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Refinements of certain hyperbolic inequalities via the Padé approximation method

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Abstract

The aim of this paper is to deal with the refinements of certain inequalities for hyperbolic functions using Padé approximation method. We provide a useful way of improving the inequalities for trigonometric functions and hyperbolic functions.

Keywords: Padé approximation, Taylor expansion, continued fraction, hyperbolic inequalities, refinement.
2010 MSC: 41A21, 26D05, 26D15, 33B10, 11J10.

1. Introduction

The famous Wilker inequality for trigonometric functions asserts that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad 0 < x < \frac{\pi}{2},$$

while the Huygens trigonometric inequality states

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \quad 0 < x < \frac{\pi}{2}.$$

The following double inequality

$$\sqrt[3]{\cos x} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad 0 < x < \frac{\pi}{2}, \tag{1.1}$$

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has also attracted the attention of many authors in the recent past.

The left-hand side of (1.1) was discovered by Lazarević (see, [6, p.238]), while the right-hand side of (1.1) is the trigonometric Cusa-Huygens inequality (see [12]).

The hyperbolic counterparts of the Wilker and Huygens trigonometric inequalities have been introduced by Zhu [25, 26], Neumann and Sándor [11] as follows:

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2, \quad x \neq 0,$$

$$2\left(\frac{\sinh x}{x}\right) + \frac{\tanh x}{x} > 3, \quad x \neq 0.$$

The hyperbolic Cusa-Huygens inequality [11] and the hyperbolic Lazarević inequality [8] state that

$$\sqrt[3]{\cosh x} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3}, \quad x \neq 0. \tag{1.2}$$

These inequalities were proved by using the variation of some functions and their derivatives. Recently, some of the above inequalities have been improved by using the Taylor's expansion for hyperbolic functions (see [9]) as follows:

$$2\left(\frac{\sinh x}{x}\right) + \frac{\tanh x}{x} > 3 + \frac{3}{20}x^4 - \frac{3}{56}x^6, \quad x > 0,$$

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \frac{\sin x}{x} + \left(\frac{\tan \frac{x}{2}}{\frac{x}{2}}\right)^2 > 2,$$

and its hyperbolic counterpart (see [10])

$$\frac{\sinh x}{x} + \left(\frac{\tanh \frac{x}{2}}{\frac{x}{2}}\right)^2 > 2. \tag{1.3}$$

The following inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \in \mathbb{R}, \tag{1.4}$$

is known as Redheffer's inequality (see [13]).

By using the mathematical induction and the infinite product representation of $\sinh x$ and $\cosh x$, Chen et al. [2] found the hyperbolic analogue of inequality (1.4), by showing that

$$\frac{\sinh x}{x} \leq \frac{\pi^2 + x^2}{\pi^2 - x^2}, \quad |x| < \pi. \tag{1.5}$$

They also proved that

$$\cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad |x| \leq \frac{\pi}{2}. \tag{1.6}$$

Recently, Sándor and Bhayo [14] improved the inequality (1.5), by showing that

$$\frac{\sinh x}{x} \leq \frac{12 + x^2}{12 - x^2}, \quad |x| < \pi. \tag{1.7}$$

For more relevant papers on the topic, we refer the interested reader to [1, 3–5, 7, 15–24].

The aim of this paper is to prove and refine the aforesaid hyperbolic inequalities by using Padé approximation method.

It is known that a Padé approximation is the best approximation of a function by a rational function of given order. The rational approximation is particularly good for series with alternating terms and poor polynomial convergence. Optimal rational polynomials are frequently used in computer calculations, because they provide a good compromise in accuracy, size and speed.

The Padé approximation [L/M] corresponds to the Taylor series. When it exists, the [L/M] Padé approximation to any power series $A(x) = \sum_{j=0}^{\infty} a_j x^j$ is unique. If $A(x)$ is a transcendental function, then the terms are given by the Taylor series about x_0, $a_n = \frac{1}{n!} A^{(n)}(x_0)$.

