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Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions

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Abstract

In this paper, we investigate a new class of mixed initial value problems of Hadamard and Riemann-Liouville fractional integro-differential inclusions. The existence of solutions for convex valued (the upper semicontinuous) case is established by means of Krasnoselskii's fixed point theorem for multivalued maps and nonlinear alternative criterion, while the existence result for non-convex valued maps (the Lipschitz case) relies on a fixed point theorem due to Covitz and Nadler. Illustrative examples are also included.

Keywords: Fractional differential inclusions, Hadamard derivative, Riemann-Liouville derivative, fixed point theorem.
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1. Introduction and preliminaries

Fractional calculus remained a theoretical field of research until the last three decades. Afterwards, this branch of mathematics has found extensive applications in different fields of engineering and applied

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sciences such as electrical circuit [31], rotor-bearing system [10], finance system [29], biological system [35], thermoelectric system [17], reaction-diffusion system [18], etc. The memory (hereditary) property of fractional-order calculus provides a novel and better approach to model real-world phenomena than integer-order one such as viscoelasticity and diffusion processes [34, 43]. Fractional order systems and controllers have also drawn considerable attention [16] and the concept of designing controllers based on fractional-order systems has been developed, for example, sliding mode control of fractional-order chaotic systems [1, 2, 23, 39, 42]. Fractional order controllers contain some extra parameters for tuning, which can lead to a better closed loop performance. In [32], two novel nonlinear fractional-order sliding mode controllers for power angle response improvement of multi-machine power systems have been presented. Active disturbance rejection control for nonlinear fractional-order systems has been discussed in [19].

Initial and boundary value problems of fractional differential equations have been extensively studied by several researchers in recent years. A significant development in the theory and applications of fractional order differential inclusions has also been observed. Differential inclusions are regarded as generalization of differential equations and inequalities, and have very important and interesting applications in optimal control theory and stochastic processes [28]. In fact, the tools of differential inclusions facilitate the investigation of dynamical systems having velocities not uniquely determined by the state of the system. For details and examples of fractional order differential inclusions, we refer the reader to the works [3–7, 12, 14, 20, 33, 37, 38, 40, 41] and the references cited therein.

In this paper, motivated by recent interest in fractional order differential inclusions, we study a new class of mixed initial value inclusion problem involving Hadamard derivative and Riemann-Liouville fractional integrals. Precisely, we consider the following problem:

$$\begin{cases} D^{\alpha}\left(x(t)-\displaystyle\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))\right)\in F(t,x(t),Kx(t)), \quad t\in J:=[1,T], \\ x(1)=0, \end{cases} \tag{1.1}$$

where D^{α} denotes the Hadamard fractional derivative of order α, $0<\alpha\le 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\dots,\beta_m\}$, $F: J\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$), $h_i\in C(J\times\mathbb{R},\mathbb{R})$ with $h_i(1,0)=0$, $i=1,2,\dots,m$, and $Kx(t)=\int_1^t\varphi(t,s)x(s)ds$, $\varphi(t,s)\in C(J^2,\mathbb{R})$.

The main objective of the present work is to obtain sufficient criteria for existence of solutions for convex and non-convex multivalued maps involved in problem (1.1). Our main results are based on multivalued version of Krasnoselskii's fixed point theorem, nonlinear alternative criteria and a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. Our results are new and well-supported with examples.

Before proceeding further, we quickly recall some basic definitions [26].

The Hadamard derivative of fractional order q for a function $g:[1,\infty)\to\mathbb{R}$ is defined as

$$D^q g(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds,\quad n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot)=\log_e(\cdot)$.

The Hadamard fractional integral of order q for a function g is defined as

$$\mathcal{I}^q g(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds,\quad q>0,$$

provided the integral exists.

The Riemann-Liouville fractional integral of order $p>0$ of a continuous function $f:(1,\infty)\to\mathbb{R}$ is defined by

$$I^p f(t)=\frac{1}{\Gamma(p)}\int_1^t(t-s)^{p-1}f(s)ds,$$

provided the right-hand side is pointwise defined on $(1, \infty)$.

