,

$$\left.\begin{aligned} &\frac{1}{2}D^*(x,T(x,x')) \le d(x,y), \\ &\frac{1}{2}D^*(x',T(x',x)) \le d(x',y'), \\ &d(x,y) = d(x',y') = 0, \text{ or} \\ &d(x,y) > 0, d(x',y') > 0, \end{aligned}\right\} \Rightarrow \alpha(x,y) + H(T(x,x'),T(y,y')) \le \psi(M(x,y,x',y')), \tag{2.1}$$

where

$$D^*(x,T(x,x')) = D(x,T(x,x')) - \operatorname{dist}(A,B), \tag{2.2}$$

and

$$\begin{aligned} M(x,y,x',y') = \max\Bigg\{&d(x,y), \frac{D(x,T(x,x'))+D(y,T(y,y'))}{2} - \operatorname{dist}(A,B), \\ &\frac{D(y,T(x,x'))+D(x,T(y,y'))}{2} - \operatorname{dist}(A,B)\Bigg\}. \end{aligned}$$

Remark 2.7. It is worth noting in Definition 2.6 that if

$$\alpha(x,y) + H(T(x,x'),T(y,y')) \le \psi(M(x,y,x',y'))$$

holds, then from the symmetries of x and x', y and y', obviously,

$$\alpha(x',y')+H(T(x',x),T(y',y))\leq\psi(M(x',y',x,y))$$

is true.

Lemma 2.8 ([14]). *Let B be nonempty closed subsets of a metric space (X,d). Then, for each $x\in X$ with $D(x,B)>0$ and $q>1$, there exists an element $b\in B$ such that*

$$d(x,b)<qD(x,B).$$

First, we state an existence theorem for the coupled best proximity points of two-variable Suzuki type α^+-ψ- proximal multivalued mappings.

Theorem 2.9. *Let (X,d) be a complete metric space and $A,B\in\mathcal{CL}(X)$ with $A_0\neq\emptyset$, $\psi\in\Omega$ is strictly increasing and $T:A\times A\to\mathcal{CL}(X)$ is a two-variable Suzuki type α^+-ψ- proximal multivalued mapping. Suppose that the following conditions hold:*

(i) *$T(x,y)\subseteq B_0$ for $(x,y)\in A_0\times A_0$ and (A,B) satisfies the weak P-property;*

(ii) *T is α^+-proximal admissible;*

(iii) *there exist elements $(x_0,y_0),(x_1,y_1)\in(A_0\times A_0)$ and $u_1\in T(x_0,y_0),v_1\in T(y_0,x_0)$ such that*

$$d(x_1,u_1)=\operatorname{dist}(A,B),\alpha(x_0,x_1)\geq 0,$$
$$d(y_1,v_1)=\operatorname{dist}(A,B),\alpha(y_0,y_1)\geq 0;$$

(iv) *if $D(x,T(x,y))=0$ or $D(y,T(y,x))=0$ for any $x,y\in A_0$, then $D(x,T(x,y))=D(y,T(y,x))=0$;*

(v) *T is continuous, or*

(vi) *if $\{x_n\}_{n=0}^{\infty}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\geq 0$ and $x_n\to x^*\in A$ as $n\to\infty$, then $\alpha(x_n,x^*)\geq 0$ for all $n\in\mathbb{N}$.*

Then there exists $(x^*,y^*)\in(A_0\times A_0)$ such that

$$D(x^*,T(x^*,y^*))=D(y^*,T(y^*,x^*))=\operatorname{dist}(A,B).$$

Proof. By condition (iii), there exist elements $(x_0,y_0),(x_1,y_1)\in(A_0\times A_0)$ and $u_1\in T(x_0,y_0),v_1\in T(y_0,x_0)$ such that

$$\begin{aligned}d(x_1,u_1)&=\operatorname{dist}(A,B),\alpha(x_0,x_1)\geq 0,\\ d(y_1,v_1)&=\operatorname{dist}(A,B),\alpha(y_0,y_1)\geq 0.\end{aligned}\tag{2.3}$$

We consider the following four cases:

(a) $u_1\in T(x_1,y_1),v_1\notin T(y_1,x_1)$;

(b) $u_1\notin T(x_1,y_1),v_1\in T(y_1,x_1)$;

(c) $u_1\in T(x_1,y_1),v_1\in T(y_1,x_1)$;

(d) $u_1\notin T(x_1,y_1),v_1\notin T(y_1,x_1)$.

Case (a): Since $u_1\in A_0,T(x_1,y_1)\subseteq B_0$, then

$$0=D(u_1,T(x_1,y_1))\geq\operatorname{dist}(A_0,B_0)\geq\operatorname{dist}(A,B)\geq 0,$$

hence,

$$D(u_1,T(x_1,y_1))=\operatorname{dist}(A,B)=0.$$

From (2.3), we get

$$d(x_1, u_1) = 0, \qquad d(y_1, v_1) = 0,$$

that is, $x_1 = u_1, y_1 = v_1$, hence, applying (iv), we have

$$\begin{aligned} D(x_1, T(x_1, y_1)) &= D(u_1, T(x_1, y_1)) = \operatorname{dist}(A, B) = 0, \\ D(y_1, T(y_1, x_1)) &= D(v_1, T(y_1, x_1)) = \operatorname{dist}(A, B) = 0. \end{aligned}$$

On the other hand, since $v_1 \notin T(y_1, x_1)$ and $T(y_1, x_1) \in \mathcal{CL}(B)$, therefore

$$D(y_1, T(y_1, x_1)) = D(v_1, T(y_1, x_1)) > 0,$$

which contradicts to $D(y_1, T(y_1, x_1)) = 0$, thus, Case (a) is not true.

Similarly, Case (b) is not true, too.

Case (c)**:** When $u_1 \in T(x_1, y_1), v_1 \in T(y_1, x_1)$, as proved above, we can get that

$$\begin{aligned} D(x_1, T(x_1, y_1)) &= D(u_1, T(x_1, y_1)) = \operatorname{dist}(A, B) = 0, \\ D(y_1, T(x_1, y_1)) &= D(v_1, T(y_1, x_1)) = \operatorname{dist}(A, B) = 0, \end{aligned}$$

which imply that (x_1, y_1) is the coupled best proximity point. So we only consider the following case.

