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Hybrid shrinking iterative solutions to convex feasibility problems and a system of generalized mixed equilibrium problems

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Abstract

The purpose of this paper is to propose a new hybrid shrinking iterative scheme for approximating common elements of the set of solutions to convex feasibility problems for countable families of weak relatively nonexpansive mappings of a set of solutions to a system of generalized mixed equilibrium problems. A strong convergence theorem is established in the framework of Banach spaces. The results extend those of other authors, in which the involved mappings consist of just finitely many ones.

Keywords: Weak relatively nonexpansive mappings, relatively nonexpansive mappings, hybrid iteration scheme, convex feasibility problems, generalized mixed equilibrium problems.
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1. Introduction

Let E be a real Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The duality mapping J has the following properties:

(1) if E is smooth, then J is single-valued;
(2) if E is strictly convex, then J is one-to-one;
(3) if E is reflexive, then J is surjective;

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(4) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
(5) if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is single-valued and also one-to-one (see [6, 13, 15, 18]).

Let E be a smooth Banach space with the dual E^*. The functional $\phi : E \times E \to R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive ([3, 17, 19, 22]) if $F(T) = \widehat{F}(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [3, 13, 15, 18].

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced in 1953 by Mann [12] which well-known as Mann's iteration process and is defined as follows:

$$\begin{cases} x_0 \quad \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \ge 0, \end{cases} \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is chosen in [0, 1]. Fourteen years later, Halpern [8] proposed the new innovation iteration process which resemble in Mann's iteration (1.1), it is defined by

$$\begin{cases} x_0 \quad \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \ge 0, \end{cases} \tag{1.2}$$

where the element $u \in C$ is fixed. Seven years later, Ishikawa [9] enlarged and improved Mann's iteration (1.1) to the new iteration method, it is often cited as Ishikawa's iteration process which is defined recursively by

$$\begin{cases} x_0 \quad \text{chosen arbitrarily,} \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad n \ge 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

In both Hilbert space [19] and uniformly smooth Banach space [22] the iteration process(1.2) has been proved to be strongly convergent if the sequence $\{\alpha_n\}$ satisfies the following conditions:

(i) $\alpha_n \to 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

By the restriction of condition (ii), it is widely believed that Halpern's iteration process (1.2) have slow convergence though the rate of convergence has not been determined. Halpern [8] proved that conditions (i) and (ii) are necessary in the strong convergence of (1.2) for a nonexpansive mapping T on a closed convex subset C of a Hilbert space H. Moreover, Wittmann [19] showed that (1.2) converges strongly to $P_{F(T)}u$ when $\{\alpha_n\}$ satisfies (i), (ii) and (iii), where $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

Both iteration processes (1.1) and (1.3) have only weak convergence, in general Banach space (see [7] for more details). As a matter of fact, process (1.1) may fail to converge while process (1.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space (see [4]). For example, Reich [16] proved that if E is a uniformly convex Banach space with Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that

$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However, we note that Mann's iteration process (1.1) has only weak convergence even in a Hilbert space (see [7]).

Some attempts to modify the Mann iteration method so that the strong convergence is guaranteed have recently been made. Nakajo and Takahashi [14] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a closed convex subset of H, P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}(x_0)$ where $F(T)$ denotes the set of fixed points of T.

The ideas to generalize the process (1.4) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [13] presented their ideas as the following method for a single relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.5}$$

They proved the following convergence theorem.

Theorem 1.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.5), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

In 2007, Plubtieng and Ungchittrakool [15] proposed the following hybrid algorithms for two relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

Theorem 1.2. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let T, S be two relatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x_0) \end{cases} \tag{1.6}$$

with the following restrictions:

(i) $0 \le \alpha_n < 1$, $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $0 \le \beta_n^{(1)},\ \beta_n^{(1)},\ \beta_n^{(3)} \le 1,\ \lim_{n\to\infty} \beta_n^{(1)} = 0,\ \liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0.$

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Theorem 1.3. *Let* E *be a uniformly convex and uniformly smooth Banach space, let* C *be a nonempty closed convex subset of* E*, let* T, S *be two relatively nonexpansive mappings from* C *into itself with* $F := F(T)\cap F(S)$ *is nonempty. Let a sequence* $\{x_n\}$ *be defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n), \\ z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x_0) \end{cases} \tag{1.7}$$

with the following restrictions:

(i) $0 < \alpha_n < 1,\ \limsup_{n\to\infty} \alpha_n < 1$*;*

(ii) $0 \le \beta_n^{(1)},\ \beta_n^{(1)},\ \beta_n^{(3)} \le 1,\ \lim_{n\to\infty} \beta_n^{(1)} = 0,\ \liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0.$

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Recently, Wei-Qi Deng and Shanguang Qian [21] proposed a new hybrid shrinking iterative scheme for approximating common elements of the set of solutions to convex feasibility problems for countable families of relatively nonexpansive mappings of a set of solutions to a system of generalized mixed equilibrium problems. They proved the following convergence theorem.