The coefficients are found by setting $A(x) = \frac{p_0 + p_1 x + ... + p_L x^L}{1 + q_1 x + ... + q_M x^M}$. These give the set of equations

$$\begin{cases} p_0 = a_0, \\ p_1 = a_0 q_1 + a_1, \\ p_2 = a_0 q_2 + a_1 q_1 + a_2, \\ \vdots \\ p_L = a_0 q_L + ... + a_{L-1} q_1 + a_L, \\ 0 = a_{L-M+1} q_M + ... + a_L q_1 + a_{L+1}, \\ 0 = a_L q_M + ... + a_{L+M-1} q_1 + a_{L+M}. \end{cases}$$

For example, let us consider the Taylor series for cosh : $\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320} + O\left(x^{10}\right)$ and its associate polynomial: $1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320}$. The Padé approximation

$$\cosh_{[6/4]}(x) = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + p_6 x^6}{1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4},$$

satisfies

$$\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320}\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4\right) = p_0 + p_1 + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + p_6 x^6.$$

We find

$$p_1 = p_3 = p_5 = q_1 = q_3 = 0,\ q_2 = -\frac{1}{26},\ q_4 = \frac{1}{1456},\ p_0 = 1,\ p_2 = \frac{6}{13},\ p_4 = \frac{101}{4368},\ p_6 = \frac{17}{131040}.$$

Therefore we obtain

$$\begin{aligned} \cosh_{[6/4]}(x) &= \frac{1 + \frac{6}{13}x^2 + \frac{101}{4368}x^4 + \frac{17}{131040}x^6}{1 - \frac{1}{26}x^2 + \frac{1}{1456}x^4} \\ &= \frac{131040 + 60480x^2 + 3030x^4 + 17x^6}{131040 - 5040x^2 + 90x^4}. \end{aligned}$$

Similar calculations lead us to the following results

$$\tanh_{[3/2]}(x) = \frac{x^3 + 15x}{6x^2 + 15}, \quad \text{and} \quad \sinh_{[3/2]}(x) = \frac{7x^3 + 60x}{-3x^2 + 60}.$$

2. Some lemmas

In order to obtain our main results, we first prove several lemmas.

Lemma 2.1. *For every $x \in \mathbb{R}$, one has*

$$\cosh x > \frac{17x^6 + 3030x^4 + 60480x^2 + 131040}{90x^4 - 5040x^2 + 131040}. \tag{2.1}$$

Proof. First we remark that the denominator $90x^4 - 5040x^2 + 131040$ is positive for all $x \in \mathbb{R}$. We consider the function

$$f : (0,\infty) \to \mathbb{R}, \quad f(x) = \left(90x^4 - 5040x^2 + 131040\right)\cosh x - 17x^6 - 3030x^4 - 60480x^2 - 131040.$$

Elementary calculations reveal that

$$\begin{aligned}
f'(x) &= \left(360x^3 - 10080x\right)\cosh x + \left(90x^4 - 5040x^2 + 131040\right)\sinh x - 102x^5 - 12120x^3 - 120960x,\\
f^{(2)}(x) &= \left(90x^4 - 3960x^2 + 120960\right)\cosh x + \left(720x^3 - 20160x\right)\sinh x - 510x^4 - 36360x^2 - 120960,\\
f^{(3)}(x) &= \left(1080x^3 - 28080x\right)\cosh x + \left(90x^4 - 1800x^2 + 100800\right)\sinh x - 2040x^3 - 72720x,\\
f^{(4)}(x) &= \left(90x^4 + 1440x^2 + 72720\right)\cosh x + \left(1440x^3 - 31680x\right)\sinh x - 6120x^2 - 72720,\\
f^{(5)}(x) &= \left(1800x^3 - 28800x\right)\cosh x + \left(90x^4 + 5760x^2 + 41040\right)\sinh x - 12240x,\\
f^{(6)}(x) &= \left(90x^4 + 11160x^2 + 12240\right)\cosh x + \left(2160x^3 - 17280x\right)\sinh x - 12240,\\
f^{(7)}(x) &= \left(2520x^3 + 5040x\right)\cosh x + \left(90x^4 + 17640x^2 - 5040\right)\sinh x,\\
f^{(8)}(x) &= \left(90x^4 + 25200\right)\cosh x + \left(2880x^3 + 40320x\right)\sinh x.
\end{aligned}$$