Case (d)**:** Let $u_1 \notin T(x_1, y_1), v_1 \notin T(y_1, x_1)$. Because $T(x_1, y_1)$ and $T(y_1, x_1)$ are closed in B, therefore

$$D(u_1, T(x_1, y_1)) > 0, \qquad D(v_1, T(y_1, x_1)) > 0.$$

Since $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$, it is obvious that

$$0 < D(u_1, T(x_1, y_1)) \le H(T(x_0, y_0), T(x_1, y_1)), \tag{2.4}$$

and

$$0 < D(v_1, T(y_1, x_1)) \le H(T(y_0, x_0), T(y_1, x_1)). \tag{2.5}$$

Applying Lemma 2.8, we obtain that for $q_0 > 1, q_0' > 1$, there exist $u_2 \in T(x_1, y_1), v_2 \in T(y_1, x_1)$ such that

$$0 < d(u_1, u_2) < q_0 D(u_1, T(x_1, y_1)), \qquad 0 < d(v_1, v_2) < q_0' D(v_1, T(y_1, x_1)). \tag{2.6}$$

On the other hand, as $u_2 \in T(x_1, y_1) \subseteq B_0, v_2 \in T(y_1, x_1) \subseteq B_0$, there exist $x_2 \ne x_1, y_2 \ne y_1 \in A_0$, for otherwise (x_1, y_1) is the coupled best proximity point, such that

$$d(x_2, u_2) = d(y_2, v_2) = \operatorname{dist}(A, B). \tag{2.7}$$

Since T is an α^+-proximal admissible, $u_2 \in T(x_1, y_1), v_2 \in T(y_1, x_1)$ and $\alpha(x_0, x_1) \ge 0$, $\alpha(y_0, y_1) \ge 0$, and using (2.3), we obtain $\alpha(x_1, x_2) \ge 0$ and $\alpha(y_1, y_2) \ge 0$, that is,

$$\begin{gathered} d(x_2, u_2) = \operatorname{dist}(A, B), \alpha(x_1, x_2) \ge 0, \\ d(y_2, v_2) = \operatorname{dist}(A, B), \alpha(y_1, y_2) \ge 0. \end{gathered} \tag{2.8}$$

Because (A, B) satisfies the weak P-property and in combination with (2.3), (2.7), we have

$$d(x_1, x_2) \le d(u_1, u_2), \qquad d(y_1, y_2) \le d(v_1, v_2). \tag{2.9}$$

From (2.4), (2.5), (2.6), and (2.9), we derive

$$\begin{aligned} d(x_1, x_2) \le d(u_1, u_2) < q_0 D(u_1, T(x_1, y_1)) \le q_0 H(T(x_0, y_0), T(x_1, y_1)), \\ d(y_1, y_2) \le d(v_1, v_2) < q_0' D(v_1, T(y_1, x_1)) \le q_0' H(T(y_0, x_0), T(y_1, x_1)). \end{aligned}$$

Likewise, assume that $u_2 \notin T(x_2, y_2), v_2 \notin T(y_2, x_2)$; for otherwise, condition (iv) is not true or (x_2, y_2) is the coupled best proximity point. Because $T(x_2, y_2)$ and $T(y_2, x_2)$ are closed in B, therefore

$$D(u_2, T(x_2, y_2)) > 0, \qquad D(v_2, T(y_2, v_2)) > 0.$$

Thus, by $u_2 \in (x_1, y_1), v_2 \in (y_1, x_1)$, we have

$$\begin{aligned} 0 < D(u_2, T(x_2, y_2)) \leq H(T(x_1, y_1), T(x_2, y_2)),\\ 0 < D(v_2, T(y_2, x_2)) \leq H(T(y_1, x_1), T(y_2, x_2)). \end{aligned} \tag{2.10}$$

Applying Lemma 2.8, we obtain that for $q_1 > 1, q_1' > 1$ and there exist $u_3 \in T(x_2, y_2), v_3 \in T(y_2, x_2)$ such that

$$0 < d(u_2, u_3) < q_1 D(u_2, T(x_2, y_2)),\ 0 < d(v_2, v_3) < q_1' D(v_2, T(y_2, x_2)). \tag{2.11}$$

On the other hand, as $u_3 \in T(x_2, y_2) \subseteq B_0, v_3 \in T(y_2, x_2) \subseteq B_0$, there exist $x_3 \neq x_2, y_3 \neq y_2 \in A_0$. For otherwise (x_2, y_2) is the coupled best proximity point, such that

$$d(x_3, u_3) = d(y_3, v_3) = \operatorname{dist}(A, B).$$

Again, since T is an α^+-proximal admissible, $u_3 \in T(x_2, y_2), v_3 \in T(y_2, x_2)$ and $\alpha(x_1, x_2) \geq 0$ and $\alpha(y_1, y_2) \geq 0$, we obtain $\alpha(x_2, x_3) \geq 0$ and $\alpha(y_2, y_3) \geq 0$, that is,

$$\begin{aligned} d(x_3, u_3) = \operatorname{dist}(A, B), \alpha(x_2, x_3) \geq 0,\\ d(y_3, v_3) = \operatorname{dist}(A, B), \alpha(y_2, y_3) \geq 0. \end{aligned} \tag{2.12}$$

Because (A, B) satisfies the weak P-property and in combination with (2.8), (2.12), we have

$$d(x_2, x_3) \leq d(u_2, u_3), \qquad d(y_2, y_3) \leq d(v_2, v_3). \tag{2.13}$$

From (2.10), (2.11), and (2.13), we have

$$\begin{aligned} d(x_2, x_3) \leq d(u_2, u_3) < q_1 D(u_2, T(x_2, y_2)) \leq q_1 H(T(x_1, y_1), T(x_2, y_2)),\\ d(y_2, y_3) \leq d(v_2, v_3) < q_1' D(v_2, T(y_2, x_2)) \leq q_1' H(T(y_1, x_1), T(y_2, x_2)) \end{aligned}$$

for all $n \in \mathbb{N}\backslash\{0\}$. Inductively, we can obtain sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} \subseteq A_0$ and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty} \subseteq B_0$ satisfying $D(u_n, T(x_n, y_n)) > 0$, $D(v_n, T(y_n, x_n)) > 0$, $u_{n+1} \in T(x_n, y_n)$ and $v_{n+1} \in T(y_n, x_n)$, for $n \in \mathbb{N}$ such that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) = \operatorname{dist}(A, B), \alpha(x_n, x_{n+1}) \geq 0,\\ d(y_{n+1}, v_{n+1}) = \operatorname{dist}(A, B), \alpha(y_n, y_{n+1}) \geq 0 \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$, and

$$\begin{aligned} d(x_n, x_{n+1}) \leq d(u_n, u_{n+1}),\\ d(y_n, y_{n+1}) \leq d(v_n, v_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N}$, and

$$\begin{aligned} d(x_n, x_{n+1}) \leq d(u_n, u_{n+1}) < q_{n-1} D(u_n, T(x_n, y_n)) \leq q_{n-1} H(T(x_{n-1}, y_{n-1}), T(x_n, y_n)),\\ d(y_n, y_{n+1}) \leq d(v_n, v_{n+1}) < q_{n-1}' D(v_n, T(y_n, x_n)) \leq q_{n-1}' H(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \end{aligned} \tag{2.15}$$