For further details of Hadamard fractional derivative and integral, we refer the reader to [8, 9, 22, 25].

In relation to problem (1.1), we need the following lemma [26].

Lemma 1.1. *Let $g : J \to \mathbb{R}$ and $h_i : J \times \mathbb{R} \to \mathbb{R}$ be continuous functions, $0 < \alpha \le 1$, $\beta_i > 0$, $i = 1, \dots, m$. Then the unique solution of the fractional initial value problem*

$$\begin{cases} D^\alpha \left(x(t) - \displaystyle\sum_{i=1}^m I^{\beta_i} h_i(t, x(t)) \right) = g(t), \quad t \in J, \\ x(1) = 0, \end{cases}$$

is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha - 1} g(s) \frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)), \ \ t \in J.$$

Now we outline some basic concepts of multivalued analysis [15, 24].

For a normed space $(\mathcal{A}, \| \cdot \|)$, let $\mathcal{P}_{cl}(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is closed}\}$, $\mathcal{P}_b(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is bounded}\}$, $\mathcal{P}_{cp}(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(\mathcal{A}) = \{\mathcal{A}_1 \in \mathcal{P}(\mathcal{A}) : \mathcal{A}_1 \text{ is compact and convex}\}$.

A multivalued map $G : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ is convex (closed) valued if $G(a)$ is convex (closed) for all $a \in \mathcal{A}$.

The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in $\mathcal{A}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathcal{A})$ (i.e., $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on $\mathcal{A}$ if for each $a_0 \in \mathcal{A}$, the set $G(a_0)$ is a nonempty closed subset of $\mathcal{A}$, and if for each open set $\mathcal{N}$ of $\mathcal{A}$ containing $G(a_0)$, there exists an open neighborhood $\mathcal{N}_0$ of a_0 such that $G(\mathcal{N}_0) \subseteq N$.

G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{A})$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $a_n \to a_*$, $b_n \to b_*$, $b_n \in G(a_n)$ imply $b_* \in G(a_*)$.

G has a fixed point if there is $a \in \mathcal{A}$ such that $a \in G(a)$. The fixed point set of the multivalued operator G will be denoted by $\text{Fix}(G)$.

A multivalued map $G : J \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $b \in \mathbb{R}$, the function $t \longmapsto d(b, G(t)) = \inf\{|b - c| : c \in G(t)\}$ is measurable.

We denote by $C(J, \mathbb{R})$ the Banach space of continuous functions from J into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in J} |x(t)|$. Let $L^1(J, \mathbb{R})$ be the Banach space of measurable functions $x : J \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_1^T |x(t)| dt$.

For each $y \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t), Kx(t)) \text{ for a.e. } t \in J\}.$$

Definition 1.2. A multivalued map $F : J \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$;

(ii) $x \longmapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in J$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x, y)\| = \sup\{|v| : v \in F(t, x, y)\} \le \varphi_\rho(t),$$

for all $|x|, |y| \le \rho$ and for a.e. $t \in J$.

2. Existence results

Definition 2.1. A function $x \in C^1(J,\mathbb{R})$ is called a solution of problem (1.1) if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,x(t),Kx(t))$, for a.e. $t \in J$ such that $x(1) = 0$ and

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t,x(t)), \ \ t \in J.$$

2.1. Convex valued case

This subsection is devoted to the case when the multivalued map in problem (1.1) is convex valued. We prove two existence results which rely on Krasnoselskii's fixed point theorem and nonlinear alternative criteria.

In the forthcoming analysis, we need the following known results.

Lemma 2.2 ([30])**.** *Let X be a Banach space. Let $F : J \times X \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_{F,x} : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X)), \quad x \mapsto (\Theta \circ S_{F,x})(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2.3 ([36, Krasnoselskii's fixed point theorem])**.** *Let X be a Banach space, $Y \in \mathcal{P}_{b,cl,c}(X)$ and $A, B : Y \to \mathcal{P}_{cp,c}(X)$ two multivalued operators. If the following conditions are satisfied*

(i) *$Ay + By \subset Y$ for all $y \in Y$;*

(ii) *A is contraction;*

(iii) *B is u.s.c and compact,*

then, there exists $y \in Y$ such that $y \in Ay + By$.