Theorem 1.4. *Let* E *be a real uniformly smooth and strictly convex Banach space, and* C *be a nonempty closed convex subset of* E*, let* $\{T_i\}, \{S_i\} : C \to C$ *be two sequences of relatively nonexpansive mappings with* $F := (\cap_{i=1}^{\infty} F(T_i)) \bigcap (\cap_{i=1}^{\infty} F(S_i)) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by:*

$$\begin{cases} x_0 = x \in C \quad H_{-1} = W_{-1} = C, \\ y_n = J^{-1}\left[\lambda_n J x_n + 1 - \lambda_n J z_n\right], \\ z_n = J^{-1}\left(\alpha_n J x_n + \beta_n J T_{i_n} x_n + \gamma_n J S_{i_n} x_n\right), \\ H_n = \left\{z \in H_{n-1} \bigcap W_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\right\}, \\ W_n = \left\{z \in H_{n-1} \bigcap W_{n-1} : \langle x_n - z, Jx - Jx_n\rangle \ge 0\right\}, \\ x_{n+1} = P_{H_n \cap W_n}(x), \end{cases} \tag{1.8}$$

where $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *satisfying*

(1) $0 \le \lambda_n < 1,\ \forall\ n = 0, 1, 2, 3, \ldots,\ \limsup_{n\to\infty} \lambda_n > 0$*;*

(2) $\alpha_n + \beta_n + \gamma_n = 1,\ \lim_{n\to\infty} \alpha_n = 0,\ \limsup_{n\to\infty} \beta_n\gamma_n > 0;$

and i_n *is the solution to the positive integer equation* $n = i_n + \frac{(m_n-1)m_n}{2}, (m_n \ge i_n, n = 1, 2, 3, \ldots)$*, that is, for each* $n \ge 1$*, there exists a unique* i_n *such that*

$$i_1 = 1, i_2 = 1, i_3 = 2, i_4 = 1, i_5 = 2, i_6 = 3, i_7 = 1, i_8 = 2, i_9 = 3, i_{10} = 4, i_{11} = 1.$$

Then $\{x_n\}$ *converges strongly to* $P_F x$*, where* $P_F x$ *is the generalized projection from* C *onto* F*.*

The purpose of this paper is to propose a new hybrid shrinking iterative scheme for approximating common elements of the set of solutions to convex feasibility problems for countable families of weak relatively nonexpansive mappings of a set of solutions to a system of generalized mixed equilibrium problems. A strong convergence theorem is established in the framework of Banach spaces. The results extend those of other authors, in which the involved mappings consist of just finitely many ones. In addition, the concept of cycle of the sequence of mappings was presented in this paper. The results of this article modify and improve the results of Deng and Qian [21], it also in some sense, improves some results of [23–27, 29].

2. Preliminaries

Let E be a smooth Banach space with the dual E^*. The functional $\phi : E \times E \to R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.1}$$

for all $x, y \in E$. Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2,\ x, y \in H$.

Recall that if C is a nonempty, closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This is true only when H is a real Hilbert space. In this connection, Alber [1] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \tag{2.2}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the functional ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(y, x) \le (\|y\|^2 + \|x\|^2) \tag{2.3}$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) - 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for all $x, y \in E$ (see [12] for more details).

This section collects some definitions and lemmas which will be used in the proofs of the main results in the next section. Some of them are known; others are not hard to derive.

Remark 2.1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, then we have $x = y$ (see [6, 19, 20] for more details).

Let E be a Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a mapping. We use $F(T)$ to denote the set of fixed points of a mapping T. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Definition 2.2. A mapping T is said to be relatively nonexpansive mapping if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u, x),\ \forall\ u \in F(T), x \in C$;
(3) $\widehat{F}(T) = F(T)$.

If the above conditions (1) and (2) are satisfied, the mapping T is said to be quasi-ϕ-nonexpansive mapping. The relative study for the quasi-ϕ-nonexpansive mappings, we can see [20].

Definition 2.3. A mapping T is said to be weak relatively nonexpansive mapping if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u, x),\ \forall\ u \in F(T), x \in C$;

(3) $\widetilde{F}(T) = F(T)$.

In [28], the author gives an example which is a weak relatively nonexpansive mapping but not a relatively nonexpansive mapping.

We need the following Lemmas to prove our main results.

Lemma 2.4 ([12]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.5 ([1, 2, 10, 12]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \textit{for all} \;\; y \in C.$$

Lemma 2.6 ([1, 2, 10]). *Let E be a reflexive, strictly convex and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_c x) + \phi(\Pi_c x, x) \leq \phi(y, x) \quad \textit{for all } y \in C.$$

Lemma 2.7 ([5]). *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|) \tag{2.5}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

It is easy to prove the following results.