for all $n \in \mathbb{N}\backslash\{0\}$. Since $x_n \in A_0, T(x_{n-1}, y_{n-1}) \subseteq B_0$ and $A_0 \subseteq A, B_0 \subseteq B$, from the definition of D and (2.14), we have

$$\begin{aligned} \operatorname{dist}(A, B) = d(x_n, u_n) \geq D(x_n, T(x_{n-1}, y_{n-1})) \geq \operatorname{dist}(A_0, B_0) \geq \operatorname{dist}(A, B),\\ \operatorname{dist}(A, B) = d(y_n, v_n) \geq D(y_n, T(y_{n-1}, x_{n-1})) \geq \operatorname{dist}(A_0, B_0) \geq \operatorname{dist}(A, B) \end{aligned}$$

for all $n \in \mathbb{N}\backslash\{0\}$, hence,

$$\begin{aligned} d(x_n, u_n) &= D(x_n, T(x_{n-1}, y_{n-1})) = \operatorname{dist}(A, B), \\ d(y_n, v_n) &= D(y_n, T(y_{n-1}, x_{n-1})) = \operatorname{dist}(A, B) \end{aligned} \tag{2.16}$$

for all $n \in \mathbb{N}\backslash\{0\}$. In addition, we deduce that

$$\begin{aligned} \frac{1}{2} D^*(x_{n-1}, T(x_{n-1}, y_{n-1})) &= \frac{1}{2}\big[D(x_{n-1}, T(x_{n-1}, y_{n-1})) - \operatorname{dist}(A, B)\big] \\ &\le \frac{1}{2}\big[d(x_{n-1}, x_n) + D(x_n, T(x_{n-1}, y_{n-1})) - \operatorname{dist}(A, B)\big] \\ &= \frac{1}{2} d(x_{n-1}, x_n) \\ &\le d(x_{n-1}, x_n), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \frac{1}{2} D^*(y_{n-1}, T(y_{n-1}, x_{n-1})) &= \frac{1}{2}\big[D(y_{n-1}, T(y_{n-1}, x_{n-1})) - \operatorname{dist}(A, B)\big] \\ &\le \frac{1}{2}\big[d(y_{n-1}, y_n) + D(y_n, T(y_{n-1}, x_{n-1})) - \operatorname{dist}(A, B)\big] \\ &= \frac{1}{2} d(y_{n-1}, y_n) \\ &\le d(y_{n-1}, y_n) \end{aligned} \tag{2.18}$$

for all $n \in \mathbb{N}\backslash\{0\}$. If for some $n_0 \in \mathbb{N}$, $d(x_{n_0-1}, x_{n_0}) = 0$ and $d(y_{n_0-1}, y_{n_0}) = 0$, then $x_{n_0-1} = x_{n_0}$, $y_{n_0-1} = y_{n_0}$, thus, from (2.16), we obtain

$$\begin{aligned} D(x_{n_0}, T(x_{n_0}, y_{n_0})) &= D(x_{n_0}, T(x_{n_0-1}, y_{n_0-1})) = \operatorname{dist}(A, B), \\ D(y_{n_0}, T(y_{n_0}, x_{n_0})) &= D(y_{n_0}, T(y_{n_0-1}, x_{n_0-1})) = \operatorname{dist}(A, B), \end{aligned}$$

that is, (x_{n_0}, y_{n_0}) is a coupled best proximity point. So, we can suppose that

$$d(x_{n-1}, x_n) > 0, \qquad d(y_{n-1}, y_n) > 0$$

for all $n \in \mathbb{N}\backslash\{0\}$. Since $T : A \times A \to \mathcal{CL}(X)$ is a two-variable Suzuki type α^+-ψ-proximal multivalued mapping, then inequalities (2.17), (2.18) imply that

$$\begin{aligned} H(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) &\le \alpha(x_{n-1}, x_n) + H(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\le \psi(M(x_{n-1}, x_n, y_{n-1}, y_n)), \\ H(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) &\le \alpha(y_{n-1}, y_n) + H(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\ &\le \psi(M(y_{n-1}, y_n, x_{n-1}, x_n)). \end{aligned} \tag{2.19}$$

In combination with inequalities (2.15) and (2.19), we obtain that for $q_{n-1} > 1, q'_{n-1} > 1$,

$$0 < d(x_n, x_{n+1}) \le d(u_n, u_{n+1}) < q_{n-1} D(u_n, T(x_n, y_n)) \le q_{n-1}\psi(M(x_{n-1}, x_n, y_{n-1}, y_n)),$$

$$0 < d(y_n, y_{n+1}) \le d(v_n, v_{n+1}) < q'_{n-1} D(v_n, T(y_n, x_n)) \le q'_{n-1}\psi(M(y_{n-1}, y_n, x_{n-1}, x_n))$$

for all $n \in \mathbb{N}\backslash\{0\}$. We check

$$M(x_{n-1}, x_n, y_{n-1}, y_n) = \max\Big\{d(x_{n-1}, x_n), \frac{D(x_{n-1}, T(x_{n-1}, y_{n-1})) + D(x_n, T(x_n, y_n))}{2}$$

$$
\begin{aligned}
&- \operatorname{dist}(A,B), \frac{D(x_n, T(x_{n-1}, y_{n-1})) + D(x_{n-1}, T(x_n, y_n))}{2} - \operatorname{dist}(A,B)\Big\} \\
&\leq \max\Big\{ d(x_{n-1}, x_n), \frac{1}{2}\big[d(x_{n-1}, x_n) + D(x_n, T(x_{n-1}, y_{n-1}))\big] \\
&\quad + \frac{1}{2}\big[d(x_n, x_{n+1}) + D(x_{n+1}, T(x_n, y_n))\big] - \operatorname{dist}(A,B), \qquad (2.20)\\
&\quad \frac{1}{2}\big[\operatorname{dist}(A,B) + d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + D(x_{n+1}, T(x_n, y_n))\big] - \operatorname{dist}(A,B)\Big\} \\
&\leq \max\Big\{ d(x_{n-1}, x_n), \frac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big], \frac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big]\Big\} \\
&\leq \max\Big\{ d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}
\end{aligned}
$$

for all $n \in \mathbb{N}\backslash\{0\}$. Similarly, we get

$$M(y_{n-1}, y_n, x_{n-1}, x_n) \leq \max\big\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\big\}. \tag{2.21}$$