Lemma 2.4 ([21, Nonlinear alternative for Kakutani maps])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Our first result is based on Krasnoselskii's fixed point theorem for multivalued maps.

Theorem 2.5. *Assume that*

(A_1) *there exists a constant $L_0 > 0$ such that*

$$|h_i(t,x(t)) - h_i(t,y(t))| \le L_0|x(t) - y(t)|,$$

for $t \in J$ and $x, y \in \mathbb{R}$, $i = 1, 2, \ldots, m$;

(A_2) *there exist functions $\nu, \mu \in C(J,\mathbb{R}^+)$ such that*

$$\|F(t,x,y)\| \le \nu(t) + \mu(t)|y|,$$

for all $(t,x,y) \in J \times \mathbb{R}^2$ with $\varphi_0\|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] < 1$, where $\varphi_0 = \sup\{|\varphi(t,s)|\,;\,(t,s) \in J \times J\}$, and

$$\gamma = T\int_0^{\log T} u^{\alpha-1}e^{-u}du;$$

(A_3) *there exist functions* $\theta_i \in C(J, \mathbb{R}^+)$, $i = 1, 2, \ldots, m$, *such that*

$$|h_i(t, x)| \leq \theta_i(t), \quad \forall (t, x) \in J \times \mathbb{R}.$$

Then the problem (1.1) *has at least one solution on* J, *provided that*

$$L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} < 1.$$

Proof. Define an operator $\Omega_F : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ by

$$\Omega_F(x) = \left\{ \begin{array}{l} h \in C(J, \mathbb{R}) : \\ h(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} v(s) \frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t)), v \in S_{F,x}. \end{array} \right\} \tag{2.1}$$

We consider $B_R = \{x \in C(J, \mathbb{R}) : \|x\| \leq R\}$, where

$$R \geq \left(\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} \|\theta_i\| + \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} \|\nu\| \right) \Big/ \left(1 - \varphi_0 \|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} \right] \right).$$

We define an operator $\mathcal{Q} : B_R \to C(J, \mathbb{R})$ by

$$\mathcal{Q}x(t) = \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} h_i(s, x(s)) ds, \ \ t \in J,$$

and a multivalued operator $\mathcal{T} : B_R \to \mathcal{P}(C(J, \mathbb{R}))$ by

$$\mathcal{T}x(t) = \left\{ h \in C(J, \mathbb{R}) : h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} v(s) \frac{ds}{s}, \ v \in S_{F,x} \right\}.$$

In this way, the fractional differential inclusion (1.1) is equivalent to the inclusion problem $u \in \mathcal{Q}u + \mathcal{T}u$. We show that the operators $\mathcal{Q}$ and $\mathcal{T}$ satisfy the conditions of Lemma 2.3 on B_R.

First, we show that the operator $\mathcal{T}$ defines the multivalued operator $\mathcal{T} : B_R \to \mathcal{P}_{cp,c}(C(J, \mathbb{R}))$. Note that the operator $\mathcal{T}$ is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1(J, \mathbb{R})$ into $C(J, \mathbb{R})$, defined by

$$\mathcal{L}(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} v(s) \frac{ds}{s}.$$

Suppose that $x \in B_R$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t), Kx(t))$ for almost all $t \in J$. Since $F(t, x(t), Kx(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ is pointwise on J.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equicontinuous sequence. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then we have

$$\begin{aligned} |\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha - 1} v_n(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha - 1} v_n(s) \frac{ds}{s} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha - 1} - \left(\log \frac{t_1}{s}\right)^{\alpha - 1} \right] |v_n(s)| \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_1}{s}\right)^{\alpha-1}|v_n(s)|\frac{ds}{s}\\
&\le \frac{\|\nu\|+\|\mu\|\varphi_0 R(T-1)}{\Gamma(\alpha+1)}[(\log t_2)^\alpha-(\log t_1)^\alpha|+2(\log(t_2/t_1))^\alpha].
\end{aligned}
$$

We see that the right hand side of the above inequality tends to zero as $t_1 \to t_2$ independent of v_n. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equicontinuous and by using the Arzelá-Ascoli theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that, $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{T}(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{T}(x)$ is compact.

