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The threshold behavior and periodic solution of stochastic SIR epidemic model with saturated incidence

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Abstract

We investigate degenerate stochastic SIR epidemic model with saturated incidence. For the constant coefficients case, we achieve a threshold which determines the extinction and persistence of the epidemic by utilizing Markov semigroup theory. Furthermore, we conclude that environmental white noise plays a positive effect in the control of infectious disease in some sense comparing to the corresponding deterministic system. For the stochastic non-autonomous system, we prove the existence of periodic solution.

Keywords: SIR epidemic model, Markov semigroups, asymptotic stability, threshold, periodic solution.
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1. Introduction

In recent years, mathematical modeling has been widely used to analyze the spread of infectious diseases. The classical SIR epidemic model is our familiar model, and it has been studied in many literatures

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([2],[9],[21],[28],[29]). The incidence of a disease is vital to guarantee whether the model gives a reasonable qualitative description of the disease dynamics ([6],[12]). In most classical disease transmission models, the incidence rate is assumed to be bilinear incidence rate βSI ([9],[28]). However, with the number of susceptible individuals increasing, the number of susceptible individuals with every infective contacts within a certain time are limited, it is likely to be unreasonable to consider the bilinear incidence rate. Capasso et al. ([7]) used a saturated incidence rate to prevent the unboundedness of contact rate. Liu et al. ([18],[19]) used a non-linear incidence rate to discuss the effect of behavioral changes in epidemic models. Compared with bilinear incidence, saturation incidence may be more appropriate for many cases ([25],[29]).

The classic autonomous SIR epidemic model takes the form

$$\begin{cases} \dfrac{dS_t}{dt} = \Lambda - \dfrac{\beta S_t I_t}{1+\alpha I_t} - \mu S_t, \\ \dfrac{dI_t}{dt} = \dfrac{\beta S_t I_t}{1+\alpha I_t} - (\mu+\varepsilon+\gamma) I_t, \\ \dfrac{dR_t}{dt} = \gamma I_t - \mu R_t, \end{cases} \tag{1.1}$$

where S_t and I_t represent the number of susceptible individuals and infected individuals at time t, respectively, Λ is the influx of individuals into the susceptibles, β and ε are the disease transmission coefficient and the disease related death rate, respectively, R_t represents the number of removed individuals with permanently immunity at time t and the recovery rate is given by γ, μ is the natural death rate, which is assumed to be equal for every group. The parameters in (1.1) are considered as positive constants. The basic reproduction number $R_0 = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\gamma)}$ is the threshold of the system (1.1) for an epidemic to occur.

As various stochastic disturbances appear in real life, the deterministic dynamics system can be altered by them. Here we show some beautiful results about stochastic version of system (1.1). Liu et al. [20] have studied the asymptotic behavior of globally positive solution for SIR epidemic

$$\begin{cases} dS_t = \left(\Lambda - \dfrac{\beta S_t I_t}{1+\alpha I_t} - \mu S_t\right) dt + \sigma_1 S_t dB_1(t), \\ dI_t = \left(\dfrac{\beta S_t I_t}{1+\alpha I_t} - (\mu+\varepsilon+\gamma) I_t\right) dt + \sigma_2 I_t dB_2(t), \\ dR_t = \gamma I_t - \mu R_t + \sigma_3 R_t dB_3(t), \end{cases} \tag{1.2}$$

where $B_i, i = 1, 2, 3$ are independent standard Brownian motions. Besides, Yang et al. [29] utilized stochastic Lyapunov functions to show that under some conditions, the solution of system (1.2) has the ergodic property as $R_0 > 1$, while exponential stability as $R_0 \le 1$.

In this paper, we suppose that the random perturbation for three populations is related, which means the system is influenced by the same factor, such as other epidemic disease weather and so on. Then the corresponding stochastic system becomes

$$\begin{cases} dS_t = \left(\Lambda - \dfrac{\beta S_t I_t}{1+\alpha I_t} - \mu S_t\right) dt + \sigma_1 S_t dB_t, \\ dI_t = \left(\dfrac{\beta S_t I_t}{1+\alpha I_t} - (\mu+\varepsilon+\gamma) I_t\right) dt + \sigma_2 I_t dB_t, \\ dR_t = \gamma I_t - \mu R_t + \sigma_3 R_t dB_t. \end{cases} \tag{1.3}$$

As the population R has no influence on the disease transmission dynamics, we can leave it out and only consider

$$\begin{cases} dS_t = \left(\Lambda - \dfrac{\beta S_t I_t}{1+\alpha I_t} - \mu S_t\right) dt + \sigma_1 S_t dB_t, \\ dI_t = \left(\dfrac{\beta S_t I_t}{1+\alpha I_t} - (\mu+\varepsilon+\gamma) I_t\right) dt + \sigma_2 I_t dB_t. \end{cases} \tag{1.4}$$

Throughout this article, we suppose $\sigma_i > 0$, $i = 1, 2, 3$. Note that the existence of positive solution of system (1.3) can be obtain by [20], because the independence of B_1 and B_2 plays an unimportant role in the proof. However, the idea in [9],[10],[29] to acquire the asymptotic behavior of system (1.2) is unavailable for system (1.4) because the Fokker−Planck equation corresponding to system (1.4) is of degenerate type. In this paper, one of our aims is to study the stationary distribution of system (1.4) by applying Markov semigroup theory ([13],[14],[16],[17],[23],[24]) which is different from the idea in [8] and [30].

However, perturbations that biological populations suffer often appear periodic phenomena, such as seasonal effect, individual lifecycle and so on. In order to better description of ecological systems, it is vital to research on the periodic solution of stochastic non-autonomous system. Literatures [4],[15],[27] recently have done study on the periodic solution of SIR epidemic model with bilinear incidence, besides Lin et al. [9] obtained the threshold for the epidemic to occur. Inspired by these work, we discuss the stochastic periodic system

$$\begin{cases} dS_t = \left[\Lambda(t) - \dfrac{\beta(t)S(t)I(t)}{1+\alpha(t)I(t)} - \mu(t)S(t)\right]dt + \sigma_1(t)S(t)dB(t), \\ dI_t = \left[\dfrac{\beta(t)S(t)I(t)}{1+\alpha(t)I(t)} - (\mu(t)+\varepsilon(t)+\gamma(t))I(t)\right]dt + \sigma_2(t)I(t)dB(t). \end{cases} \tag{1.5}$$

Here we assume that the coefficients $\Lambda(t)$, $\beta(t)$, $\alpha(t)$, $\mu(t)$, $\varepsilon(t)$, $\gamma(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are positive ω-periodic continuous functions and ω is a positive constant. We will prove that the ω-periodic solution of system (1.5) exists by applying periodic theory in [11].

The rest of this article is organized as follows. In Section 2, we present the asymptotic stability of system (2.1) and the condition of the disease extinct. Furthermore, we obtain the threshold of the epidemic to occur. In Section 3, we prove the existence of ω-periodic solution of system (1.5). In Section 4, we give a brief analysis and the interesting work which will be done in the future. In Appendix we present some auxiliary results about Markov semigroup theory. For convenience we let

$$\check{f} = \sup_{t\in[0,\infty)} f(t),\ \hat{f} = \inf_{t\in[0,\infty)} f(t),\ \langle f\rangle_t = \frac{1}{t}\int_0^t f(s)ds,$$

where f is a continuous bounded function on $[0, +\infty)$.