We see that $f^{(8)}(x) > 0$ for all $x > 0$. Then $f^{(7)}$ is strictly increasing on $(0,\infty)$. As $f^{(7)}(0) = 0$, we get $f^{(7)} > 0$ on $(0,\infty)$. Continuing the algorithm, finally we obtain $f(x) > 0$ for all $x \in (0,\infty)$. Due to the form of function f, it follows that the inequality $f(x) > 0$ holds also for $x < 0$. The proof is completed. □

Lemma 2.2. *For every $x \neq 0$, one has*

$$\frac{10x^2 + 105}{x^4 + 45x^2 + 105} < \frac{\tanh x}{x} < \frac{x^2 + 15}{6x^2 + 15}. \tag{2.2}$$

Proof. We introduce the function $g : (0,\infty) \to \mathbb{R}$, $g(x) = \left(6x^2 + 15\right)\sinh x - \left(x^3 + 15x\right)\cosh x$.

Its derivative is

$$g'(x) = x\left[\left(-x^2 - 3\right)\sinh x + 3x\cosh x\right].$$

Then we consider the function $r : (0,\infty) \to \mathbb{R}$, $r(x) = \left(-x^2 - 3\right)\sinh x + 3x\cosh x$ and its derivative

$$r'(x) = x\left(\sinh x - x\cosh x\right).$$

The function $p(x) = \sinh x - x\cosh x$ has the derivative $p'(x) = -x\sinh x$, therefore $p'(x) < 0$ for all $x \in (0,\infty)$. Then p is strictly decreasing on $(0,\infty)$. As $p(0) = 0$, it follows $p < 0$ on $(0,\infty)$, hence $r'(x) < 0$ for every $x \in (0,\infty)$. Due to similar arguments, finally it results that $g(x) < 0$ for all $x \in (0,\infty)$.

For proving the first part of Lemma 2.2, we consider the function

$$s : (0,\infty) \to \mathbb{R}, \quad s(x) = \left(x^4 + 45x^2 + 105\right)\sinh x - \left(10x^3 + 105x\right)\cosh x,$$

and its derivative

$$\begin{aligned}
s'(x) &= x\left[\left(-6x^2 - 15\right)\sinh x + \left(x^3 + 15x\right)\cosh x\right]\\
&= -xg(x).
\end{aligned}$$

Since $g(x) < 0$ for all $x \in (0,\infty)$, it follows that $s' > 0$ on $(0,\infty)$, therefore s is strictly increasing on $(0,\infty)$. As $s(0) = 0$, we get $s(x) > 0$ for all $x \in (0,\infty)$.

We remark that if the inequality (2.2) is true for $x > 0$, then it holds clearly also for $x < 0$. This completes the proof. □

Remark 2.3. The idea to compare the function $\tanh x$ with the function $\dfrac{10x^3+105x}{x^4+45x^2+105}$ is given by the continued fraction representation of the hyperbolic tangent function:

$$\tanh x = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\dots}}}}.$$

We also have

Lemma 2.4.

(i) *For every $x \neq 0$, the inequality*

$$\frac{\sinh x}{x} > \frac{170x^8+32085x^6+922950x^4+7660800x^2+13759200}{90x^8-990x^6-86310x^4+5367600x^2+13759200},$$

holds.

(ii) *For every $|x| < \sqrt{20}$, one has*

$$\frac{\sinh x}{x} < \frac{7x^2+60}{-3x^2+60}. \tag{2.3}$$

Proof.

(i) By multiplying the inequalities of positive functions (2.1) and (2.2), we obtain the desired lower rational bound for the hyperbolic sine function.