Assume that

$$
\begin{aligned}
\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} = d(x_n, x_{n+1}),\\
\max\big\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\big\} = d(y_n, y_{n+1}),
\end{aligned}
$$

since $\psi(t) < t$ for $t > 0$, then from (2.15), (2.19), (2.20), and (2.21), we deduce that for $q_{n-1} > 1, q'_{n-1} > 1$,

$$
\begin{aligned}
0 < d(x_n, x_{n+1}) \leq d(u_n, u_{n+1}) < q_{n-1}\psi(d(x_n, x_{n+1})) < q_{n-1} d(x_n, x_{n+1}),\\
0 < d(y_n, y_{n+1}) \leq d(v_n, v_{n+1}) < q'_{n-1}\psi(d(y_n, y_{n+1})) < q'_{n-1} d(y_n, y_{n+1}),
\end{aligned}
$$

which is a contradiction. Thus

$$
\begin{aligned}
\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} = d(x_{n-1}, x_n),\\
\max\big\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\big\} = d(y_{n-1}, y_n)
\end{aligned}
\tag{2.22}
$$

for all $n \in \mathbb{N}\backslash\{0\}$, hence,

$$
\begin{aligned}
0 < d(x_n, x_{n+1}) \leq d(u_n, u_{n+1}) < q_{n-1}\psi(d(x_{n-1}, x_n)) < q_{n-1} d(x_{n-1}, x_n),\\
0 < d(y_n, y_{n+1}) \leq d(v_n, v_{n+1}) < q'_{n-1}\psi(d(y_{n-1}, y_n)) < q'_{n-1} d(y_{n-1}, y_n)
\end{aligned}
\tag{2.23}
$$

for all $n \in \mathbb{N}\backslash\{0\}$. Since ψ is strictly increasing, we have

$$0 < \psi(d(x_n, x_{n+1})) \leq \psi(d(u_n, u_{n+1})) < \psi(q_{n-1}\psi(d(x_{n-1}, x_n))),$$

and

$$0 < \psi(d(y_n, y_{n+1})) \leq \psi(d(v_n, v_{n+1})) < \psi(q'_{n-1}\psi(d(y_{n-1}, y_n)))$$

for all $n \in \mathbb{N}\backslash\{0\}$, thus,

$$\frac{\psi\big(q_{n-1}\psi(d(x_{n-1}, x_n))\big)}{\psi(d(x_n, x_{n+1}))} > 1, \qquad \frac{\psi\big(q'_{n-1}\psi(d(y_{n-1}, y_n))\big)}{\psi(d(y_n, y_{n+1}))} > 1.$$

Set

$$q_n = \frac{\psi\big(q_{n-1}\psi(d(x_{n-1}, x_n))\big)}{\psi(d(x_n, x_{n+1}))},\ q'_n = \frac{\psi\big(q'_{n-1}\psi(d(y_{n-1}, y_n))\big)}{\psi(d(y_n, y_{n+1}))},$$

then

$$
\begin{aligned}
q_n\psi(d(x_n, x_{n+1})) = \psi\big(q_{n-1}\psi(d(x_{n-1}, x_n))\big),\\
q'_n\psi(d(y_n, y_{n+1})) = \psi\big(q'_{n-1}\psi(d(y_{n-1}, y_n))\big)
\end{aligned}
\tag{2.24}
$$

for all $n \in \mathbb{N}\backslash\{0\}$. Iterating (2.24) and combining (2.23), we get

$$d(x_{n+1},x_{n+2}) \le d(u_{n+1},u_{n+2}) < \psi^n(q_0\psi(d(x_0,x_1))),$$
$$d(y_{n+1},y_{n+2}) \le d(v_{n+1},v_{n+2}) < \psi^n(q_0'\psi(d(y_0,y_1)))$$

for all $n \in \mathbb{N}$. Now, we prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Regarding the properties of the function ψ, for any $\epsilon > 0$ there exists $n(\epsilon)$ such that

$$\sum_{k \geqslant n(\epsilon)}^{n-1} \psi^k\big(q_0\psi(d(x_0,x_1))\big) < \epsilon.$$

Let $n > m > n(\epsilon)$. Applying the triangle inequality repeatedly, we get

$$\begin{aligned} d(x_m,x_n) &\le \sum_{k=m}^{n-1} d(x_k,x_{k+1}) \\ &\le \sum_{k=m}^{n-1} \psi^k\big(q_0\psi(d(x_0,x_1))\big) \\ &\le \sum_{k \geqslant n(\epsilon)}^{n-1} \psi^k\big(q_0\psi(d(x_0,x_1))\big) \\ &< \epsilon. \end{aligned}$$

Hence, we deduce that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the complete metric space (X,d). Similarly, we can also deduce that $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, and $\{v_n\}_{n=0}^{\infty}$ are Cauchy sequence in (X,d). Since A and B are closed subsets of complete metric space (X,d), thus, there exists $(x^*,y^*) \in A \times A$ such that $x_k \xrightarrow{d} x^*$ as $k \to \infty$ and $y_k \xrightarrow{d} y^*$ as $k \to \infty$. Likewise, there exists $(u^*,v^*) \in A \times A$ such that $u_k \xrightarrow{d} u^*$ as $k \to \infty$ and $v_k \xrightarrow{d} v^*$ as $k \to \infty$.

If (v) holds, then from (2.16), noting that $u_n \in T(x_{n-1},x_n)$ and $v_n \in T(y_{n-1},y_n)$, for $n \in \mathbb{N}\backslash\{0\}$, it is easy to derive that

$$d(x^*,u^*) = D(x^*,T(x^*,y^*)) = d(y^*,v^*) = D(y^*,T(y^*,x^*)) = \mathrm{dist}(A,B).$$

If (vi) holds, then $\alpha(x_n,x^*) \ge 0$ and we conclude that

$$\frac{1}{2}D^*(x_n,T(x_n,y_n)) \le d(x_n,x^*), \quad \frac{1}{2}D^*(y_n,T(y_n,x_n)) \le d(y_n,y^*), \tag{2.25}$$

or

$$\frac{1}{2}D^*(x_{n+1},T(x_{n+1},y_{n+1})) \le d(x_{n+1},x^*), \quad \frac{1}{2}D^*(y_{n+1},T(y_{n+1},x_{n+1})) \le d(y_{n+1},y^*) \tag{2.26}$$

hold for all $n \in \mathbb{N}$. In fact, assume that

$$\frac{1}{2}D^*(x_n,T(x_n,y_n)) > d(x_n,x^*), \quad \frac{1}{2}D^*(y_n,T(y_n,x_n)) > d(y_n,y^*),$$

and

$$\frac{1}{2}D^*(x_{n+1},T(x_{n+1},y_{n+1})) > d(x_{n+1},x^*), \quad \frac{1}{2}D^*(y_{n+1},T(y_{n+1},x_{n+1})) > d(y_{n+1},y^*),$$

are true for some $n \in \mathbb{N}$, then by using (2.2), (2.16) and (2.22), we derive the following contradictive inequalities

$$\begin{aligned} d(x_n,x_{n+1}) &\le d(x_n,x^*) + d(x_{n+1},x^*) \\ &< \frac{1}{2}\big[D^*(x_n,T(x_n,y_n)) + D^*(x_{n+1},T(x_{n+1},y_{n+1}))\big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big[D(x_n, T(x_n, y_n)) + D(x_{n+1}, T(x_{n+1}, y_{n+1})) - 2\mathrm{dist}(A, B)\big] \\
&= \frac{1}{2}\big[d(x_n, x_{n+1}) + D(x_{n+1}, T(x_n, y_n)) + d(x_{n+1}, x_{n+2}) \\
&\quad + D(x_{n+2}, T(x_{n+1}, y_{n+1})) - 2\mathrm{dist}(A, B)\big] \\
&\leq \frac{1}{2}\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big] \\
&< d(x_n, x_{n+1}),
\end{aligned}
$$