2. Threshold behavior of system (1.4)

By using the same method in [20], we know that system (1.4) has a unique global positive solution. Substitute $u = \ln S, v = \ln I$ in system (1.4), then we gain

$$\begin{cases} du_t = \left(\Lambda e^{-u} - \dfrac{\beta e^v}{1+\alpha e^v} - \mu - \dfrac{\sigma_1^2}{2}\right)dt + \sigma_1 dB(t), \\ dv_t = \left[\dfrac{\beta e^v}{1+\alpha e^v} - (\mu+\varepsilon+\gamma) - \dfrac{\sigma_2^2}{2}\right]dt + \sigma_2 dB(t), \end{cases} \tag{2.1}$$

and the positive constants $c_1 = \mu + \frac{\sigma_1^2}{2}, c_2 = \mu + \varepsilon + \gamma + \frac{\sigma_2^2}{2}$. In this section, we mainly study the threshold behavior of system (1.4), because of the equivalence between system (1.4) and (2.1), it is enough to focus on (2.1).

2.1. Asymptotic stability

Let $X = \mathbb{R}^2$, Σ be the σ-algebra of Borel subsets of X, and m be the Lebesgue measure on (X, Σ). $\mathscr{P}(t, x, y, A)$ is noted as the transition probability function for the diffusion process (u_t, v_t), that is,

$$\mathscr{P}(t, x, y, A) = \text{Prob}((u_t, v_t) \in A),$$

(u_t, v_t) is a solution of (2.1) with the initial condition $(u_0, v_0) = (x, y)$.

Furthermore, we know that for each $(x, y) \in \mathbb{R}^2$ and $t > 0$, the distribution of (u_t, v_t) is absolutely continuous with respect to the Lebesgue measure with the density $U(t, x, y)$, then $U(t, x, y)$ satisfies the Fokker-Planck equation:

$$\frac{\partial U}{\partial t} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 U}{\partial x^2} + \sigma_1\sigma_2 \frac{\partial^2 U}{\partial x \partial y} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 U}{\partial y^2} - \frac{\partial(f_1(x, y)U)}{\partial x} - \frac{\partial(f_2(x, y)U)}{\partial y}, \tag{2.2}$$

where $f_1(x, y) = -c_1 + \Lambda e^{-x} - \frac{\beta e^y}{1+\alpha e^y}$ and $f_2(x, y) = -c_2 + \frac{\beta e^x}{1+\alpha e^y}$.

Theorem 2.1. *Assume that (u_t, v_t) is a solution of system (2.1). The distribution of (u_t, v_t) has a density $U(t, x, y)$ for every $t > 0$. Furthermore, if $\frac{\beta\Lambda}{\mu} - (\mu + \varepsilon + \gamma) - \frac{\sigma_2^2}{2} > 0$, there exists a unique density $U_*(x, y)$ of system (2.2) satisfying*

$$\lim_{t\to\infty} \iint_{\mathbb{R}^2} |U(t, x, y) - U_*(x, y)| dxdy = 0.$$

Remark 2.2. In Theorem 2.1, we have:

(1) If $\sigma_2 < \sigma_1$ or $\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha} > 0$, then

$$\text{supp } u_* = E = \mathbb{R}^2,$$

where supp u_* is defined as follows:

$$\text{supp } u_* = \left\{(x, y) \in \mathbb{R}^2 : u_*(x, y) \neq 0\right\}.$$

(2) If $\sigma_2 \geq \sigma_1$ and $\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha} \leq 0$, then

$$\text{supp } u_* = E(M_0) = \left\{(x, y) : y < \frac{\sigma_2}{\sigma_1}x + M_0\right\}, \tag{2.3}$$

where M_0 is the smallest number such that $f(x, y)(\sigma_2, -\sigma_1) \geq 0$ for all $(x, y) \notin E(M_0)$, and $f = (f_1, f_2)$.

Before proving Theorem 2.1, we firstly present a Markov semigroup related to (2.2). Let $P(t)V(x, y) = U(t, x, y)$ for any $V(x, y) \in D$. The definition of D is as follows:

$$D = \{f \in L^1 : f \geq 0, \|f\| = 1\}. \tag{2.4}$$

Since $P(t)$ is a contract on D, it can be extended to a contraction on $L^1(X, \Sigma, m)$. So the operators $\{P(t)\}_{t\geq 0}$ form a Markov semigroup. Let $\mathscr{A}$ be the infinitesimal generator of the semigroup $\{P(t)\}_{t\geq 0}$, that is,

$$\mathscr{A}V = \frac{1}{2}\sigma_1^2 \frac{\partial^2 V}{\partial x^2} + \sigma_1\sigma_2 \frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 V}{\partial y^2} - \frac{\partial(f_1 V)}{\partial x} - \frac{\partial(f_2 V)}{\partial y}.$$

The adjoint operator of $\mathscr{A}$ has the following form

$$\mathscr{A}^* V = \frac{1}{2}\sigma_1^2 \frac{\partial^2 V}{\partial x^2} + \sigma_1\sigma_2 \frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 V}{\partial y^2} + f_1 \frac{\partial V}{\partial x} + f_2 \frac{\partial V}{\partial y}. \tag{2.5}$$

We divide the proof of Theorem 2.1 into five lemmas.

Lemma 2.3. *The semigroup $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup and the transition function of the process (u_t, v_t) has a continuous density $k(t, x, y; x_0, y_0)$.*

Proof. We will use Hörmander condition [5] to prove this result. If $a(x)$ and $b(x)$ are vector fields on $\mathbb{R}^2$, then the Lie bracket $[a,b]$ is a vector field given by

$$[a,b]_j(x)=\sum_{k=1}^{d}\left(a_k\frac{\partial b_j}{\partial x_k}(x)-b_k\frac{\partial a_j}{\partial x_k}(x)\right),\quad j=1,2,\ldots,d.$$

For the sake of simplicity, we use the following marks:

$$b_1:=[a,b],\ b_2=[[a,b],b]=[b_1,b],\ldots,\ b_n=[b_{n-1},b],\ldots.$$

Let $a(u,v)=\big(-c_1+\Lambda e^{-u}-\frac{\beta e^v}{1+\alpha e^v},-c_2+\frac{\beta e^u}{1+\alpha e^v}\big)^T$, $b(u,v)=(\sigma_1,\sigma_2)^T$. Assume $p(x)=\frac{1}{1+\alpha e^x}$, then $a(u,v)=\big(-c_1+\Lambda e^{-u}-\beta e^v p(v),-c_2+\beta e^u p(v)\big)^T$. Then, we have,

$$[a,b]=\big(\sigma_1\Lambda e^{-u}+\sigma_2\beta e^v(p(v)+p'(v)),-\beta e^u(\sigma_1 p(v)+\sigma_2 p'(v))\big)^T.$$

Suppose $A(x)=e^x p(x)$, $B(x)=e^{\frac{\sigma_1}{\sigma_2}x}p(x)$, then

$$\begin{aligned}
b_1&=[a,b]=\big(\sigma_1\Lambda e^{-u}+\beta\sigma_2 A^{(1)}(v),-\beta e^{u-\frac{\sigma_1}{\sigma_2}v}\sigma_2 B^{(1)}(v)\big)^T;\\
b_2&=[[a,b],b]=(\sigma_1^2\Lambda e^{-u}-\beta\sigma_2^2 A^{(2)}(v),\beta e^{u-\frac{\sigma_1}{\sigma_2}v}\sigma_2^2 B^{(2)}(v))^T;\\
b_3&=[[[a,b],b],b]=\big(\sigma_1^3\Lambda e^{-u}+\beta\sigma_2^3 A^{(3)}(v),-\beta e^{u-\frac{\sigma_1}{\sigma_2}v}\sigma_2^3 B^{(3)}(v)\big)^T;
\end{aligned}$$

and so on. We summarize b_n by using the induction method

$$b_n=\big(\sigma_1^n\Lambda e^{-u}+(-1)^{n-1}\beta\sigma_2^n A^{(n)}(v),\beta e^{u-\frac{\sigma_1}{\sigma_2}v}(-1)^n\sigma_2^n B^{(n)}(v)\big)^T,n=1,2,.... \tag{2.6}$$