(ii) We consider the function $t : \left(0,\sqrt{20}\right) \to \mathbb{R},\ t\left(x\right) = \left(-3x^2+60\right)\sinh x - 7x^3 - 60x$.

Its derivatives are

$$\begin{aligned}
t'\left(x\right) &= -6x\sinh x + \left(-3x^2+60\right)\cosh x - 21x^2 - 60,\\
t^{(2)}\left(x\right) &= \left(54-3x^2\right)\sinh x - 12x\cosh x - 42x,\\
t^{(3)}\left(x\right) &= -18x\sinh x + \left(42-3x^2\right)\cosh x - 42,\\
t^{(4)}\left(x\right) &= \left(24-3x^2\right)\sinh x - 24x\cosh x,\\
t^{(5)}\left(x\right) &= -30x\sinh x - 3x^2\cosh x.
\end{aligned}$$

We notice that $t^{(5)} < 0$ on $(0,\infty)$. Then the function $t^{(4)}$ is strictly decreasing for all $x \in (0,\infty)$. As $t^{(4)}\left(0\right) = 0$, we have $t^{(4)} < 0$ on $(0,\infty)$. By using the same arguments, finally we conclude that $t\left(x\right) < 0$ for all $x \in (0,\infty)$.

Since the inequality (2.3) is true for $x \in \left(0,\sqrt{20}\right)$, then it holds also for $x \in \left(-\sqrt{20},0\right)$. The proof is completed.

□

3. Main results

In this section we will formulate and prove the rational refinements of the aforesaid hyperbolic inequalities.

Firstly, we will refine the Mortici's improved version of hyperbolic Huygens inequality as follows:

Theorem 3.1. *For every $x \neq 0$, the following inequality holds true*

$$2\left(\frac{\sinh x}{x}\right)+\frac{\tanh x}{x}>\frac{P\left(x\right)}{Q\left(x\right)}>3+\frac{3}{20}x^{4}-\frac{3}{56}x^{6},$$

where

$$P\left(x\right)=170x^{12}+40185x^{10}+2384400x^{8}+52078950x^{6}+477711675x^{4}+1774143000x^{2}+2167074000,$$

and

$$Q\left(x\right)=45x^{12}+1530x^{10}-60705x^{8}+689850x^{6}+123119325x^{4}+591381000x^{2}+722358000.$$

Proof. We remark that if the inequality is true for $x>0$, then it holds clearly also for $x<0$, so it is sufficient to show only for $x>0$.

The first inequality is an easy consequence of Lemmas 2.2 and 2.4. The second inequality has the equivalent form

$$x^{8}\left(\frac{135}{56}x^{10}+\frac{1053}{14}x^{8}-\frac{194967}{56}x^{6}+46097x^{4}+\frac{52222365}{8}x^{2}+\frac{63118965}{4}\right)>0.$$

The polynomial function from the left-hand side has no real non-zero roots, hence the last inequality holds true for every $x \neq 0$. □

By using Padé approximation method, we also improve the hyperbolic version of Wilker inequality as follows:

Theorem 3.2. *For every $x \neq 0$, one has*

$$\left(\frac{\sinh x}{x}\right)^{2}+\frac{\tanh x}{x}>\frac{A\left(x\right)}{B\left(x\right)}>2,$$

where

$$\begin{aligned}A\left(x\right)=\;&28900x^{20}+12290400x^{18}+1836253725x^{16}+123257800125x^{14}\\&+4281358717125x^{12}+82293438024000x^{10}+892105608933000x^{8}\\&+5736412528560000x^{6}+22757784325920000x^{4}\\&+48057033024000000x^{2}+39756272774400000\end{aligned}$$

and

$$\begin{aligned}B\left(x\right)=\;&8100x^{20}+186300x^{18}-21724200x^{16}+463344300x^{14}\\&+48937656600x^{12}-865986754500x^{10}-16558594573500x^{8}\\&+24028516512000000x^{2}+19878136387200000.\end{aligned}$$