Lemma 2.8. *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a weak relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.9. *Let $\{T_n\}_{n=1}^{\infty}$, $\{T_n^*\}_{n=1}^{\infty}$ be two sequences of mappings such that*

$$\{\{T_n\}_{n=1}^{\infty}\} \supset \{\{T_n^*\}_{n=1}^{\infty}\},$$

if for each $i = 1, 2, 3, ...$, $\{T_n^*\}_{n=1}^{\infty}$ contains a subsequence $\{T_{n_k}^*\}_{k=1}^{\infty}$ such that $T_{n_k}^* = T_i$ for all $k = 1, 2, 3, \ldots$. Then $\{T_n^*\}_{n=1}^{\infty}$ is said to be a cycle of the $\{T_n\}_{n=1}^{\infty}$.

Example 2.10. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings, the following sequences are some cycles of the $\{T_n\}_{n=1}^{\infty}$:

$$T_1, T_1, T_2, T_1, T_2, T_3, T_1, T_2, T_3, T_4, T_1, T_2, T_3, T_4, T_5, \ldots, \tag{cycle 1}$$

$$T_1, T_2, T_1, T_3, T_2, T_1, T_4, T_3, T_2, T_1, T_5, T_4, T_3, T_2, T_1, \ldots, \tag{cycle 2}$$

$$T_1, T_2, T_1, T_1, T_2, T_3, T_2, T_1, T_1, T_2, T_3, T_4, T_3, T_2, T_1, \ldots. \tag{cycle 3}$$

3. Main results

Now we prove our convergence theorems as follows.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}, \{S_n\}$ be two sequences of weak relatively nonexpansive mappings from C into itself such that $F := (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C = C_0 \;\; \textit{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \;\; n = 0, 1, 2, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0,\ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0,\ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(iii) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0, 1)$,

where $\{T_n^*\}, \{S_n^*\}$ *are the cycles of the* $\{T_n\}, \{S_n\}$ *respectively. Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection from* C *onto* F.

Proof. We first show that C_n is closed and convex for each $n \ge 0$. From the definition of C_n, it is obvious that C_n is closed for each $n \ge 0$. We show that C_n is convex for any $n \ge 0$. Since

$$\phi(z, y_n) \le \phi(z, x_n)$$

is equivalent to

$$2\langle z, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2,$$

it follows that C_n is convex.

Next, we show that $F \subset C_n$ for all $n \ge 0$. Observe that

$$z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n).$$

Hence from the definition of $\phi(x, y)$ and the convexity of $\|\cdot\|^2$ we have, for all $p \in F$ that

$$\begin{aligned}\phi(p, z_n) =&\phi\left(p, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n)\right)\\ =&\|p\|^2 - 2\left\langle p,\ \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\right\rangle\\ &+ \left\|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\right\|^2\\ \le&\beta_n^{(1)}\phi(p, x_n) + \ \beta_n^{(2)}\phi(p, T_n^* x_n) + \beta_n^{(3)}\phi(p, S_n^* x_n)\\ \le&\beta_n^{(1)}\phi(p, x_n) + \ \beta_n^{(2)}\phi(p, x_n) + \beta_n^{(3)}\phi(p, x_n)\\ =&\phi(p, x_n).\end{aligned}$$

By the similar reason we have, for all $p \in F$ that

$$y_n = J^{-1}(\alpha_n Jz_n + (1 - \alpha_n)Jx_n),$$

$$\begin{aligned}\phi(p, y_n) =&\phi\left(p, J^{-1}(\alpha_n Jz_n + (1 - \alpha_n)Jx_n)\right)\\ \le&\|p\|^2 - 2\left\langle p, \alpha_n Jz_n + (1 - \alpha_n)Jx_n\right\rangle\\ &+ \|\alpha_n Jz_n + (1 - \alpha_n)Jx_n\|^2\\ \le&\alpha_n\phi(p, z_n) + (1 - \alpha_n)_n\phi(p, x_n)\\ \le&\alpha_n\phi(p, x_n) + (1 - \alpha_n)_n\phi(p, x_n)\\ =&\phi(p, x_n).\end{aligned}$$

That is, $p \in C_n$ for all $n \ge 0$.

Since $x_{n+1} = \Pi_{C_n} x_0$ and $C_n \subset C_{n-1}$ for all $n \ge 1$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \tag{3.2}$$

for all $n \ge 0$. Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. In addition, it follows from Lemma 2.6 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0)$$

for each $p \in F \subset Q_n$ and for all $n \ge 0$. Therefore, $\phi(x_n, x_0)$ is bounded, this together with (3.2) implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Put

$$\lim_{n\to\infty} \phi(x_n, x_0) = d. \tag{3.3}$$

From Lemma 2.6, we have, for any positive integer m, that

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+m}, x_0) - \phi(x_n, x_0)\end{aligned} \tag{3.4}$$

for all $n \geq 0$. Therefore

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0. \tag{3.5}$$

From (3.5) and (2.3), we know that $\{x_n\}$ is bounded and Lemma 2.7 together with (3.5) implies

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0.$$

Then $\{x_n\}$ is a Cauchy sequence, hence there exists a point $x^* \in C$ such that $\{x_n\}$ converges strongly to x^*.