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leq d(y_n, y^*) + d(y_{n+1}, x^*) \\
&< \frac{1}{2}\big[D^*(y_n, T(y_n, x_n)) + D^*(y_{n+1}, T(y_{n+1}, x_{n+1}))\big] \\
&= \frac{1}{2}\big[D(y_n, T(y_n, x_n)) + D(y_{n+1}, T(y_{n+1}, x_{n+1})) - 2\mathrm{dist}(A, B)\big] \\
&= \frac{1}{2}\big[d(y_n, y_{n+1}) + D(y_{n+1}, T(y_n, x_n)) + d(y_{n+1}, y_{n+2}) \\
&\quad + D(y_{n+2}, T(y_{n+1}, x_{n+1})) - 2\mathrm{dist}(A, B)\big] \\
&\leq \frac{1}{2}\big[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})\big] \\
&< d(y_n, y_{n+1}),
\end{aligned}
$$

hence, either (2.25) or (2.26) holds. Notice that $\{x_{n+1}\}_{n=0}^{\infty}$ is a subsequence of $\{x_n\}_{n=0}^{\infty}$, consequently, we can verify that there exists at least a subsequence $\{x_{n_k}\}$ of $\{x_n\}_{n=0}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$
\frac{1}{2}D^*(x_{n_k}, T(x_{n_k}, y_{n_k})) \leq d(x_{n_k}, x^*), \quad \frac{1}{2}D^*(y_{n_k}, T(y_{n_k}, x_{n_k})) \leq d(y_{n_k}, y^*),
$$

and $\alpha(x_{n_k}, x^*) \geq 0, \alpha(y_{n_k}, y^*) \geq 0$ hold. From (2.1), we obtain

$$
\begin{aligned}
H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) &\leq \alpha(x_{n_k}, x^*) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) \\
&\leq \psi(M(x_{n_k}, x^*, y_{n_k}, y^*)), \\
H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) &\leq \alpha(y_{n_k}, y^*) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) \\
&\leq \psi(M(y_{n_k}, y^*, x_{n_k}, x^*)).
\end{aligned} \tag{2.27}
$$

Moreover,

$$
\begin{aligned}
M(x_{n_k}, x^*, y_{n_k}, y^*) &= \max\Big\{d(x_{n_k}, x^*), \frac{D(x_{n_k}, T(x_{n_k}, y_{n_k})) + D(x^*, T(x^*, y^*))}{2} \\
&\quad - \mathrm{dist}(A, B), \frac{D(x^*, T(x_{n_k}, y_{n_k})) + D(x_{n_k}, T(x^*, y^*))}{2} - \mathrm{dist}(A, B)\Big\} \\
&\leq \max\Big\{d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + D(x_{n_k+1}, T(x_{n_k}, y_{n_k})) + D(x^*, T(x^*, y^*))}{2} \\
&\quad - \mathrm{dist}(A, B), \frac{d(x^*, x_{n_k+1}) + D(x_{n_k+1}, T(x_{n_k}, y_{n_k}))}{2} \\
&\quad + \frac{D(x_{n_k}, x^*) + D(x^*, T(x^*, y^*))}{2} - \mathrm{dist}(A, B)\Big\} \\
&\leq \max\Big\{d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + \mathrm{dist}(A, B) + D(x^*, T(x^*, y^*))}{2} - \mathrm{dist}(A, B), \\
&\quad \frac{d(x^*, x_{n_k+1}) + \mathrm{dist}(A, B) + D(x_{n_k}, x^*) + D(x^*, T(x^*, y^*))}{2} - \mathrm{dist}(A, B)\Big\},
\end{aligned}
$$

which implies that

$$\lim_{k\to\infty} M(x_{n_k}, x^*, y_{n_k}, y^*) \le \frac{D(x^*, T(x^*, y^*)) - \mathrm{dist}(A,B)}{2}.$$

Similarly, we get

$$\lim_{k\to\infty} M(y_{n_k}, y^*, x_{n_k}, x^*) \le \frac{D(y^*, T(y^*, x^*)) - \mathrm{dist}(A,B)}{2}.$$

Additionally, we have

$$\begin{aligned} D(x^*, T(x^*, y^*)) &\le d(x^*, x_{n_k}) + d(x_{n_k}, u_{n_k}) + D(u_{n_k}, T(x^*, y^*)) \\ &\le d(x^*, x_{n_k}) + d(x_{n_k}, u_{n_k}) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) \\ &\le d(x^*, x_{n_k}) + \mathrm{dist}(A,B) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)), \end{aligned}$$

and

$$\begin{aligned} D(y^*, T(y^*, x^*)) &\le d(y^*, y_{n_k}) + d(y_{n_k}, v_{n_k}) + D(v_{n_k}, T(y^*, x^*)) \\ &\le d(y^*, y_{n_k}) + d(y_{n_k}, v_{n_k}) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) \\ &\le d(y^*, y_{n_k}) + \mathrm{dist}(A,B) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)), \end{aligned}$$

which implies

$$D(x^*, T(x^*, y^*)) - \mathrm{dist}(A,B) \le d(x^*, x_{n_k}) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)),$$

and

$$D(y^*, T(y^*, x^*)) - \mathrm{dist}(A,B) \le d(y^*, y_{n_k}) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)).$$

Using (2.27), then we obtain

$$\begin{aligned} D(x^*, T(x^*, y^*)) - \mathrm{dist}(A,B) &\le d(x^*, x_{n_k}) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) \\ &\le d(x^*, x_{n_k}) + \psi(M(x_{n_k}, x^*, y_{n_k}, y^*)) \\ &\le d(x^*, x_{n_k}) + M(x_{n_k}, x^*, y_{n_k}, y^*), \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} D(y^*, T(y^*, x^*)) - \mathrm{dist}(A,B) &\le d(y^*, y_{n_k}) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) \\ &\le d(y^*, y_{n_k}) + \psi(M(y_{n_k}, y^*, x_{n_k}, x^*)) \\ &\le d(y^*, y_{n_k}) + M(y_{n_k}, y^*, x_{n_k}, x^*). \end{aligned} \tag{2.29}$$