At present, we show that vector $b(u,v),b_1(u,v),b_2(u,v),b_3(u,v),...$ span the space R^2 for every $(u,v)\in R^2$ by reduction to absurdity. By the contrary, if this result is not true, then for every $(u,v)\in R^2$,

$$\left|\begin{matrix} b & b_n\end{matrix}\right|=\left|\begin{matrix}\sigma_1 & \sigma_1^n\Lambda e^{-u}+(-1)^{n-1}\beta\sigma_2^n A^{(n)}(v)\\ \sigma_2 & \beta e^{u-\frac{\sigma_1}{\sigma_2}v}(-1)^n\sigma_2^n B^{(n)}(v)\end{matrix}\right|=0,\qquad n=1,2,....$$

That is,

$$\sigma_2\sigma_1^n\Lambda e^{-u}+\sigma_2(-1)^n\beta\sigma_2^n A^{(n)}(v)-\sigma_1\beta e^{u-\frac{\sigma_1}{\sigma_2}v}(-1)^n\sigma_2^n B^{(n)}(v)=0,\qquad n=1,2,....$$

Therefore, we have

$$\beta A^{(n)}(v)+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}B^{(n)}(v)=\frac{\sigma_1}{\sigma_2}^n(-1)^n\Lambda e^{-u},\qquad n=1,2,....$$

Since functions $A(x)$ and $B(x)$ are analytic in the field $K=\{x\in\mathbb{C}:\|e^x\|<\frac{1}{\alpha}\}$, by Taylor expansion in the region K, we get

$$A(x)=\sum_{n=0}^{\infty}\frac{A^{(n)}(v)(x-v)^n}{n!}\quad\text{and}\quad B(x)=\sum_{n=0}^{\infty}\frac{B^{(n)}(v)(x-v)^n}{n!}.$$

Thus

$$\begin{aligned}
&\beta A(x)+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}B(x)\\
&=\beta\sum_{n=0}^{\infty}\frac{A^{(n)}(v)(x-v)^n}{n!}+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}\sum_{n=0}^{\infty}\frac{B^{(n)}(v)(x-v)^n}{n!}\\
&=\beta A(v)+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}B(v)+\sum_{n=1}^{\infty}\frac{(x-v)^n}{n!}\Big[\beta A^{(n)}(v)+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}B^{(n)}(v)\Big]\\
&=\beta A(v)+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}B(v)+\sum_{n=1}^{\infty}\frac{(x-v)^n}{n!}\left(\frac{\sigma_1}{\sigma_2}\right)^n(-1)^n\Lambda e^{-u}\\
&=\beta A(v)+\frac{\sigma_1}{\sigma_2}\beta e^{u-\frac{\sigma_1}{\sigma_2}v}B(v)+\Lambda e^{-u}\Big[-1+e^{\frac{\sigma_1}{\sigma_2}(x-v)}\Big].
\end{aligned}\tag{2.7}$$

Particularly, first letting $x \to -\infty$, we obtain $0 = \beta A(v) + \frac{\sigma_1}{\sigma_2}\beta e^u/(1+\alpha e^v) - \Lambda e^{-u}$ from (2.7). Then letting $v \to -\infty$, we get $0 = \beta e^u - \Lambda e^{-u}$, when $u \to -\infty$, the right close to positive infinity, which contradicts to the hypothesis. Thus, our claim holds. That is to say, for every $(u,v) \in \mathbb{R}^2$, vector $b(u,v), b_1(u,v), b_2(u,v), b_3(u,v), \ldots$ span the space $\mathbb{R}^2$. So the transition probability function $\mathscr{P}(t, x_0, y_0, A)$ has a density $k(t,x,y;x_0,y_0)$ and $k \in C^\infty\left((0,\infty)\times\mathbb{R}^2\times\mathbb{R}^2\right)$ by Hörmander Theorem [5].

As the transition probability function $\mathscr{P}(t, x_0, y_0, A)$ has a density $k(t,x,y;x_0,y_0)$, thus for every $f \in D$, we have

$$P(t)f(x,y) = \iint_{\mathbb{R}^2} k(t,x,y;u,v)f(u,v)dudv$$

and the semigroup $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup. □

In the proof of Lemma 2.5, we will apply support theorems [1, 3, 26]. Now we briefly represent the method based on it. These thesis will enable us to attest the continuous density k is positive. Fix a point $(x_0, y_0) \in \mathbb{R}^2$ and a function $\phi \in L^2([0,T];\mathbb{R})$, consider the following system of integral equations:

$$x_\phi(t) = x_0 + \int_0^t [f_1(x_\phi(s), y_\phi(s)) + \sigma_1\phi]ds, \tag{2.8}$$

$$y_\phi(t) = y_0 + \int_0^t [f_2(x_\phi(s), y_\phi(s)) + \sigma_2\phi]ds, \tag{2.9}$$

where $f_1(x,y) = -c_1 + \Lambda e^{-x} - \frac{\beta e^y}{1+\alpha e^y}$, and $f_2(x,y) = -c_2 + \frac{\beta e^x}{1+\alpha e^y}$.

Remark 2.4 ([24]). If there exists some $\phi \in L^2([0,T];\mathbb{R})$ such that the derivative $D_{x_0,y_0;\phi}$ has rank 2, then $k(T,x,y;x_0,y_0) > 0$ for $x = x_\phi(T)$ and $y = y_\phi(T)$. Here $D_{x_0,y_0;\phi}$ is the Frechét derivative of the function $h \mapsto \mathbf{x}_{\phi+h}(T) = \begin{bmatrix} x_{\phi+h} \\ y_{\phi+h} \end{bmatrix}$ from $L^2([0,T];\mathbb{R})$ to $\mathbb{R}^2$, and the derivative $D_{x_0,y_0;\phi}$ can be obtained by means of the perturbation method for ordinary differential equations. That is to say, let $\Gamma(t) = \mathbf{f}'(x_\phi(t), y_\phi(t))$, where $\mathbf{f}'$ is the Jacobians of $\mathbf{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$. Let $Q(t,t_0)$ for $T \geq t \geq t_0 \geq 0$, be a matrix function such that $Q(t_0,t_0) = I$, $\partial Q(t,t_0)/\partial t = \Gamma(t)Q(t,t_0)$ and $\mathbf{v} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$. Then we obtain

$$D_{x_0,y_0;\phi}h = \int_0^T Q(T,s)\mathbf{v}h(s)ds.$$

Lemma 2.5. *Let $E = \mathbb{R}^2$ while $\sigma_2 < \sigma_1$ or $\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha} > 0$; and $E = E(M_0)$ while $\sigma_2 \geq \sigma_1$ and $\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha} \leq 0$. Then for each $(x_0,y_0) \in E$ and $(x,y) \in E$, there exists $T > 0$ such that $k(T,x,y;x_0,y_0) > 0$.*