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$, from the definition of C_n, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\phi(x_{n+1}, y_n) \to 0.$$

By using Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

and hence $y_n \to x^*$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} (1-\alpha_n)\|Jz_n - Jx_n\| = \lim_{n\to\infty} \|Jy_n - Jx_n\| = 0.$$

Since $0 \leq \alpha_n \leq \alpha < 1$, then

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous bounded sets, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0,$$

so that $z_n \to x^*$ as $n \to \infty$.

Since $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded so are $\{z_n\}, \{JT_n^* x_n\}$ and $\{JS_n^* x_n\}$. From the definition of $\phi(x, y)$ and

$$z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n),$$

we have, for all $p \in F$ that

$$\begin{aligned}\phi(p, z_n) =& \phi(p, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n)) \\ =& \|p\|^2 - 2\langle p,\ \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\rangle \\ &+ \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\|^2.\end{aligned}$$

Therefore, by using (2.5) in Lemma 2.7, for all $p \in F$, we have

$$\begin{aligned}\phi(p, z_n) \leq& \|p\|^2 - 2\left\langle p,\ \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n \right\rangle \\ &+ \beta_n^{(1)} \|Jx_n\|^2 + \beta_n^{(2)} \|JT_n^* x_n\|^2 + \beta_n^{(3)} \|JS_n^* x_n\|^2 - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n^* x_n\|) \\ \leq& \beta_n^{(1)} \phi(p, x_n) + \ \beta_n^{(2)} \phi(p, T_n^* x_n) + \beta_n^{(3)} \phi(p, S_n^* x_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n^* x_n\|) \\ \leq& \beta_n^{(1)} \phi(p, x_n) + \ \beta_n^{(2)} \phi(p, x_n) + \beta_n^{(3)} \phi(p, x_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n^* x_n\|) \\ =& \phi(p, x_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n^* x_n\|),\end{aligned}$$

and hence

$$\beta_n^{(1)}\beta_n^{(2)}g\left(\|Jx_n - JT_n^*x_n\|\right) \leq \phi(p, x_n) - \phi(p, z_n) \to 0,$$

as $n \to \infty$. By using the same way, we can prove that

$$\beta_n^{(1)}\beta_n^{(3)}g\left(\|Jx_n - JS_n^*x_n\|\right) \leq \phi(p, x_n) - \phi(p, z_n) \to 0,$$

as $n \to \infty$. From the properties of the mapping g, we have

$$\|Jx_n - JT_n^*x_n\| \to 0,$$

as $n \to \infty$, and

$$\|Jx_n - JS_n^*x_n\| \to 0,$$

as $n \to \infty$. Since J^{-1} is also uniformly norm-to-norm continuous on any bounded set, we have

$$\|x_n - T_n^*x_n\| \to 0,$$

as $n \to \infty$, and

$$\|x_n - S_n^*x_n\| \to 0,$$

as $n \to \infty$. Since $\{T_n^*\}_{n=0}^{\infty}, \{S_n^*\}_{n=0}^{\infty}$ are the cycles of the $\{T_n\}_{n=0}^{\infty}, \{S_n\}_{n=0}^{\infty}$, respectively, so for any $i = 0, 1, ...,$ there exist a subsequence $\{T_{i_n}^*\} \subset \{T_n^*\}$ such that $T_{i_n}^* = T_i$ for all $n = 0, 1, \ldots$. That is,

$$\|x_{i_n} - T_i x_{i_n}\| \to 0,$$

as $n \to \infty$, and $x_{i_n} \to x^*$, since T_i is a weak relatively nonexpansive mapping, then $x^* \in F(T_i)$, for all $i = 0, 1, \ldots$. By the same reason we know that $x^* \in F(S_i)$, for all $i = 0, 1, \ldots$. Hence $x^* \in F := (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$.

Finally, we prove that $x^* = \Pi_F x_0$. From Lemma 2.7, we have

$$\phi(x^*, \Pi_F x_0) + \phi(\Pi_F x_0, x_0) \leq \phi(x^*, x_0).$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}x_0}$ and $C_n \supset F$, for all n, we get from Lemma 2.9 that,

$$\phi(\Pi_F x_0, x_{n+1}) + \phi(x_{n+1}, x_0) \leq \phi(\Pi_F x_0, x_0).$$

By the definition of $\phi(x, y)$, it follows that both $\phi(x^*, x_0) \leq \phi(\Pi_F x_0, x_0)$ and $\phi(x^*, x_0) \geq \phi(\Pi_F x_0, x_0)$, whence $\phi(x^*, x_0) = \phi(\Pi_F x_0, x_0)$. Therefore, it follows from the uniqueness of $\Pi_F x_0$ that $x^* = \Pi_F x_0$. This completes the proof. □