Letting $k \to \infty$ in (2.28) and (2.29), we derive

$$\begin{aligned} D(x^*, T(x^*, y^*)) - \mathrm{dist}(A,B) &\le \lim_{k\to\infty} [d(x^*, x_{n_k}) + M(x_{n_k}, x^*, y_{n_k}, y^*)] \\ &= \lim_{k\to\infty} M(x_{n_k}, x^*, y_{n_k}, y^*) \\ &\le \frac{D(x^*, T(x^*, y^*)) - \mathrm{dist}(A,B)}{2}, \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} D(y^*, T(y^*, x^*)) - \mathrm{dist}(A,B) &\le \lim_{k\to\infty} [d(y^*, y_{n_k}) + M(y_{n_k}, y^*, x_{n_k}, x^*)] \\ &= \lim_{k\to\infty} M(y_{n_k}, y^*, x_{n_k}, x^*) \\ &\le \frac{D(y^*, T(y^*, x^*)) - \mathrm{dist}(A,B)}{2}. \end{aligned} \tag{2.31}$$

Because $x^*, y^* \in A, T(x^*, y^*), T(y^*, x^*) \in \mathcal{CL}(B)$, therefore

$$D(x^*, T(x^*, y^*)) - \operatorname{dist}(A, B) \geq 0, \quad D(y^*, T(y^*, x^*)) - \operatorname{dist}(A, B) \geq 0.$$

Equations (2.30) and (2.31) imply that

$$D(x^*, T(x^*, y^*)) - \operatorname{dist}(A, B) = D(y^*, T(y^*, x^*)) - \operatorname{dist}(A, B) = 0,$$

that is,

$$D(x^*, T(x^*, y^*)) = \operatorname{dist}(A, B), \quad D(y^*, T(y^*, x^*)) = \operatorname{dist}(A, B).$$

This completes the proof. □

Addressing to Theorem 2.9, we give the following example to support it.

Example 2.10. Let $X = R^2$ be equipped with the metric

$$d((p_1, p_1'), (q_1, q_1')) = |p_1 - q_1| + |p_1' - q_1'|$$

for any $(p_1, p_1'), (q_1, q_1') \in X$. $A = A_1 \cup A_2$, $B = \{(p, p') : 1 \leq p \leq 2, \frac{1}{6} \leq p' \leq \frac{1}{3}\}$, where

$$A_1 = \{(p, p') : p = \frac{1}{2}, 0 \leq p' \leq \frac{1}{6}\} \cup \{(p, p') : p = \frac{1}{2}, \frac{1}{3} \leq p' \leq \frac{1}{2}\}, \qquad A_2 = \{(p, p') : p = \frac{1}{4}, 0 \leq p' \leq \frac{1}{4}\}.$$

Obviously, $\operatorname{dist}(A, B) = \frac{1}{2}$, $A_0 = \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{1}{3})\}$, $B_0 = \{(1, \frac{1}{6}), (1, \frac{1}{3})\}$. Define the mapping $T : A \times A \to \mathcal{CL}(B)$, by

$$T((p_1, p_1'), (q_1, q_1')) = \begin{cases} \Big\{(p, q) : p = 1, \frac{1}{4} \leq q < \frac{1}{3}\Big\}, & \text{if } p_1' > \frac{1}{3}; \\ \Big\{(p, q) : p = 1, q = \frac{1}{3}\Big\}, & \text{if } p_1' = \frac{1}{3}; \\ \Big\{(p, q) : p = 1, \frac{1}{6} < q \leq \frac{1}{4}\Big\}, & \text{if } p_1' \leq \frac{1}{4}, p_1' \neq \frac{1}{6}; \\ \Big\{(p, q) : p = 1, q = \frac{1}{6}\Big\}, & \text{if } p_1' = \frac{1}{6}, \end{cases}$$

for $(p_1, p_1'), (q_1, q_1') \in A$. It is clear that $T(A_0 \times A_0) \subseteq B_0$. Let $P_1 = (p_1, p_1'), P_2 = (p_2, p_2') \in A_0$, $Q_1 = (q_1, q_1')$, $Q_2 = (q_2, q_2') \in B_0$. If $d(P_1, Q_1) = (P_2, Q_2) = \operatorname{dist}(A, B) = \frac{1}{2}$, then

$$\begin{aligned} d(P_1, Q_1) &= |p_1 - q_1| + |p_1' - q_1'| \\ &= |\frac{1}{2} - 1| + |p_1' - q_1'| \\ &= \frac{1}{2}, \end{aligned}$$

hence, $p_1' = q_1'$. Similarly, $p_2' = q_2'$, subsequently,

$$\begin{aligned} d(P_1, P_2) &= |p_1 - p_2| + |p_1' - p_2'| \\ &= |p_1' - p_2'| \\ &= |q_1' - q_2'| \\ &= |q_1 - q_2| + |q_1' - q_2'| \\ &= d(Q_1, Q_2). \end{aligned}$$

Consequently, $d(P_1, P_2) \leq d(Q_1, Q_2)$ for $P_1, P_2 \in A_0$, $Q_1, Q_2 \in B_0$. So (A, B) satisfies the weak P-property.

Define functions $\psi : [0, +\infty) \to [0, +\infty)$ and $\alpha : A \times A \to [-\infty, +\infty)$ by

$$\psi(t) := \frac{1}{2}t, \ \alpha(x, y) := \begin{cases} 0, & x, y \in \big\{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\big\}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Suppose that for any $x_1, x_2, w_1, w_2, w_1', w_2', y_1, y_2 \in A$ and $u_1 \in T(x_1, y_1), u_2 \in T(x_2, y_2)$, $v_1 \in T(y_1, x_1)$, $v_2 \in T(y_2, x_2)$, we have

$$\begin{cases} \alpha(x_1, x_2) \geq 0, \\ d(w_1, u_1) = \operatorname{dist}(A, B), \\ d(w_2, u_2) = \operatorname{dist}(A, B), \end{cases}$$

and

$$\begin{cases} \alpha(y_1, y_2) \geq 0, \\ d(w_1', v_1) = \operatorname{dist}(A, B), \\ d(w_2', v_2) = \operatorname{dist}(A, B). \end{cases}$$