Proof. Since we consider a continuous control function ϕ, the system (2.8), (2.9) can be replaced by the following system of differential equations:

$$x'_\phi = f_1(x_\phi, y_\phi) + \sigma_1\phi, \tag{2.10}$$

$$y'_\phi = f_2(x_\phi, y_\phi) + \sigma_2\phi. \tag{2.11}$$

First, we show that the rank of $D_{x_0,y_0;\phi}$ is 2 for almost every $(x,y) \in \mathbb{R}^2$. Let $\delta \in (0,T)$ and $h(t) = \mathbf{1}_{[T-\delta,T]}$, $t \in [0,T]$, where $\mathbf{1}_{[T-\delta,T]}$ is the characteristic function of interval $[T-\delta,T]$. By Taylor expansion we acquire $Q(T,s) = I + \Gamma(T)(T-s) + o(T-s)$, and

$$D_{x_0,y_0;\phi}h = \delta\mathbf{v} + \frac{1}{2}\delta^2\Gamma(T)\mathbf{v} + o(\delta^2), \quad \mathbf{v} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix},$$

$$\Gamma(T)\mathbf{v} = \begin{bmatrix} -\Lambda e^{-x} & -\frac{\beta e^y}{1+\alpha e^y} + \frac{\alpha\beta e^{2y}}{(1+\alpha e^y)^2} \\ \frac{\beta e^x}{1+\alpha e^y} & -\frac{\alpha\beta e^{x+y}}{(1+\alpha e^y)^2} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$$
$$= \begin{bmatrix} -\sigma_1\Lambda e^{-x} - \frac{\sigma_2\beta e^y}{1+\alpha e^y} + \frac{\sigma_2\alpha\beta e^{2y}}{(1+\alpha e^y)^2} \\ \frac{\sigma_1\beta e^x}{1+\alpha e^y} - \frac{\sigma_2\alpha\beta e^{2y}}{(1+\alpha e^y)^2} \end{bmatrix}.$$

So

$$\left| \, \Gamma(T)\mathbf{v} \quad \mathbf{v} \, \right| = e^{-x}\Big[-\sigma_1\sigma_2\Lambda + e^x\Big(\frac{\sigma_2^2\alpha\beta e^y}{(1+\alpha e^y)^2} - \frac{\sigma_2^2\beta e^y}{1+\alpha e^y}\Big) + e^{2x}\Big(\frac{\sigma_1\sigma_2\alpha\beta e^y}{(1+\alpha e^y)^2} - \frac{\sigma_1^2\beta)}{1+\alpha e^y}\Big)\Big].$$

Denote

$$S_1 = \left\{(x,y)\in\mathbb{R}^2 : -\sigma_1\sigma_2\Lambda + e^x\Big(\frac{\sigma_2^2\alpha\beta e^y}{(1+\alpha e^y)^2} - \frac{\sigma_2^2\beta e^y}{1+\alpha e^y}\Big) + e^{2x}\Big(\frac{\sigma_1\sigma_2\alpha\beta e^y}{(1+\alpha e^y)^2} - \frac{\sigma_1^2\beta)}{1+\alpha e^y}\Big) = 0\right\},$$

then the Lebesgue measure of set S_1 is zero. Hence, $\mathbf{v}$ and $\Gamma(T)\mathbf{v}$ are linearly independent for any $(x,y)\in\mathbb{R}^2\setminus S_1$. Thus $D_{x_0,y_0;\phi}$ has rank 2 for almost every $(x,y)\in\mathbb{R}^2$.

Then, we show that there exist a control function $\phi\in L^2([0,T],\mathbb{R})$ and $T>0$ such that $x_\phi(0)=x_0$, $y_\phi(0)=y_0$, $x_\phi(T)=x$, $y_\phi(T)=y$ for any two points $(x_0,y_0)\in E$ and $(x,y)\in E$. We substitute $z_\phi = y_\phi - \frac{\sigma_2}{\sigma_1}x_\phi$. Then (2.10) and (2.11) become

$$x'_\phi = g_1(x_\phi,z_\phi)+\sigma_1\phi, \tag{2.12}$$
$$z'_\phi = g_2(x_\phi,z_\phi), \tag{2.13}$$

where

$$g_1(x,z) = -c_1 + \Lambda e^{-x} - \frac{\beta e^{z+\frac{\sigma_2}{\sigma_1}x}}{1+\alpha e^{z+\frac{\sigma_2}{\sigma_1}x}},$$
$$g_2(x,z) = \frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\beta e^x}{1+\alpha e^{z+\frac{\sigma_2}{\sigma_1}x}} - \frac{\sigma_2}{\sigma_1}\Lambda e^{-x} + \frac{\frac{\sigma_2}{\sigma_1}\beta e^{z+\frac{\sigma_2}{\sigma_1}x}}{1+\alpha e^{z+\frac{\sigma_2}{\sigma_1}x}}.$$

We divide the rest of the proof into six steps.

Step 1: For any fixed $z_0, z_1\in\mathbb{R}$, if $z_1<z_0$, we have $g_2(x,z)\to-\infty,\ x\to-\infty$, then there exist $x_0\in\mathbb{R}$ such that $g_2(x_0,z)\le -c_2/2$. Therefore, there exist $x_0\in\mathbb{R}$, ϕ and $T>0$, such that $z_\phi(0)=z_0,\ z_\phi(T)=z_1,\ x_\phi=x_0$.

Step 2: Assume that $\sigma_2<\sigma_1$, then for any fixed $z_0\in\mathbb{R}, z_1\in\mathbb{R}$, if $z_0<z_1$, we get $g_2(x,z)\to+\infty, x\to+\infty$, then there exist $x_0\in\mathbb{R}$ such that $g_2(x_0,z)\ge\delta_0/2$, where δ_0 is a positive constant. Therefore, there exist $x_0\in\mathbb{R}$, ϕ and $T>0$, such that $z_\phi(0)=z_0,\ z_\phi(T)=z_1,\ x_\phi=x_0$.

Step 3: Assume that $\sigma_2\ge\sigma_1$ and $\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha}>0$, then for every $z_0\in\mathbb{R}, z_1\in\mathbb{R}$, and $z_0<z_1$, there exists $x_0\in\mathbb{R}$ such that $g_2(x_0,z)\ge\delta_1$ for $z\in[z_1,z_0]$, where δ_1 is a positive constant. Then we find a control function ϕ such that $x_\phi=x_0,\ z_\phi(0)=z_0,\ z_\phi(T)=z_1$ for some $T>0$.

Step 4: Consider the case $\sigma_2\ge\sigma_1$ and $\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha}\le 0$. Then

$$\begin{aligned} g_2(x,z) =& \frac{1}{1+\alpha e^{z+\frac{\sigma_2}{\sigma_1}x}}\ \Big[\frac{\sigma_2}{\sigma_1}c_1 - c_2 + \beta e^x - \frac{\sigma_2}{\sigma_1}\Lambda e^{-x} \\ &+ \Big(\alpha(\frac{\sigma_2}{\sigma_1}c_1 - c_2) + \frac{\sigma_2}{\sigma_1}\beta\Big)e^{z+\frac{\sigma_2}{\sigma_1}x} - \frac{\sigma_2}{\sigma_1}\Lambda\alpha e^{z+(\frac{\sigma_2}{\sigma_1}-1)x}\Big] \\ =& \frac{-\big(\alpha(\frac{\sigma_2}{\sigma_1}c_1 - c_2) + \frac{\sigma_2}{\sigma_1}\beta\big)e^{\frac{\sigma_2}{\sigma_1}x} + \frac{\sigma_2}{\sigma_1}\Lambda\alpha e^{z+(\frac{\sigma_2}{\sigma_1}-1)x}}{1+\alpha e^{z+\frac{\sigma_2}{\sigma_1}x}} \end{aligned}$$

$$\times\Bigg[-e^{-z}+\frac{\beta e^{x}-\frac{\sigma_2}{\sigma_1}\Lambda e^{-x}+\frac{\sigma_2}{\sigma_1}c_1-c_2}{-\big(\alpha(\frac{\sigma_2}{\sigma_1}c_1-c_2)+\frac{\sigma_2}{\sigma_1}\beta\big)e^{\frac{\sigma_2}{\sigma_1}x}+\frac{\sigma_2}{\sigma_1}\Lambda\alpha e^{z+(\frac{\sigma_2}{\sigma_1}-1)x}}\Bigg].$$