Taking $\alpha_n \equiv 0$, Theorem 3.1 is reduced to the following result.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}, \{S_n\}$ be two sequences of weak relatively nonexpansive mappings from C into itself such that $F := (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C = C_0 \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \ n = 0, 1, 2, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0) \end{cases} \tag{3.7}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0, \ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0, \ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1,$

where $\{T_n^*\}, \{S_n^*\}$ are the cycles of $\{T_n\}, \{S_n\}$ respectively. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Next, we prove a convergence theorem for Halpern-type iterative algorithm.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}_{n=0}^{\infty}, \{S_n\}_{n=0}^{\infty}$ be two sequences of weak relatively nonexpansive mappings from C into itself such that $F = (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C = C_0 \quad \text{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n), \\ y_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n)Jx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le (1-\alpha_n\beta_n^{(1)})\phi(z, x_n) + \alpha_n\beta_n^{(1)}\phi(z, x_0)\},\ n = 0, 1, 2, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.8}$$

with the conditions

(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0,\ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
(ii) $\limsup_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0,\ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$,

where $\{T_n^*\}, \{S_n^*\}$ are the cycles of the $\{T_n\}, \{S_n\}$ respectively. Then $\{x_n\}$ converges to $q = \Pi_{F(T)} x_0$.

Proof. We first show that C_n is closed and convex for each $n \ge 0$. From the definition of C_n, it is obvious that C_n is closed for each $n \ge 0$. Next, we prove that C_n is convex for each $n \ge 0$. Since

$$\phi(z, y_n) \le (1-\beta_n^{(1)})\phi(z, x_n) + \beta_n^{(1)}\phi(z, x_0)$$

is equivalent to

$$2\left\langle z, (1-\beta_n^{(1)})Jx_n + \beta_n^{(1)}Jx_0 - Jy_n \right\rangle \le (1-\beta_n^{(1)})\|x_n\|^2 + \beta_n^{(1)}\|x_0\|^2.$$

It is easy to get C_n is convex for each $n \ge 0$.

Next, we show that $F \subset C_n$ for all $n \ge 0$. Observe that

$$z_n = J^{-1}\left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\right).$$

Hence from the definition of $\phi(x, y)$ and the convexity of $\|\cdot\|^2$ we have, for each $p \in F$ that

$$\begin{aligned} \phi(p, z_n) =& \phi\left(p, J^{-1}\left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\right)\right) \\ =& \|p\|^2 - 2\left\langle p,\ \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n \right\rangle \\ &+ \left\|\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n^* x_n + \beta_n^{(3)} JS_n^* x_n\right\|^2 \\ \le& \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, T_n^* x_n) + \beta_n^{(3)}\phi(p, S_n^* x_n) \\ \le& \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, x_n) + \beta_n^{(3)}\phi(p, x_n) \\ \le& \beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})\phi(p, x_n). \end{aligned}$$

By the similar reason we have, for each $p \in F$ that

$$\begin{aligned} \phi(p, y_n) =& \phi\left(p, J^{-1}\left(\alpha_n Jz_n + (1-\alpha_n)Jx_n\right)\right) \\ =& \|p\|^2 - 2\left\langle p,\ \alpha_n Jz_n + (1-\alpha_n)Jx_n \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+\|\alpha_n Jz_n+(1-\alpha_n)Jx_n\|^2\\
\le&\alpha_n\phi(p,z_n)+(1-\alpha_n)\phi(p,x_n)\\
\le&\alpha_n\left(\beta_n^{(1)}\phi(p,x_0)+(1-\beta_n^{(1)})\phi(p,x_n)\right)+(1-\alpha_n)\,\phi(p,x_n)\\
\le&\alpha_n\beta_n^{(1)}\phi(p,x_0)+\left(1-\alpha_n\beta_n^{(1)}\right)\phi(p,x_n).
\end{aligned}
$$

So, $p\in C_n$, which implies that $F\subset C_n$ for all $n\ge 0$.

Since $x_{n+1}=\Pi_{C_{n+1}}x_0\in Q_n$ and $C_n\subset C_{n-1}$, then we get

$$\phi(x_n,x_0)\le\phi(x_{n+1},x_0),\quad \text{for all}\quad n\ge 0. \tag{3.9}$$

Therefore $\{\phi(x_n,x_0)\}$ is nondecreasing. It follows from Lemma 2.6 that

$$\phi(x_n,x_0)=\phi(\Pi_{C_n}x_0,x_0)\le\phi(p,x_0)-\phi(p,x_n)\le\phi(p,x_0),$$

for each $p\in F\subset Q_n$ and for all $n\ge 0$. Therefore, $\phi(x_n,x_0)$ is bounded. This together with (3.9) implies that the limit of $\{\phi(x_n,x_0)\}$ exists. Put

$$\lim_{n\to\infty}\phi(x_n,x_0)=d. \tag{3.10}$$

From Lemma 2.6, we have, for any positive integer m, that

$$
\begin{aligned}
\phi(x_{n+m},x_n)=&\phi(x_{n+m},\Pi_{C_n}x_0)\le\phi(x_{n+m},x_0)-\phi(\Pi_{C_n}x_0,x_0)\\
=&\phi(x_{n+m},x_0)-\phi(x_n,x_0),
\end{aligned}
\tag{3.11}
$$