We deduce by the definition of α that $x_i, y_i \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}$ for $i = 1, 2$. Moreover, $d(w_i, u_i) = \operatorname{dist}(A, B)$ if and only if $w_i = (\frac{1}{2}, \frac{1}{6}), u_i = (1, \frac{1}{6})$ or $w_i = (\frac{1}{2}, \frac{1}{3}), u_i = (1, \frac{1}{3})$ for $i = 1, 2$. It follows that $\alpha(w_1, w_2) \geq 0$. Similarly, $\alpha(w_1', w_2') \geq 0$. Thus, T is a proximal α^+-admissible mapping. Assume that

$$\left.\begin{array}{l} \frac{1}{2} D^*(P, T(P, Q)) \leq d(P, U), \\ \frac{1}{2} D^*(Q, T(Q, P)) \leq d(Q, V), \\ d(P, U) = d(Q, V) = 0, \text{ or} \\ d(P, U) > 0, d(Q, V) > 0, \end{array}\right\} \text{and} \quad \alpha(P, U) \geq 0, \ \alpha(Q, V) \geq 0 \tag{2.32}$$

for all $P, Q, U, V \in A$. It is easy to verify that $P, U, Q, V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}, P \neq U$ and $Q \neq V$. All cases satisfying (2.32) are as follows:

$$\text{(i).} \begin{cases} P \in \{(\frac{1}{4}, 0)\}, \ U \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})\}, \\ Q \in \{(\frac{1}{4}, 0)\}, \ V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})\}, \end{cases} \quad \text{(ii).} \begin{cases} P \in \{(\frac{1}{4}, 0)\}, \ U \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})\}, \\ Q \in \{(\frac{1}{2}, \frac{1}{6})\}, \ V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{4}, 0)\}, \end{cases}$$

$$\text{(iii).} \begin{cases} P \in \{(\frac{1}{4}, 0)\}, \ U \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})\}, \\ Q \in \{(\frac{1}{2}, \frac{1}{3})\}, \ V \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}, \end{cases} \quad \text{(iv).} \begin{cases} P \in \{(\frac{1}{2}, \frac{1}{6})\}, \ U \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{4}, 0)\}, \\ Q \in \{(\frac{1}{2}, \frac{1}{6})\}, \ V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{4}, 0)\}, \end{cases}$$

$$\text{(v).} \begin{cases} P \in \{(\frac{1}{2}, \frac{1}{6})\}, \ U \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{4}, 0)\}, \\ Q \in \{(\frac{1}{4}, 0)\}, \ V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})\}. \end{cases} \quad \text{(vi).} \begin{cases} P \in \{(\frac{1}{2}, \frac{1}{6})\}, \ U \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{4}, 0)\}, \\ Q \in \{(\frac{1}{2}, \frac{1}{3})\}, \ V \in \{(\frac{1}{4}, 0), (\frac{1}{2}, \frac{1}{6})\}. \end{cases}$$

$$\text{(vii).} \begin{cases} P \in \{(\frac{1}{2}, \frac{1}{3})\}, \ U \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}, \\ Q \in \{(\frac{1}{2}, \frac{1}{3})\}, \ V \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}, \end{cases} \quad \text{(viii).} \begin{cases} P \in \{(\frac{1}{2}, \frac{1}{3})\}, \ U \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}, \\ Q \in \{(\frac{1}{2}, \frac{1}{6})\}, \ V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{4}, 0)\}, \end{cases}$$

$$\text{(ix).} \begin{cases} P \in \{(\frac{1}{2}, \frac{1}{3})\}, \ U \in \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{4}, 0)\}, \\ Q \in \{(\frac{1}{4}, 0)\}, \ V \in \{(\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})\}. \end{cases}$$

Case (i). $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = 0$ or $\frac{1}{12}$, $d(P, U) = \frac{5}{12}$ or $d(P, U) = \frac{7}{12}$ and $d(Q, V) = \frac{7}{12}$ or $d(Q, V) = \frac{5}{12}$.

Case (ii). $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = 0$ or $\frac{1}{12}$, $d(P, U) = \frac{7}{12}$ or $d(P, U) = \frac{5}{12}$, and $d(Q, V) = \frac{5}{12}$ or $d(Q, V) = \frac{1}{6}$.

Case (iii). $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = 0$ or $\frac{1}{12}$, $d(P, U) = \frac{7}{12}$ or $d(P, U) = \frac{5}{12}$ and $d(Q, V) = \frac{1}{6}$ or $d(Q, V) = \frac{7}{12}$.

Case (iv). $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = 0$ or $\frac{1}{12}$, $d(P, U) = \frac{1}{6}$ or $d(P, U) = \frac{5}{12}$, and $d(Q, V) = \frac{1}{6}$ or $d(Q, V) = \frac{5}{12}$.

Case (v). $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = 0$ or $\frac{1}{12}$, $d(P, U) = \frac{1}{6}$ or $d(P, U) = \frac{5}{12}$, and $d(Q, V) = \frac{5}{12}$ or $d(Q, V) = \frac{7}{12}$.

Case (vi). $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = 0$ or $\frac{1}{12}$, $d(P, U) = \frac{1}{6}$ or $d(P, U) = \frac{5}{12}$, and $d(Q, V) = \frac{1}{6}$ or $d(Q, V) = \frac{7}{12}$.

Case (vii). $H(T(P,Q),T(U,V)) = H(T(U,V),T(P,Q)) = \frac{1}{12}$, $d(P,U) = \frac{1}{6}$ or $d(P,U) = \frac{7}{12}$, and $d(Q,V) = \frac{1}{6}$ or $d(Q,V) = \frac{7}{12}$.

Case (viii). $H(T(P,Q),T(U,V)) = H(T(U,V),T(P,Q)) = \frac{1}{12}$, $d(P,U) = \frac{1}{6}$ or $d(P,U) = \frac{7}{12}$, and $d(Q,V) = \frac{1}{6}$ or $d(Q,V) = \frac{5}{12}$.

Case (ix). $H(T(P,Q),T(U,V)) = H(T(U,V),T(P,Q)) = \frac{1}{12}$, $d(P,U) = \frac{1}{6}$ or $d(P,U) = \frac{7}{12}$, and $d(Q,V) = \frac{5}{12}$ or $d(Q,V) = \frac{7}{12}$.

For all cases (i)-(ix), we observe that

$$\max\{H(T(P,Q),T(U,V))\} = \max\{H(T(U,V),T(P,Q))\} = \frac{1}{12},$$

and

$$\min\Big\{d(P,U), d(Q,V)\Big\} = \frac{1}{6},$$

thus,

$$\begin{aligned} H(T(P,Q),T(U,V)) \le \frac{1}{12} &= \max\{H(T(P,Q),T(U,V))\} \\ &= \psi\big(\min\{d(P,U), d(Q,V)\}\big) \\ &\le \psi(d(P,U)) \\ &\le \psi(M(P,Q,U,V)). \end{aligned}$$

Similarly,

$$\begin{aligned} H(T(U,V),T(P,Q)) \le \frac{1}{12} &= \max\{H(T(P,Q),T(U,V))\} \\ &= \psi\big(\min\{d(P,U), d(Q,V)\}\big) \\ &\le \psi(d(Q,V)) \\ &\le \psi(M(U,V,P,Q)). \end{aligned}$$