Proof. We need to prove only for $x>0$. The inequalities from Lemmas 2.2 and 2.4 lead us to the following rational inequality

$$\left(\frac{\sinh x}{x}\right)^{2}+\frac{\tanh x}{x}>\frac{\left(170x^{8}+32085x^{6}+922950x^{4}+7660800x^{2}+13759200\right)^{2}}{\left(90x^{8}-990x^{6}-86310x^{4}+5367600x^{2}+13759200\right)^{2}}+\frac{10x^{2}+105}{x^{4}+45x^{2}+105}.$$

The right-hand side of the above inequality takes the equivalent form $\frac{A\left(x\right)}{B\left(x\right)}$.

On the other hand, the inequality $\frac{A(x)}{B(x)} > 2$ becomes

$$\begin{aligned} &12700x^{20} + 11917800x^{18} + 1879702125x^{16} + 122331111525x^{14} \\ &\quad + 4183483403925x^{12} + 84025411533000x^{10} + 925222798080000x^{8} \\ &\quad + 3262053150720000x^{6} + 3533890913280000x^{4} > 0, \end{aligned}$$

which is obviously true for all $x \neq 0$. $\square$

Since the hyperbolic version of Lazarević's inequality (1.2) can be rewritten as

$$\left(\frac{\sinh x}{x}\right)^2 \left(\frac{\tanh x}{x}\right) > 1, \quad x \neq 0,$$

we will improve this result as follows:

Theorem 3.3. *For every $x \neq 0$, one has*

$$\left(\frac{\sinh x}{x}\right)^2 \left(\frac{\tanh x}{x}\right) > \frac{10x^6 + 120x^4 + 360x^2 + 105x^4 + 1260x^2 + 3780}{36x^4 + 1620x^2 + 3780} > 1.$$

Proof. We remark that if the inequality is true for $x > 0$, then it holds also for $x < 0$, so it is enough to prove only for $x > 0$. Arising from Taylor's expansion for the hyperbolic sine, the following estimate

$$\frac{\sinh x}{x} > 1 + \frac{x^2}{6},$$

holds for $x > 0$. By using also the estimate for hyperbolic tangent function obtained in Lemma 2.2, we have

$$\left(\frac{\sinh x}{x}\right)^2 \left(\frac{\tanh x}{x}\right) > E(x),$$

where

$$\begin{aligned} E(x) &= \left(1 + \frac{x^2}{6}\right)^2 \left(\frac{10x^2 + 105}{x^4 + 45x^2 + 105}\right) \\ &= \frac{10x^6 + 225x^4 + 1620x^2 + 3780}{36x^4 + 1620x^2 + 3780}. \end{aligned}$$

The inequality $E(x) > 1$ has the equivalent form $10x^6 + 225x^4 > 0$, which is obviously true for all $x \neq 0$. $\square$

We also will sharpen the hyperbolic Cusa-Huygens inequality as follows:

Theorem 3.4. *For every $x \neq 0$, one has*

$$\frac{\sinh x}{x} < \frac{x^2 + 15}{6x^2 + 15} \cosh x < \frac{\cosh x + 2}{3}.$$

Proof. Let us consider $x > 0$. From inequality (2.2) we have

$$\frac{\tanh x}{x} < \frac{x^2 + 15}{6x^2 + 15},$$

equivalently,

$$\frac{\sinh x}{x} < \frac{x^2 + 15}{6x^2 + 15} \cosh x.$$

We transform the inequality

$$\frac{x^2+15}{6x^2+15}\cosh x < \frac{\cosh x + 2}{3}$$

into the inequality

$$\left(10-x^2\right)\cosh x - 4x^2 - 10 < 0 \text{ for all } x > 0.$$

We introduce the function

$$h:(0,\infty)\to\mathbb{R}, \quad h(x) = \left(10-x^2\right)\cosh x - 4x^2 - 10.$$