for all $n\ge 0$. Therefore (3.11) implies

$$\lim_{n\to\infty}\phi(x_{n+m},x_n)=0. \tag{3.12}$$

Since $\{x_n\}$ is bounded, from (3.12) and by using Lemma 2.7 we have

$$\lim_{n\to\infty}\|x_{n+m}-x_n\|=0.$$

Then $\{x_n\}$ is a Cauchy sequence, hence there exists a point $x^*\in C$ such that $\{x_n\}$ converges strongly to x^*. In particular, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \tag{3.13}$$

$$\lim_{n\to\infty}\phi(x_{n+1},x_n)=0.$$

Since $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1},y_n)\to 0,\quad \phi(x_{n+1},z_n)\to 0,$$

and

$$\|x_{n+1}-y_n\|\to 0,\quad \|x_{n+1}-z_n\|\to 0. \tag{3.14}$$

as $n\to\infty$.

From the definition of $\phi(x,y)$ and

$$z_n=J^{-1}\left(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_nx_n+\beta_n^{(3)}JS_nx_n\right),$$

we have, for all $p\in F$ that

$$
\begin{aligned}
\phi(p,z_n)=&\phi\left(p,J^{-1}\left(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n\right)\right)\\
=&\|p\|^2-2\left\langle p,\ \beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n\right\rangle\\
&+\left\|\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n\right\|^2.
\end{aligned}
$$

Therefore, from the boundedness of $\{x_n\},\{z_n\},\{Tx_n\}$ and $\{Sx_n\}$, and by using (2.5) in Lemma 2.7, we have

$$\begin{aligned}\phi(p,z_n) \le&\|p\|^2-2\left\langle p,\ \beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n\right\rangle\\ &+\beta_n^{(1)}\|Jx_0\|+\beta_n^{(2)}\|JT_n^*x_n\|+\beta_n^{(3)}\|JS_n^*x_n\|^2-\beta_n^{(2)}\beta_n^{(3)}g(\|JT_n^*x_n-JS_n^*x_n\|)\\ \le&\beta_n^{(1)}\phi(p,x_0)+\beta_n^{(2)}\phi(p,T_n^*x_n)+\beta_n^{(3)}\phi(p,S_n^*x_n)-\beta_n^{(2)}\beta_n^{(3)}g(\|JT_n^*x_n-JS_n^*x_n\|)\\ \le&\beta_n^{(1)}\phi(p,x_0)+\beta_n^{(2)}\phi(p,x_n)+\beta_n^{(3)}\phi(p,x_n)-\beta_n^{(2)}\beta_n^{(3)}g(\|JT_n^*x_n-JS_n^*x_n\|)\\ \le&\beta_n^{(1)}\phi(p,x_0)+(1-\beta_n^{(1)})\phi(p,x_n)-\beta_n^{(2)}\beta_n^{(3)}g(\|JT_n^*x_n-JS_n^*x_n\|),\end{aligned}$$

which implies that

$$\beta_n^{(2)}\beta_n^{(3)}g(\|JT_n^*x_n-JS_n^*x_n\|)\le\beta_n^{(1)}\phi(p,x_0)+(1-\beta_n^{(1)})\phi(p,x_n)-\phi(p,z_n).$$

From $\lim_{n\to\infty}\beta_n^{(1)}=0$ and $x_n\to x^*,\ \ z_n\to x^*$, we have

$$\beta_n^{(2)}\beta_n^{(3)}g(\|JT_n^*x_n-JS_n^*x_n\|)\to 0,$$

as $n\to\infty$. From the properties of the mapping g, we have

$$\|JT_n^*x_n-JS_n^*x_n\|\to 0, \tag{3.15}$$

as $n\to\infty$. Since

$$z_n=J^{-1}\left(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n\right),$$

then we have

$$Jz_n=\left(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n\right).$$

Therefore

$$\begin{aligned}\|Jx_n-Jz_n\|&=\left\|Jx_n-(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n)\right\|\\ &=\left\|\beta_n^{(1)}(Jx_n-Jx_0)+\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\right\|\\ &\ge\left\|\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\|-\|\beta_n^{(1)}(Jx_n-Jx_0)\right\|,\end{aligned}$$

which leads to

$$\left\|\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\|\le\|Jx_n-Jz_n\|+\|\beta_n^{(1)}(Jx_n-Jx_0)\right\|.$$

Since $x_n\to x^*,\ z_n\to x^*$ and $\lim_{n\to\infty}\beta_n^{(1)}=0$, then from above inequality we obtain

$$\left\|\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\right\|=0. \tag{3.16}$$