Let

$$W(x)=\frac{\beta e^{x}-\frac{\sigma_2}{\sigma_1}\Lambda e^{-x}+\frac{\sigma_2}{\sigma_1}c_1-c_2}{-(\alpha(\frac{\sigma_2}{\sigma_1}c_1-c_2)+\frac{\sigma_2}{\sigma_1}\beta)e^{\frac{\sigma_2}{\sigma_1}x}+\frac{\sigma_2}{\sigma_1}\Lambda\alpha e^{z+(\frac{\sigma_2}{\sigma_1}-1)x}}$$

and $e^{M_0}=\sup_{x\in\mathbb{R}}W(x)$. Then for every $\epsilon>0$ there exists a $\delta_2>0$ having the following property. If $z_0<z_1\le M_0-\epsilon$ for every $z\in[z_0,z_1]$, then there exists x^* such that $g_2(x^*,z)\ge\delta_2$, where x^* satisfies $g_2(x^*,M_0)=0$. Then we also find a control function ϕ such that $x_\phi=x_0,\ z_\phi(0)=z_0,\ z_\phi(T)=z_1$ for some $T>0$.

Step 5 ([13]): Fix $x_0\in\mathbb{R}$, $L>0$, $A_0, A_1>A_0$ and $\epsilon>0$ such that $\epsilon<L/4$ and $\epsilon<(A_1-A_0)/4$. Let

$$M=\max\{|g_1(x,z)|+|g_2(x,z)|:x\in[x_0,x_0+L];z\in[A_0,A_1]\}$$

and $t_0=\epsilon M^{-1}$, $\phi\equiv 3\sigma_1^{-1}\epsilon^{-1}ML/4$. Thus for every $z_0\in[A_0+\epsilon,A_1-\epsilon]$, the solution of system (2.12), (2.13) with initial condition $x_\phi(0)=x_0$ and $z_\phi(0)=z_0$ has the following properties:

$$z_\phi(t)\in[z_0-\epsilon,z_0+\epsilon], for\ t\le t_0 \quad\text{and}\quad x_\phi(t_0)\in(x_0+L/2,x_0+L).$$

From the procedure above we conclude that for $(x_1,z_1)\in(x_0,x_0+L/2)\times[A_0+2\epsilon,A_1-2\epsilon]$ there exist $z_0\in[z_1-\epsilon,z_1+\epsilon]$ and $T\in[0,t_0]$ such that $x_\phi(T)=x_1$ and $z_\phi(T)=z_1$. We can use the similar proof as $x_1\in(x_0-L/2,x_0]$.

Step 6: Let $E=\mathbb{R}^2$ when $\sigma_2<\sigma_1$ or $\frac{\sigma_2}{\sigma_1}c_1-c_2+\frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha}>0$; $E=E(M_0)$ when $\sigma_2\ge\sigma_1$ and $\frac{\sigma_2}{\sigma_1}c_1-c_2+\frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha}\le 0$. Then from Step 1-5 we obtain that for any $(x_0,z_0)\in E$ and $(x_1,z_1)\in E$ there exist a control function ϕ and $T>0$ such that $x_\phi(0)=x_0,\ z_\phi(0)=z_0,\ x_\phi(T)=x_1$ and $z_\phi(T)=z_1$. If this is the case, it follows that for any two points $(x_0,y_0)\in E$ and $(x,y)\in E$ there exist a control function ϕ and $T>0$ such that $x_\phi(0)=x_0,\ y_\phi(0)=y_0,\ x_\phi(T)=x$, and $y_\phi(T)=y$. So $k(T,x,y;x_0,y_0)>0$ if $(x,y)\in\mathbb{R}^2\setminus S_1$. □

Lemma 2.6. *Assume that* $\sigma_2\ge\sigma_1$ *and* $\frac{\sigma_2}{\sigma_1}c_1-c_2+\frac{\sigma_2}{\sigma_1}\frac{\beta}{\alpha}\le 0$ *and let* $E=E(M_0)$. *Then for every density* f *we have that*

$$\lim_{t\to\infty}\iint_E P(t)f(x,y)dxdy=1.$$

Proof. The proof is similar to that of Lemma 3 in [23], so we omit it. □

Lemma 2.7. *If* $\frac{\beta\Lambda}{\mu}-(\mu+\varepsilon+\gamma)-\frac{\sigma_2^2}{2}>0$, *then the semigroup* $\{P(t)\}_{t\ge0}$ *is asymptotically stable.*

Proof. We will construct a nonnegative C^2-function V and a closed set $U\in\Sigma$ (which lies entirely in E) such that

$$\sup_{(u,v)\in E\setminus U}\mathscr{A}^*V(u,v)<0,$$

where $\mathscr{A}^*$ is defined in (2.5). Such a function V is called Khasminski function [22]. Consider the function

$$H(u,v)=-u+M\Bigg[-v-\frac{\beta}{\mu}(e^u+e^v)\Bigg]+(e^u+e^v+1)^{\theta+1},(u,v)\in E.$$

It is not hard to achieve that when (u,v) equals to $(u_0,u_0+\ln M)$, $H(u,v)$ gets its minimum value, and $H(u,v)>H(u_0,u_0+\ln M)$ for any $(u,v)\ne(u_0,u_0+\ln M)$. We define a nonnegative C^2 -function V of the following form:

$$V(u,v)=-u+M\Bigg[-v-\frac{\beta}{\mu}(e^u+e^v)\Bigg]+(e^u+e^v+1)^{\theta+1}-H(u_0,u_0+\ln M),\ (u,v)\in E,$$

where $\theta \in (0,1)$ and $M > 0$ such that $\mu - \frac{(\sigma_1 \vee \sigma_2)^2}{2}\theta > 0$ and $-\lambda M + (M_2 \vee 0) = -2$, here

$$M_2 = \sup_{(u,v)\in\mathbb{R}^2} \Bigg[\frac{\beta}{\alpha} + \mu + \frac{\sigma_1^2}{2} + (\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta - \frac{\theta+1}{2}\left(\mu - \frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\right)(e^u+e^v+1)^{\theta+1}\Bigg].$$