Now, we show that $\mathcal{T}(x)$ is convex for all $x \in C(J,\mathbb{R})$. Let $z_1, z_2 \in \mathcal{T}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$
z_i(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f_i(s)\frac{ds}{s},\quad i=1,2,
$$

for almost all $t \in J$. Let $0 \le \lambda \le 1$. Then, we have

$$
[\lambda z_1+(1-\lambda)z_2](t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[\lambda f_1(s)+(1-\lambda)f_2(s)]\frac{ds}{s}.
$$

Since F has convex values, so $S_{F,x}$ is convex and $\lambda f_1(s)+(1-\lambda)f_2(s) \in S_{F,x}$. Thus

$$
\lambda z_1+(1-\lambda)z_2 \in \mathcal{T}(x).
$$

Consequently, $\mathcal{T}$ is convex-valued. Obviously, $\mathcal{Q}$ is compact and convex-valued.

Next, we show that $\mathcal{Q}(x)+\mathcal{T}(x) \subset B_R$ for all $x \in B_R$. Suppose $x \in B_R$ and $h \in \mathcal{Q}$ are arbitrary elements. Choose $v \in S_{F,x}$ such that

$$
h(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}+\sum_{i=1}^m I^{\beta_i}h_i(t,x(t))
$$

for almost all $t \in J$. Hence we get

$$
\begin{aligned}
|h(t)| &\le \sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^t(t-s)^{\beta_i-1}|h_i(s,x(s))|ds+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|v(s)|\frac{ds}{s}\\
&\le \sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^t(t-s)^{\beta_i-1}|\theta_i(s)|ds+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(|\nu(s)|+|\mu(s)||Kx(s)|)\,\frac{ds}{s}\\
&\le \sum_{i=1}^m\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|\theta_i\|+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\|\nu\|+\varphi_0\|\mu\|R\left[\frac{\gamma}{\Gamma(\alpha)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]\\
&\le R.
\end{aligned}
$$

Hence $\|h\| \le R$, which means that $\mathcal{Q}(x)+\mathcal{T}(x) \subset B_R$ for all $x \in B_R$.

The rest of the proof consists of several steps and claims.

Step 1: We show that $\mathcal{Q}$ is a contraction on $C(J,\mathbb{R})$. This is a consequence of (A_1). Indeed, for $x, y \in C(J,\mathbb{R})$, we have

$$
\begin{aligned}
|\mathcal{Q}x(t)-\mathcal{Q}y(t)| &\le \sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^t(t-s)^{\beta_i-1}|h_i(s,x(s))-h_i(s,y(s))|ds\\
&\le L_0\|x-y\|\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^t(t-s)^{\beta_i-1}ds
\end{aligned}
$$

$$\leq L_0\|x-y\|\sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}.$$

Hence, by the given assumption, $\mathcal{Q}$ is a contraction mapping.

Step 2: $\mathcal{T}$ is compact and upper semicontinuous. This will be established in several claims.

Claim I: $\mathcal{T}$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. For each $h\in\mathcal{T}(x)$, $x\in B_R$, there exists $v\in S_{F,x}$ such that

$$h(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}.$$

Then we have

$$\|h\|\leq\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\|\nu\|+\|\mu\|\varphi_0 R\left[\frac{\gamma}{\Gamma(\alpha)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]$$

and thus the operator $\mathcal{T}(B_R)$ is uniformly bounded.