Its derivatives are

$$\begin{aligned}
h'(x) &= -2x\cosh x + \left(10-x^2\right)\sinh x - 8x,\\
h^{(2)}(x) &= \left(8-x^2\right)\cosh x - 4x\sinh x - 8,\\
h^{(3)}(x) &= \left(4-x^2\right)\sinh x - 6x\cosh x,\\
h^{(4)}(x) &= \left(-2-x^2\right)\cosh x - 8x\sinh x.
\end{aligned}$$

We have $h^{(4)} < 0$ on $(0,\infty)$, hence $h^{(3)}$ is strictly decreasing on $(0,\infty)$. As $h^{(3)}(0) = 0$, it follows that $h^{(3)} < 0$ on $(0,\infty)$. By using the same arguments, finally we conclude that $h(x) < 0$ on $(0,\infty)$.

Since the inequalities from Theorem 3.4 are true for $x > 0$, then they hold clearly also for $x < 0$. □

In order to refine the inequality (1.3), we will prove the following.

Theorem 3.5. *For all $x \neq 0$, one has*

$$\frac{\sinh x}{x} + \left(\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\right)^2 > \frac{x^{10}+366x^8+37920x^6+828960x^4+7257600x^2+33868800}{6x^8+2160x^6+214560x^4+3628800x^2+16934400} > 2.$$

Proof. It is sufficient only to prove it for $x > 0$. From Taylor's expansion for the hyperbolic sine and from Lemma 2.2, we have

$$\begin{aligned}
\frac{\sinh x}{x} + \left(\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\right)^2 &> \frac{x^2+6}{6} + \left(\frac{10\left(\frac{x}{2}\right)^2+105}{\left(\frac{x}{2}\right)^4+45\left(\frac{x}{2}\right)^2+105}\right)^2\\
&= \frac{x^{10}+366x^8+37920x^6+828960x^4+7257600x^2+33868800}{6x^8+2160x^6+214560x^4+3628800x^2+16934400}.
\end{aligned}$$

The last rational expression is greater than 2 since its equivalent inequality

$$x^{10}+354x^8+33600x^6+399840x^4 > 0,$$

is obviously true for all $x \neq 0$ □

Finally, we also will improve the Redheffer-type inequalities (1.5), (1.6) and (1.7) as follows:

Theorem 3.6.

(i) *For all $|x| < \pi$, one has*

$$\frac{\sinh x}{x} < \frac{60+7x^2}{60-3x^2} < \frac{12+x^2}{12-x^2}.$$

(ii) *For all $|x| < \frac{\pi}{2}$, one has*

$$\cosh x < \frac{10+4x^2}{10-x^2} < \frac{\pi^2+4x^2}{\pi^2-4x^2}.$$

Proof.

(i) First inequality is a consequence of Lemma 2.4. We remark that the denominator $60-3x^2$ is positive for all $|x|<\pi$. It is easy to see that the difference

$$C(x)=\frac{12+x^2}{12-x^2}-\frac{7x^2+60}{60-3x^2},$$

has the equivalent positive form $\dfrac{4x^4}{(12-x^2)(60-3x^2)}$.

(ii) In the demonstration of Theorem 3.4, we showed that

$$\left(10-x^2\right)\cosh x-4x^2-10<0,\quad \text{for all } x\in\mathbb{R}.$$

Therefore we get

$$\cosh x<\frac{10+4x^2}{10-x^2},\quad \text{for all}\quad |x|<\frac{\pi}{2}.$$

The last inequality

$$\frac{10+4x^2}{10-x^2}<\frac{\pi^2+4x^2}{\pi^2-4x^2},$$

can be rewritten as the following true inequality

$$0<12x^4+\left(80-5\pi^2\right)x^2,\quad \text{for all}\quad |x|<\frac{\pi}{2}.$$

This completes the proof of Theorem 3.6.

□

4. Final remarks

Let us emphasize that the Padé approximation method was here applied for proving the refinements of some remarkable hyperbolic inequalities. We are convinced that Padé approximation method is suitable to establish many other similar inequalities.