Comprehensively, for all cases (i)-(ix), we have

$$\left.\begin{aligned} &\frac{1}{2}D^*(P,T(P,Q)) \le d(P,U), \\ &\frac{1}{2}D^*(Q,T(Q,P)) \le d(Q,V), \\ &d(P,U) = d(Q,V) = 0, \text{ or} \\ &d(P,U) > 0, d(Q,V) > 0, \end{aligned}\right\} \Rightarrow \alpha(P,U) + H(T(P,Q),T(U,V)) \le \psi(M(P,Q,U,V)).$$

In addition, $D(P,T(P,Q)) \neq 0$ for all cases, hence, the condition (iv) holds in Theorem 2.9. Thus all the assumptions of Theorem 2.9 are satisfied and there exists $\big((\frac{1}{2},\frac{1}{3}),(\frac{1}{2},\frac{1}{6})\big) \in A_0 \times A_0$ such that

$$D\Big((\tfrac{1}{2},\tfrac{1}{3}), T\big((\tfrac{1}{2},\tfrac{1}{3}),(\tfrac{1}{2},\tfrac{1}{6})\big)\Big) = D\Big((\tfrac{1}{2},\tfrac{1}{6}), T\big((\tfrac{1}{2},\tfrac{1}{6}),(\tfrac{1}{2},\tfrac{1}{3})\big)\Big) = \tfrac{1}{2} = \text{dist}(A,B),$$

that is, $\text{CBpp}(T) = \big\{\big((\frac{1}{2},\frac{1}{3}),(\frac{1}{2},\frac{1}{6})\big)\big\}$.

If we replace the condition that T is a two-variable Suzuki type α^+-ψ-proximal multivalued mapping with

$$\alpha(x,y) + H(T(x,x'),T(y,y')) \le \psi(M(x,y,x',y')),$$

then the next result can be deduced easily from Theorem 2.9.

Corollary 2.11. *Let X, A, A_0, and B be as in Theorem 2.9. Assume that $T : A \times A \to \mathcal{CL}(B)$ satisfies the assertions (i)-(vi) in Theorem 2.9, $\psi \in \Omega$ is strictly increasing and*

$$\alpha(x,y) + H(T(x,x'),T(y,y')) \le \psi(M(x,y,x',y'))$$

holds for all $x, y, x', y\prime \in A$, where $\alpha : A \times A \to [-\infty, +\infty)$. Then $\text{CBpp}(T)$ is nonempty.

Remark 2.12. Especially, letting $\alpha(x, y) = 0$ for all $x, y \in A$ in Theorem 2.9, obviously, (ii), (iii) and (vi) hold in Theorem 2.9, thus we can state the following new result.

Corollary 2.13. *Suppose that* X, A, A_0, *and* B *are as defined in Theorem* 2.9 *and* $T : A \times A \to \mathcal{CL}(B)$ *satisfies the following assertions:*

(i) $T(x, y) \subseteq B_0$ *for* $(x, y) \in A_0 \times A_0$ *and* (A, B) *satisfies the weak* P*-property;*

(ii)
$$\left.\begin{array}{l} \frac{1}{2}D^*(x, T(x, x')) \le d(x, y), \\ \frac{1}{2}D^*(x', T(x', x)) \le d(x', y'), \\ d(x, y) = d(x', y') = 0, \text{ or} \\ d(x, y) > 0, d(x', y') > 0, \end{array}\right\} \Rightarrow H(T(x, x'), T(y, y')) \le \psi(M(x, y, x', y'))$$

holds for all $x, y, x', y\prime \in A$;

(iii) *if* $D(x, T(x, y)) = 0$ *or* $D(y, T(y, x)) = 0$ *for any* $x, y \in A_0$, *then* $D(x, T(x, y)) = D(y, T(y, x)) = 0$.

Then $\mathrm{CBpp}(T)$ *is nonempty.*

Remark 2.14. If we replace the two-variable mapping $T : A \times A \to \mathcal{CL}(B)$ on A with $T : A \to \mathcal{CL}(B)$ in Theorem 2.9, Corollary 2.11, and Corollary 2.13, respectively, applying Definitions 2.2, 2.3 of one-variable mapping T, taking the similar proof processes to Theorem 2.9, Corollary 2.11 and Corollary 2.13, and simplifying them to one-variable version of T, it is easy to see that the corresponding one-variable results are still true, in addition, it is worth noting that the condition corresponding to condition (iv) of Theorem 2.9 do not need to be kept in one-variable results, thus, we state the following results with omitting proofs.

Theorem 2.15. *Let* (X, d) *be a complete metric space and* $A, B \in \mathcal{CL}(X)$ *with* $A_0 \ne \emptyset$. $\psi \in \Omega$ *is strictly increasing and* $T : A \to \mathcal{CL}(X)$ *is a one-variable Suzuki type* α^+-ψ*-proximal multivalued mapping. Suppose that the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the weak* P*-property;*

(ii) T *is* α^+*-proximal admissible;*

(iii) *there exist elements* $x_0, x_1 \in A_0$, *and* $u_1 \in Tx_0$ *such that*
$$d(x_1, u_1) = \mathrm{dist}(A, B), \quad \alpha(x_0, x_1) \ge 0;$$

(iv) T *is continuous, or*

(v) *if* $\{x_n\}_{n=0}^{\infty}$ *is a sequence in* A *such that* $\alpha(x_n, x_{n+1}) \ge 0$ *and* $x_n \to x^* \in A$ *as* $n \to \infty$, *then* $\alpha(x_n, x^*) \ge 0$ *for all* $n \in \mathbb{N}$.

Then there exists an element $x^* \in A_0$ *such that*
$$D(x^*, Tx^*) = \mathrm{dist}(A, B).$$

Corollary 2.16. *Let* X, A, A_0, *and* B *be as in Theorem* 2.15. *Assume that* $T : A \times A \to \mathcal{CL}(B)$ *satisfies the assertions* (i)-(v) *in Theorem* 2.15, $\psi \in \Omega$ *is strictly increasing and*
$$\frac{1}{2}D^*(x, Tx) \le d(x, y) \Rightarrow \alpha(x, y) + H(Tx, Ty) \le \psi(M(x, y))$$
holds for all $x, y \in A$, *where* $\alpha : A \times A \to [-\infty, +\infty)$. *Then* $\mathrm{Bpp}(T)$ *is nonempty.*

Corollary 2.17. *Suppose that X, A, A_0, and B are as defined in Theorem* 2.15 *and $T : A \to \mathcal{CL}(B)$ satisfies the following assertions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;*

(ii) *$\frac{1}{2}D^*(x, Tx) \leq d(x, y) \Rightarrow H(Tx, Ty) \leq \psi(M(x, y))$, holds for all $x, y \in A$.*

Then $\mathrm{Bpp}(T)$ *is nonempty.*

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