Claim II: $\mathcal{T}$ maps bounded sets into equicontinuous sets. Let $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and $x\in B_R$. Then we have

$$\begin{aligned}|\mathcal{T}x(\tau_2)-\mathcal{T}x(\tau_1)|&=\left|\frac{1}{\Gamma(\alpha)}\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}\right|\\&\leq\frac{\|\nu\|+\|\mu\|\varphi_0R(T-1)}{\Gamma(\alpha+1)}[(\log\tau_2)^\alpha-(\log\tau_1)^\alpha|+2(\log(\tau_2/\tau_1))^\alpha],\end{aligned}$$

which is independent of x and tends to zero as $\tau_2-\tau_1\to 0$. Thus, $\mathcal{T}$ is equicontinuous. So $\mathcal{T}$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{T}$ is compact on B_R.

Claim III: $\mathcal{T}$ has a closed graph. Let $x_n\to x_*, h_n\in\mathcal{T}(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\mathcal{T}(x_*)$. Associated with $h_n\in\mathcal{T}(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in J$,

$$h_n(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v_n(s)\frac{ds}{s}.$$

Thus it suffices to show that there exists $v_*\in S_{F,x_*}$ such that for each $t\in J$,

$$h_*(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v_*(s)\frac{ds}{s}.$$

Let us consider the linear operator $\Theta: L^1(J,\mathbb{R})\to C(J,\mathbb{R})$ given by

$$f\mapsto\Theta(f)(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}.$$

Observe that

$$\|h_n(t)-h_*(t)\|=\left\|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(v_n(s)-v_*(s))\frac{ds}{s}\right\|\to 0,$$

as $n\to\infty$. Thus, it follows by Lemma 2.2 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$h_*(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}v_*(s)\frac{ds}{s},$$

for some $v_*\in S_{F,x_*}$. Hence $\mathcal{T}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{T}$ is upper semicontinuous.

Thus, the operators $\mathcal{Q}$ and $\mathcal{T}$ satisfy all the conditions of Lemma 2.3 and hence its conclusion implies that $x\in\mathcal{Q}(x)+\mathcal{T}(x)$ has a solution in B_R. Therefore the boundary value problem (1.1) has a solution in B_r and the proof is completed. □

Example 2.6. Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential equation

$$\begin{cases} D^{1/2}\left(x(t) - \sum_{i=1}^{3} I^{(2i+1)/2} h_i(t, x(t))\right) \in F(t, x(t), Kx(t)), \quad t \in [1, e], \\ x(1) = 0, \end{cases} \tag{2.2}$$

where

$$h_1(t,x) = \frac{\log t}{4}\frac{|x|}{1+|x|}, \quad h_2(t,x) = \frac{\tan^{-1}|x|}{5(1+\log t)}, \quad h_3(t,x) = \frac{2e^{-t}}{3}\sin|x|.$$

(i) Cosider the multivalued map $F : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned} x \to F(t, x, Kx) = \Bigg[&(t^2+1)\frac{|x|}{3+|x|} + \frac{e^{-t}}{4}\int_1^t \frac{\cos^2(t-s)}{2}x(s)ds \\ &, \left(\sqrt{t}+\frac{1}{2}\right)e^{-x^2} + \frac{1}{2+\log t}\int_1^t \frac{\cos^2(t-s)}{2}x(s)ds\Bigg]. \end{aligned} \tag{2.3}$$

Here $\alpha = 1/2$, $\beta_1 = 3/2$, $\beta_2 = 5/2$, $\beta_3 = 7/2$, $m = 3$, $T = e$. With the given data, we find that $\varphi_0 = 1/2$, $|h_i(t, x) - h_i(t, y)| \le (1/4)|x - y|$, $i = 1, 2, 3$, which satisfies (A_1) with $L_0 = 1/4$. Since $\int_0^1 u^{-1/2}e^{-u}du = \sqrt{\pi}\text{erf}(1)$, where $\text{erf}(\cdot)$ is the Gauss error function, we have $\gamma = 4.06015694$. For $f \in F$, we have

$$\begin{aligned} |f| &\le \max\left((t^2+1)\frac{|x|}{3+|x|} + \frac{e^{-t}}{4}K|x|, \left(\sqrt{t}+\frac{1}{2}\right)e^{-x^2} + \frac{1}{2+\log t}K|x|\right) \\ &\le t^2 + 1 + \frac{|Kx|}{2+\log t}. \end{aligned}$$