Denote $V_1 := -u$, $V_2 := M\Big[-v - \frac{\beta}{\mu}(e^u+e^v)\Big]$, $V_3 := (e^u+e^v+1)^{\theta+1}$, then $\mathscr{A}^*V = \mathscr{A}^*V_1 + \mathscr{A}^*V_2 + \mathscr{A}^*V_3$. It is easy to calculate that

$$\mathscr{A}^*V_1 = -\Lambda e^{-u} + \frac{\beta e^v}{1+\alpha e^v} + \mu + \frac{\sigma_1^2}{2},$$
$$\mathscr{A}^*V_2 = M\Bigg[-\left(\frac{\beta\Lambda}{\mu} - (\mu+\varepsilon+\gamma) - \frac{\sigma_2^2}{2}\right) + \frac{\alpha\beta e^{u+v}}{1+\alpha e^v} + \frac{\beta(\mu+\varepsilon+\gamma)}{\mu}e^v\Bigg]$$

and

$$\begin{aligned}\mathscr{A}^*V_3 &= (\theta+1)(e^u+e^v+1)^\theta\big(\Lambda - \mu e^u - (\mu+\varepsilon+\gamma)e^v\big) \\ &\quad + \frac{\theta(\theta+1)}{2}(e^u+e^v+1)^{\theta-1}(\sigma_1 e^u + \sigma_2 e^v)^2 \\ &\le (\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta - (\theta+1)\left(\mu - \frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\right)(e^u+e^v+1)^{\theta+1}.\end{aligned}$$

Define a closed set

$$U_\epsilon(u,v) = \left\{(u,v)\in\mathbb{R}^2 : e^u \ge \epsilon, e^v \ge \epsilon, e^u+e^v \le \frac{1}{\epsilon}\right\},$$

where $\epsilon > 0$ is a sufficiently small number satisfying the following conditions:

$$\frac{M\beta(\mu+\varepsilon+\gamma)}{\mu}\epsilon < 1,$$

$$M\beta\alpha\epsilon - \frac{\theta+1}{2}(\mu - \frac{(\sigma_1\vee\sigma_2)^2}{2}\theta) < 0, \tag{2.14}$$

$$-\frac{\Lambda}{\epsilon} + \frac{\beta}{\alpha} + \mu + \frac{\sigma_1^2}{2} + M_1 < -1, \tag{2.15}$$

$$\frac{\theta+1}{2}(\mu - \frac{(\sigma_1\vee\sigma_2)^2}{2}\theta)\frac{1}{\epsilon^{\theta+1}} + M_3 < -1, \tag{2.16}$$

where $M_1,\ M_3$ are constants which can be found from Case 1 and Case 3 below. Let $\lambda = \frac{\beta\Lambda}{\mu} - (\mu+\varepsilon+\gamma) - \frac{\sigma_2^2}{2}$, and

$$\begin{aligned} D_\epsilon^1 &= \left\{(u,v)\in\mathbb{R}^2 : 0 < e^u < \epsilon\right\}, \\ D_\epsilon^2 &= \left\{(u,v)\in\mathbb{R}^2 : 0 < e^v < \epsilon\right\}, \\ D_\epsilon^3 &= \left\{(u,v)\in\mathbb{R}^2 : e^u+e^v > \frac{1}{\epsilon}\right\}.\end{aligned}$$

Then $E \setminus U_\epsilon = D_\epsilon^1 \bigcup D_\epsilon^2 \bigcup D_\epsilon^3$. So we consider $\mathscr{A}^*V$ in three regions respectively as follows:

Case 1: When $(u,v)\in D_\epsilon^1$, we have $\mathscr{A}^*V_1 < -\frac{\Lambda}{\epsilon}+\frac{\beta}{\alpha}+\mu+\frac{\sigma_1^2}{2}$, and

$$\begin{aligned}\mathscr{A}^*V_2+\mathscr{A}^*V_3 &< M\big[\beta e^u+\frac{\beta(\mu+\varepsilon+\gamma)}{\mu}e^v\big]+(\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta\\ &\quad -(\theta+1)\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)(e^u+e^v+1)^{\theta+1}\\ &\le M_1,\end{aligned}$$

where

$$\begin{aligned}M_1=\sup_{(u,v)\in\mathbb{R}^2}\big[M[\beta e^u+\frac{\beta(\mu+\varepsilon+\gamma)}{\mu}e^v\big]+(\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta\\ -(\theta+1)\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)(e^u+e^v+1)^{\theta+1}].\end{aligned}$$

In view of (2.16), we get $\mathscr{A}^*V<-1$ on D_ϵ^1.

Case 2: On D_ϵ^2, $\mathscr{A}^*V_1<\frac{\beta}{\alpha}+\mu+\frac{\sigma_1^2}{2}$, $\mathscr{A}^*V_2<-\lambda M+\frac{M\beta(\mu+\varepsilon+\gamma)}{\mu}\epsilon+M\beta\alpha\epsilon$, and

$$\mathscr{A}^*V_3<(\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta-\frac{\theta+1}{2}\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)(e^u+e^v+1)^{\theta+1}-\frac{\theta+1}{2}\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big).$$

According to (2.14) and (2.15), it follows that

$$\mathscr{A}^*V\le-\lambda M+(M_2\vee 0)+\frac{M\beta(\mu+\varepsilon+\gamma)}{\mu}\epsilon+M\beta\alpha\epsilon-\frac{\theta+1}{2}\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)<-1$$

on D_ϵ^2.

Case 3: For any $(u,v)\in D_\epsilon^3$, we have $\mathscr{A}^*V_1<\frac{\beta}{\alpha}+\mu+\frac{\sigma_1^2}{2}$, $\mathscr{A}^*V_2<\frac{M\beta(\mu+\varepsilon+\gamma)}{\mu}e^v+M\beta e^u$, and

$$\begin{aligned}\mathscr{A}^*V_3<(\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta-\frac{\theta+1}{2}\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)(e^u+e^v+1)^{\theta+1}\\ -\frac{\theta+1}{2}(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)\frac{1}{\epsilon^{\theta+1}}.\end{aligned}$$

Denote

$$\begin{aligned}M_3=\sup_{(u,v)\in\mathbb{R}^2}[\frac{\beta}{\alpha}+\mu+\frac{\sigma_1^2}{2}+\frac{M\beta(\mu+\varepsilon+\gamma)}{\mu}e^v+M\beta e^u+(\Lambda+\mu)(\theta+1)(e^u+e^v+1)^\theta\\ -\frac{\theta+1}{2}\big(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta\big)(e^u+e^v+1)^{\theta+1}].\end{aligned}$$

From (2.16), we have $\mathscr{A}^*V\le-\frac{\theta+1}{2}(\mu-\frac{(\sigma_1\vee\sigma_2)^2}{2}\theta)\frac{1}{\epsilon^{\theta+1}}+M_3<-1$ on D_ϵ^3. Summarizing the results above, we get

$$\sup_{(u,v)\in E\setminus U_\epsilon}\mathscr{A}^*V(u,v)<-1.$$

So the semigroup is not sweeping from the set U_ϵ by the theory in [22]. According to Lemma 5.3 in Appendix A, the semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable. □

2.2. Extinction

Theorem 2.8. *If $\frac{\beta\Lambda}{\mu}-(\mu+\varepsilon+\gamma)-\frac{\sigma_2^2}{2}<0$, then $\lim_{t\to\infty}v(t)=-\infty$ a.e. and the distribution of $u(t)$ converges weakly to the measure which has the density*

$$f_*(x)=C\exp(2[-c_1x-\Lambda e^{-x}]/\sigma_1^2),$$

where $C=[(2\Lambda/\sigma_1^2)^{-2c_1/\sigma_1^2}\Gamma(2c_1/\sigma_1^2)]^{-1}$.

Remark 2.9. The proof of Theorem 2.8 is almost the same to the proof of Lemma 3.5 in [13] by using ergodic theorem and comparison theorem. The reader may refer to [13] for details.