On the other hand, by using the property of norm $\|\cdot\|$, we have

$$\begin{aligned}&\left\|\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\right\|\\ &=\|\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\\ &\quad+\beta_n^{(3)}(Jx_n-JT_n^*x_n)-\beta_n^{(3)}(Jx_n-JT_n^*x_n)\|\\ &=\left\|(\beta_n^{(2)}+\beta_n^{(3)})(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(JT_n^*x_n-JS_n^*x_n)\right\|\\ &\ge\left\|(\beta_n^{(2)}+\beta_n^{(3)})(Jx_n-JT_n^*x_n)\|-\|\beta_n^{(3)}(JT_n^*x_n-JS_n^*x_n)\right\|,\end{aligned}$$

which leads to the following inequality

$$\left\|(\beta_n^{(2)}+\beta_n^{(3)})(Jx_n-JT_n^*x_n)\right\| \leq \left\|\beta_n^{(2)}(Jx_n-JT_n^*x_n)+\beta_n^{(3)}(Jx_n-JS_n^*x_n)\right\|+\beta_n^{(3)}\left\|JT_n^*x_n-JS_n^*x_n\right\|.$$

Therefore, by using (3.15) and (3.16) we have

$$\left\|(\beta_n^{(2)}+\beta_n^{(3)})(Jx_n-JT_n^*x_n)\right\| \to 0.$$

This together with the condition (ii) of Theorem 3.3 implies that

$$\|Jx_n-JT_n^*x_n\| \to 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, then we have

$$\|x_n-T_n^*x_n\| \to 0,$$

as $n\to\infty$. Since $\{T_n^*\}_{n=0}^{\infty},\{S_n^*\}_{n=0}^{\infty}$ are the cycles of $\{T_n\}_{n=0}^{\infty},\{S_n\}_{n=0}^{\infty}$, respectively, so for any $i=0,1,\ldots$, there exist a subsequence $\{T_{i_n}^*\}\subset\{T_n^*\}$ such that $T_{i_n}^*=T_i$ for all $n=0,1,\ldots$. That is,

$$\|x_{i_n}-T_ix_{i_n}\| \to 0,$$

as $n\to\infty$, and $x_{i_n}\to x^*$, since T_i is a weak relatively nonexpansive mapping, then $x^*\in F(T_i)$, for all $i=0,1,2,\ldots$. By the same reason we know that $x^*\in F(S_i)$, for all $i=0,1,2,\ldots$. Hence $x^*\in F:=(\cap_{n=0}^{\infty}F(T_n))\cap(\cap_{n=0}^{\infty}F(S_n))$.

Finally, we prove that $x^*=\Pi_Fx_0$. From Lemma 2.8, we have

$$\phi(x^*,\Pi_Fx_0)+\phi(\Pi_Fx_0,x_0)\leq\phi(x^*,x_0).$$

On the other hand, since $x_{n+1}=\Pi_{C_{n+1}}$ and $C_n\supset F$, for all n, we get from Lemma 2.6 that,

$$\phi(\Pi_Fx_0,x_{n+1})+\phi(x_{n+1},x_0)\leq\phi(\Pi_Fx_0,x_0).$$

By the definition of $\phi(x,y)$, it follows that both $\phi(x^*,x_0)\leq\phi(\Pi_Fx_0,x_0)$ and $\phi(x^*,x_0)\geq\phi(\Pi_Fx_0,x_0)$, whence $\phi(x^*,x_0)=\phi(\Pi_Fx_0,x_0)$. Therefore, it follows from the uniqueness of Π_Fx_0 that $x^*=\Pi_Fx_0$. This completes the proof. □

Taking $\alpha_n\equiv1$, Theorem 3.3 is reduced to the following result.

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}_{n=0}^{\infty},\{S_n\}_{n=0}^{\infty}$ be two sequences of weak relatively nonexpansive mappings from C into itself such that $F=(\cap_{n=0}^{\infty}F(T_n))\bigcap(\cap_{n=0}^{\infty}F(S_n))\neq\emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0\in C=C_0 \quad \textit{chosen arbitrarily},\\ y_n=J^{-1}(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_n^*x_n+\beta_n^{(3)}JS_n^*x_n),\\ C_{n+1}=\{z\in C_n:\phi(z,y_n)\leq\phi(z,x_n)\},\ n=0,1,2,\ldots,\\ x_{n+1}=\Pi_{C_{n+1}}x_0,\end{cases} \tag{3.17}$$

with the conditions

(i) $\lim_{n\to\infty}\beta_n^{(1)}=0,\ \beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)}=1$;

(ii) $\limsup_{n\to\infty}\beta_n^{(2)}\beta_n^{(3)}>0,\ \beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)}=1$,

where $\{T_n^*\},\{S_n^*\}$ are the cycles of $\{T_n\},\{S_n\}$, respectively. Then $\{x_n\}$ converges to $q=\Pi_{F(T)}x_0$.

4. Applications

The so-called convex feasibility problem for a family of mappings $\{T_n\}$ is to find a point in the nonempty intersection $\cap_{n=1}^{\infty} F(T_n)$.

Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty, closed, convex subset of E. Let $\{B_i\}_{i=1}^{\infty} : C \to E^*$ be a sequence of β_i-inverse strongly monotone mappings, $\{\psi_i\}_{i=1}^{\infty} : C \to R^1$ a sequence of lower semi-continuous and convex functions, and $\{\theta_i\}_{i=1}^{\infty} : C \to R^1$ a sequence of bifunctions satisfying the conditions:

(A1) $\theta(x,x) = 0$;

(A2) θ is monotone, i.e., $\theta(x,y) + \theta(y,x) \le 0$;

(A3) $\limsup_{n\to\infty} \theta(x + t(z-x), y) \le \theta(x,y)$;

(A4) the mapping $y \mapsto \theta(x,y)$ is convex and lower semi-continuous.