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References

[1] Á. Baricz, *Redheffer type inequality for Bessel functions*, JIPAM. J. Inequal. Pure Appl. Math., **8** (2007), 6 pages. 1

[2] C. P. Chen, J. W. Zhao, F. Qi, *Three inequalities involving hyperbolically trigonometric functions*, RGMIA res. rep. coll., **6** (2003), 6 pages. 1

[3] H. H. Chu, Z. H. Yang, Y. M. Chu, W. Zhang, *Generalized Wilker-type inequalities with two parameters*, J. Inequal. Appl., **2016** (2016), 13 pages. 1

[4] W. D. Jiang, Q. M. Luo, F. Qi, *Refinements and sharpening of some Huygens and Wilker type inequalities*, Turkish J. Anal. Number Theory, **2** (2014), 134–139.

[5] R. Klén, M. Visuri, M. Vuorinen, *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl., **2010** (2010), 14 pages. 1

[6] I. Lazarević, *Sur une inégalité de Lochs*, (French) Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No., **1968** (1968), 55–56. 1

[7] Y. Lv, G. Wang, Y. Chu, *A note on Jordan type inequalities for hyperbolic functions*, Appl. Math. Lett., **25** (2012), 505–508. 1

[8] D. S. Mitrinović, *Analytic Inequalities*, In cooperation with P. M. Vasić, Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, New York-Berlin, (1970). 1

[9] C. Mortici, *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl., **14** (2011), 535–541. 1

[10] E. Neuman, *On Wilker and Huygens type inequalities*, Math. Inequal. Appl., **15** (2012), 271–279. 1

[11] E. Neuman, J. Sándor, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*, Math. Inequal. Appl., **13** (2010), 715–723. 1

[12] F. Qi, D. W. Niu, B. N. Guo, *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl., **2009** (2009), 52 pages. 1

[13] R. Redheffer, *Problem 5642*, Amer. Math. Monthly, **76** (1969), 422. 1

[14] J. Sándor, B. A. Bhayo, *On an inequality of Redheffer*, Miskolc Math. Notes, **16** (2015), 475–482. 1

[15] S. H. Wu, *On extension and refinement of Wilker's inequality*, Rocky Mountain J. Math., **39** (2009), 683–687. 1

[16] S. H. Wu, Á. Baricz, *Generalizations of Mitrinović, Adamović and Lazarević's inequalities and their applications*, Publ. Math. Debrecen, **75** (2009), 447–458.

[17] S. H. Wu, L. Debnath, *Wilker-type inequalities for hyperbolic functions*, Appl. Math. Lett., **25** (2012), 837–842.

[18] S. H. Wu, H. M. Srivastava, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transforms Spec. Funct., **18** (2007), 529–535.

[19] S. H. Wu, H. M. Srivastava, *A further refinement of Wilker's inequality*, Integral Transforms Spec. Funct., **19** (2008), 757–765.

[20] Z. H. Yang, Y. M. Chu, *Sharp Wilker-type inequalities with applications*, J. Inequal. Appl., **2014** (2014), 17 pages.

[21] Z. H. Yang, Y. M. Chu, *Jordan type inequalities for hyperbolic functions and their applications*, J. Funct. Spaces, **2015** (2015), 4 pages.

[22] Z. H. Yang, Y. M. Chu, *Lazarević and Cusa type inequalities for hyperbolic functions with two parameters and their applications*, J. Inequal. Appl., **2015** (2015), 19 pages.

[23] Z. H. Yang, Y. M. Chu, *A sharp double inequality involving trigonometric functions and its applications*, J. Math. Inequal., **10** (2016), 423–432.

[24] Z. H. Yang, Y. M. Chu, X. H. Zhang, *Sharp Cusa type inequalities with two parameters and their applications*, Appl. Math. Comput., **268** (2015), 1177–1198. 1

[25] L. Zhu, *On Wilker type inequalities*, Math. Inequal. Appl., **10** (2007), 727–731. 1

[26] L. Zhu, *Some new Wilker-type inequalities for circular and hyperbolic functions*, Abstr. Appl. Anal., **2009** (2009), 9 pages. 1