3. Existence of the positive periodic solution of system (1.5)

3.1. Existence and uniqueness of the global positive solution

Theorem 3.1. *For any given initial value $(S_0, I_0) \in \mathbb{R}^2_+$, there is a unique solution $(S(t), I(t))$ to (1.5) and the solution will remain in $\mathbb{R}^2_+$ with probability 1.*

In the proof of the existence and uniqueness of the positive solution in [29], the independence of $B_i, i = 1, 2, 3$ is not essential, and the procedure is standard, so we omit the proof here.

3.2. Existence of ω-periodic solution

In this section, on account of the biological significance of the model, we will discuss the existence of ω-periodic solution of system (1.5) in $\mathbb{R}^2_+$. In the first place, we present some useful definitions and lemmas about the existence of periodic Markov process.

Definition 3.2 ([11])**.** An stochastic process $\xi(t) = \xi(t,\omega)(-\infty < t < +\infty)$ is said to be periodic with period θ if for every finite sequence of numbers $t_1, t_2, ..., t_n$ the joint distribution of random variables $\xi(t_1 + h), ..., \xi(t_n + h)$ is independent of h, where $h = k\theta$ $(k = \pm1, \pm2, ...)$.

Remark 3.3. Khasminskii [11] shows that a Markov process $z(t)$ is θ-periodic if and only if its transition probability function is θ-periodic and the function $P_0(t, A) = P\{z(t) \in A\}$ satisfies the equation

$$P_0(s, A) = \int_{\mathbf{R}^l} P_0(s, dz)P(s, z, s + \theta, A) \equiv P_0(s + \theta, A),$$

where $A \in \mathcal{B}$ and $\mathcal{B}$ is σ-algebra consisting of all Borel measurable sets.

Consider the following equation

$$X(t) = X(t_0) + \int_{t_0}^{t} b(s, X(s))ds + \sum_{r=1}^{k} \int_{t_0}^{t} \sigma_r(s, X(s))d\xi_r(s). \tag{3.1}$$

Lemma 3.4 ([11])**.** *Suppose that the coefficients of* (3.1) *are continuous and θ-periodic in t and satisfy the conditions below (here B is a constant):*

$$|b(s, z) - b(s, \bar{z})| + \sum_{r=1}^{k} |\sigma_r(s, z) - \sigma_r(s, \bar{z})| \le |z - \bar{z}|, |b(s, z)| + \sum_{r=1}^{k} |\sigma_r(s, z)| \le B(1 + |z|)$$

in every cylinder $I \times U$, where U is an open set. Suppose further that there exists a function $V(t, z) \in C^2$ in $\mathbb{R}^l$ which is θ-periodic in t and satisfies the following conditions:

$$\inf_{|z|>R} V(t, z) \to \infty \ as\ R \to \infty, \tag{3.2}$$

$$\mathcal{L}V(t, z) \le -1 \ outside\ of\ some\ compact\ set, \tag{3.3}$$

where the differential operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{k=1}^{l} f_k(z, t)\frac{\partial}{\partial z_k} + \frac{1}{2}\sum_{k,j=1}^{l}\sum_{r=1}^{k} \sigma_r^k(t, z)\sigma_r^j(t, z)\frac{\partial^2}{\partial z_k z_j}.$$

Then there exists a solution of (3.1) *which is a θ-periodic Markov process.*

Remark 3.5. Assume $\psi(t)$ is the unique positive ω-periodic solution of equation $\psi'(t) = \mu(t)\psi(t) - \beta(t)$, it follows from [15] that

$$\psi(t) = \frac{\int_t^{t+\omega} exp\{\int_s^t \mu(\tau)d\tau\}\beta(s)ds}{1 - exp\{-\int_0^\omega \mu(\tau)d\tau\}}, \quad t \geq 0.$$

Let

$$\lambda_0(t) = \Lambda(t)\psi(t) - \left(\mu(t) + \varepsilon(t) + \gamma(t) + \frac{\sigma_2^2(t)}{2}\right).$$

Then $\lambda_0(t)$ is also an ω-periodic function.

Theorem 3.6. *If $\langle\lambda_0\rangle_\omega > 0$, then the system* (1.5) *has a positive ω-periodic solution.*

Proof. Since the coefficients of (1.5) are continuous bounded positive periodic functions, they satisfy the local Lipschitz condition. So we only need to show conditions (3.2) and (3.3) hold. Define a C^2-function $V : [0, +\infty) \times \mathbb{R}_+^2 \to \mathbb{R}$ by

$$V(t, S, I) = -\log S + M^*[-\log I - \psi(t)(S + I) - \rho(t)] + (S + I + 1)^{\vartheta+1},$$

we choose $\vartheta \in (0, 1)$ and $M^* > 0$ such that

$$\hat{\mu} - \frac{(\check{\sigma}_1 \vee \check{\sigma}_2)^2}{2}\vartheta > 0,$$

$$\check{f} - M^*\langle\lambda_0\rangle_\omega + 2^\vartheta(\vartheta + 1)\left[\check{\Lambda} + \check{\mu} - 2(\hat{\mu} - \frac{(\check{\sigma}_1 \vee \check{\sigma}_2)^2}{2}\vartheta)\right] \leq -2, \tag{3.4}$$

and here f is given in the passage and we let

$$\rho'(t) = \langle\lambda_0\rangle_\omega - \lambda_0(t). \tag{3.5}$$

By integrating (3.5) from t to $t + \omega$, we get

$$\begin{aligned}\rho(t + \omega) - \rho(w) &= \int_t^{t+\omega} \rho'(s)ds = \int_t^{t+\omega} (\langle\lambda_0\rangle_\omega - \lambda_0(s))ds \\ &= \int_0^\omega \lambda_0(s)ds - \int_t^{t+\omega} \lambda_0(s)ds = 0.\end{aligned}$$

Obviously, $\rho(t)$ is an ω-periodic function on $[0, +\infty)$. So $V(t, S, I)$ is ω-periodic in t and the condition (3.2) in Lemma 3.4 holds.

Let $V_4 = -\log S$, $V_5 = M^*[-\log I - \psi(t)(S + I) - \rho(t)]$, $V_6 = (S + I + 1)^{\vartheta+1}$. Now we prove (3.3) in Lemma 3.4. In the same way as in Lemma 2.7, by directly calculating we have

$$\mathcal{L}V_4 = -\frac{\Lambda}{S} + \frac{\beta I}{1 + \alpha I} + \mu + \frac{\sigma_1^2}{2},$$

$$\begin{aligned}\mathcal{L}V_5 &= M^*\left[\frac{-\beta S}{1 + \alpha I} + (\mu + \varepsilon + \gamma) + \frac{\sigma_2^2}{2} - \psi(\Lambda - \mu S - (\mu + \varepsilon + \gamma)I) - \psi'(S + I) - \rho'\right] \\ &= M^*\left[\frac{-\beta S}{1 + \alpha I} + \beta S + [\beta + \psi(\varepsilon + \gamma)]I - \langle\lambda_0\rangle_\omega\right],\end{aligned}$$

$$\mathcal{L}V_6 = (\vartheta + 1)(S + I + 1)^\vartheta(\Lambda - \mu S - (\mu + \varepsilon + \gamma)I) + \frac{\vartheta(\vartheta + 1)}{2}(S + I + 1)^{\vartheta-1}(\sigma_1 S + \sigma_2 I)^2$$

$$
\begin{aligned}
&\le (\Lambda+\mu)(\vartheta+1)(S+I+1)^{\vartheta} - (\vartheta+1)\left(\mu - \frac{(\sigma_1\vee\sigma_2)^2}{2}\vartheta\right)(S+I+1)^{\vartheta+1}\\
&\le 2^{\vartheta}(\Lambda+\mu)(\vartheta+1)\Big[S^{\vartheta}+(I+1)^{\vartheta}\Big] - 2^{\vartheta+1}\left(\vartheta+1)(\mu - \frac{(\sigma_1\vee\sigma_2)^2}{2}\vartheta\right)\Big[S^{\vartheta+1}+(I+1)^{\vartheta+1}\Big].
\end{aligned}
$$