A system of generalized mixed equilibrium problems (GMEP) for $\{B_i\}_{i=1}^{\infty}, \{\psi_i\}_{i=1}^{\infty}$ and $\{\theta_i\}_{i=1}^{\infty}$ is to find an $x^* \in C$ such that

$$\theta(x^*, y) + \langle y - x^*, B_i x^* \rangle + \psi_i(y) - \psi_i(x^*) \ge 0, \quad \forall\, y \in C, \ i = 1, 2, \dots, \tag{4.1}$$

whose set of common solutions is denoted by $\Omega = \cap_{i=1}^{\infty} \Omega_i$, where Ω_i indicates the set of solutions to generalized mixed equilibrium problem for B_i, θ_i and ψ_i.

Define a countable family of mappings $\{S_{r,i}\}_{i=1}^{\infty} : E \to C$ with $r > 0$ as follows:

$$S_{r,i}(x) = \{z \in C : \tau_i(x,y) + \frac{1}{r}\langle y - z, Jz - Jy \rangle \ge 0, \ y \in C\}, \ \forall\, i = 1, 2, \dots, \tag{4.2}$$

where $\tau_i(x,y) = \theta(x,y) + \langle y - x, B_i x \rangle + \psi_i(y) - \psi_i(x)$. It has been shown by Zhang [28] that

(1) $\{S_{r,i}\}_{i=1}^{\infty}$ is a sequence of single-valued mappings;

(2) $\{S_{r,i}\}_{i=1}^{\infty}$ is a sequence of relatively nonexpansive mappings;

(3) $\bigcap_{i=1}^{\infty} F(S_{r,i}) = \Omega$.

By using Theorems 3.3 and 3.4 we can get the following results.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}$ be a sequences of weak relatively nonexpansive mappings from C into itself and $\{S_{r,i}\}_{i=1}^{\infty}$ be a sequence of mappings defined by (4.2) with $F := (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_{r,i})) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C = C_0 \ \ \text{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n^* x_n + \beta_n^{(3)} J S_{r,n}^* x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \ n = 0, 1, 2, \dots, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{4.3}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0, \ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0, \ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(iii) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$,

where $\{T_n^*\}, \{S_n^*\}$ are the cycles of the $\{T_n\}, \{S_{r,n}\}$, respectively. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, which is some common solution to the convex feasibility problem for $\{T_n\}$ and a system of generalized mixed equilibrium problems for $\{S_{r,n}\}$ where Π_F is the generalized projection from C onto F.

Theorem 4.2. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}$ be a sequences of weak relatively nonexpansive mappings from C into itself and $\{S_{r,i}\}_{i=1}^{\infty}$ be a sequence of mappings defined by (4.2) with $F := (\cap_{n=0}^{\infty} F(T_n)) \bigcap (\cap_{n=0}^{\infty} F(S_{r,i})) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C = C_0 \quad \textit{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n^* x_n + \beta_n^{(3)} J S_{r,n}^* x_n), \\ y_n = J^{-1}(\alpha_n J z_n + (1-\alpha_n) J x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq (1 - \alpha_n \beta_n^{(1)}) \phi(z, x_n) + \alpha_n \beta_n^{(1)} \phi(z, x_0)\},\ n = 0, 1, 2, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.4}$$

with the conditions

(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0,\ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(ii) $\limsup_{n\to\infty} \beta_n^{(2)} \beta_n^{(3)} > 0,\ \beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$,

where $\{T_n^*\}, \{S_{r,n}^*\}$ are the cycles of the $\{T_n\}, \{S_{r,n}\}$, respectively. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, which is some common solution to the convex feasibility problem for $\{T_n\}$ and a system of generalized mixed equilibrium problems for $\{S_{r,n}\}$ where Π_F is the generalized projection from C onto F.

Remark 4.3.

(1) In [21], the sequences $\{T_{i_n}\}_{n=0}^{\infty}$ and $\{S_{i_n}\}_{n=0}^{\infty}$ are namely the “cycle 1” of the $\{T_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$, respectively.

(2) In [21], the conditions of Theorem 3.1 and Theorem 4.1 are not sufficient, in fact, in order to use Lemma 2.6 and Lemma 2.7, the condition “Let E be a uniformly convex and uniformly smooth Banach space”, is needed.

(3) In [21], the proof of Theorem 3.1 is relatively complex. In fact, we can easily prove the iterative sequence $\{x_n\}$ is a Cauchy sequence without using Lemma 2.6.

(4) In [21], page 11, line 14, the sentence “$\{S_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed relatively nonexpansive mappings” should be “$\{S_{r,i}\}_{i=1}^{\infty}$ is a sequence of relatively nonexpansive mappings”, since the relatively nonexpansive mapping must be closed.

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