The above inequality generates

$$
\begin{aligned}
\mathcal{L}V \le& -\frac{\hat{\Lambda}}{S} + \check{\beta} I + \check{\mu} + \frac{\check{\sigma}_1^2}{2} + M^*\frac{\check{\alpha}\check{\beta}SI}{1+\hat{\alpha}I} + M^*[(\check{\beta}+\check{\psi}(\check{\varepsilon}+\check{\gamma}))I - \langle\lambda_0\rangle_\omega]\\
&+2^{\vartheta}(\check{\Lambda}+\check{\mu})(\vartheta+1)[S^{\vartheta}+(I+1)^{\vartheta}] - 2^{\vartheta+1}(\vartheta+1)\left(\hat{\mu} - \frac{(\check{\sigma}_1\vee\check{\sigma}_2)^2}{2}\vartheta\right)\left[S^{\vartheta+1}+(I+1)^{\vartheta+1}\right]\\
=& f(S)+g(I)+h(S,I),
\end{aligned}
$$

where

$$
f(S) = -\frac{\hat{\Lambda}}{S} + \check{\mu} + \frac{\check{\sigma}_1^2}{2} + 2^{\vartheta}(\check{\Lambda}+\check{\mu})(\vartheta+1)S^{\vartheta} - 2^{\vartheta+1}(\vartheta+1)\left(\hat{\mu} - \frac{(\check{\sigma}_1\vee\check{\sigma}_2)^2}{2}\vartheta\right)S^{\vartheta+1},
$$

$$
\begin{aligned}
g(I) =& \check{\beta} I + M^*[(\check{\beta}+\check{\psi}(\check{\varepsilon}+\check{\gamma}))I - \langle\lambda_0\rangle_\omega] + 2^{\vartheta}(\check{\Lambda}+\check{\mu})(\vartheta+1)(I+1)^{\vartheta}\\
&- 2^{\vartheta+1}(\vartheta+1)\left(\hat{\mu} - \frac{(\check{\sigma}_1\vee\check{\sigma}_2)^2}{2}\vartheta\right)(I+1)^{\vartheta+1},
\end{aligned}
$$

and $h(S,I) = M^*\frac{\check{\alpha}\check{\beta}SI}{1+\hat{\alpha}I}$. Define a closed set

$$
\mathcal{D} = \left\{(S,I)\in\mathbb{R}_+^2 : \epsilon \le S \le 1/\epsilon, \epsilon \le I \le 1/\epsilon\right\},
$$

where $\epsilon > 0$ is a sufficiently small number. By direct analysis, we get the following results:

Case 1. while $S \to 0$, we have $h(S,I)\to 0$, and $f(S)+\check{g} \to -\infty$;

Case 2. as $I \to 0$, then $h(S,I)\to 0$, and $\check{f}+g(I) \to \check{f} - M^*\langle\lambda_0\rangle_\omega + 2^{\vartheta}(\vartheta+1)[\check{\Lambda}+\check{\mu} - 2(\hat{\mu} - \frac{(\check{\sigma}_1\vee\check{\sigma}_2)^2}{2}\vartheta)]$;

Case 3. when $S\to+\infty$, we get $h(S,I) \le M^*\check{\beta}S$, moreover, $f(S) + M^*\check{\beta}S + \check{g} \to -\infty$;

Case 4. if $I \to +\infty$, then $h(S,I) \le M^*\check{\alpha}\check{\beta}S/\hat{\alpha}$. Let $f^*(S) = M^*\check{\alpha}\check{\beta}S/\hat{\alpha} + f(S)$, then $\check{f}^* + g(I) \to -\infty$;

Based on the discussion above and (3.4), we obtain that

$$
\mathcal{L}V(t,S,I) \le -1, \quad \text{for all} \quad (t,S,I)\in[0,+\infty)\times\mathcal{D}^{\mathcal{C}}.
$$

Therefor (3.3) in Lemma 3.4 is satisfied. Thus the system (1.5) has a positive ω-periodic solution from Lemma 3.4 and the proof is completed. □

4. Analysis

The threshold of deterministic SIR epidemic model (1.1) is $R_0 = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\gamma)}$. We denote

$$
R_0^s = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\gamma)} - \frac{\sigma_2^2}{2(\mu+\varepsilon+\gamma)}.
$$

For our stochastic SIR model (1.4), from Theorem 2.1 and Theorem 2.8 we obtain that if $R_0^s > 1$, the disease prevail; if $R_0^s < 1$, the disease extinct. The results are the same to the corresponding deterministic system. Besides, the system (1.4) has a stationary distribution. Without imposing any extra restricted condition, we obtain the threshold of system (1.3), which are beautiful results.

Non-autonomous system (1.5) is a more general type of (1.3), when we choose $\psi(t) = \beta/\mu$, and the coefficients are constant, the results of two systems are consistent. Furthermore, $\langle\lambda_0\rangle_\omega$ similarly determines the persistence or extinction of disease I in system (1.5). This theory can be used to investigate the other stochastic epidemic models.

It is worth to discuss the system with generalized non-linear incidence, but the support is a hard problem. In the future, we will try best to investigate this problem.

5. Appendix A

We will show some auxiliary definitions and results about Markov semigroups ([23],[24]). Assume the triple (X, Σ, m) is a σ-finite measure space. D is the subset of the space $L^1 = L^1(X, \Sigma, m)$, its definition can be found in (2.4). A linear mapping $P : L^1 \to L^1$ is called a Markov operator if $P(D) \subset D$.

Definition 5.1 ([23]). If there exists a measurable function $k : X \times X \to [0, \infty)$ such that

$$\int_X k(x, y)m(dx) = 1 \tag{5.1}$$

for all $y \in X$ and

$$Pf(x) = \int_X k(x, y)f(y)m(dy)$$

for every density f, then the Markov operator P is called an integral or kernel operator.

Definition 5.2 ([23]). A family $\{P(t)\}_{t\geq 0}$ of Markov operators which satisfies conditions:

(a) $P(0) = \mathrm{Id}$,
(b) $P(t + s) = P(t)P(s)$ for $s, t \geq 0$,
(c) for each $f \in L^1$ the function $t \mapsto P(t)f$ is continuous with respect to the L^1-norm,

is called a Markov semigroup. A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called integral, if for each $t > 0$, the operator $P(t)$ is an integral Markov operator.

There are some definitions about the asymptotic behaviour of a Markov semigroup. A density f_* is called invariant if $P(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for } f \in D.$$

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if for every $f \in D$

$$\lim_{t\to\infty} \int_A P(t)f(x)m(dx) = 0.$$

In Lemma 5.3, we will give the conclusion about asymptotic stability and sweeping.

Lemma 5.3 ([23]). *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be an integral Markov semigroup with a continuous kernel $k(t, x, y)$ for $t > 0$, which satisfies* (5.1) *for all $y \in X$. We assume that for every $f \in D$ we have*

$$\int_0^\infty P(t)f dt > 0 \quad a.e..$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

Remark 5.4 ([23]). The property that a Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping for a sufficiently large family of sets (e.g. for all compact sets) is called the Foguel alternative.

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