Thus

$$\|F(t, x, y)\| \le t^2 + 1 + \frac{|y|}{2+\log t},$$

for all $(t, x, y) \in [1, e] \times \mathbb{R}^2$ with $\nu(t) = t^2 + 1$, $\mu(t) = 1/(2 + \log t)$. Then, we have

$$\varphi_0\|\mu\|\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] = 0.8547693548 < 1.$$

Hence, (A_2) is satisfied. It is easy to verify that $|h_1(t, x)| \le (\log t)/4$, $|h_2(t, x)| \le \pi/(10(1 + \log t))$ and $|h_3(t, x)| \le 2e^{-t}/3$. In addition, we can show that

$$L_0\sum_{i=1}^{3}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} = 0.8576592205 < 1.$$

Thus all conditions of Theorem 2.5 are satisfied. Therefore, by the conclusion of Theorem 2.5, the problem (2.2) with the $F(t, x, Kx)$ is given by (2.3) has at least one solution on $[1, e]$.

Now we make use of nonlinear alternative theorem to show the existence of solutions for problem (1.1).

Theorem 2.7. *Assume that:*

(H_0) $F : J \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ *is* L^1*-Carathéodory and has nonempty compact and convex values;*

(H_1) *there exist functions* $p_1, p_2 \in C(J, \mathbb{R}^+)$, *and* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *nondecreasing such that*

$$\|F(t, x, y)\| \le p_1(t)\psi(|x|) + p_2(t)|y|,$$

for each $(t, x, y) \in J \times \mathbb{R}^2$;

(H_2) *there exist functions* $q_i \in C(J,\mathbb{R}^+)$, *and* $\Omega_i : \mathbb{R}^+ \to \mathbb{R}^+$ *nondecreasing such that*

$$|h_i(t,x)| \le q_i(t)\Omega_i(|x|) \quad \textit{for each} \ (t,x) \in J \times \mathbb{R},\ i = 1,2,\dots,m;$$

(H_3) *there exists a number* $M_0 > 0$ *such that*

$$\frac{\left(1 - \|p_2\|\varphi_0\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]\right)M_0}{\sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(M_0) + \|p_1\|\psi(M_0)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}} > 1,$$

where $\gamma = T\int_0^{\log T} u^{\alpha-1}e^{-u}du$ *and* $\|p_2\|\varphi_0\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] < 1.$

Then the boundary value problem (1.1) *has at least one solution on* J.

Proof. Consider the operator $\Omega_F : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by (2.1). We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω_F is convex for each $x \in C(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F maps bounded sets (balls) into bounded sets in $C(J,\mathbb{R})$. For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C(J,\mathbb{R})$. Then, for each $h \in \Omega_F(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t)).$$

Then for $t \in J$ we have

$$\begin{aligned}
|h(t)| &\le \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^t (t-s)^{\beta_i-1}|h_i(s,x(s))|ds + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|v(s)|\frac{ds}{s}\\
&\le \sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(\|x\|)\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left[p_1(s)\psi(\|x\|) + p_2(s)\left|\int_1^s \phi(s,\tau)x(\tau)d\tau\right|\right]\frac{ds}{s}\\
&\le \sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(\|x\|) + \|p_1\|\psi(\|x\|)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \|p_2\|\varphi_0\|x\|\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].
\end{aligned}$$

Thus,

$$\|h\| \le \sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(r) + \|p_1\|\psi(r)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \|p_2\|\varphi_0 r\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].$$

Now we show that Ω_F maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. For each $h \in \Omega_F(x)$, we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| \le &\left|\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^{t_2}(t_2-s)^{\beta_1-1}h_i(s,x(s))ds - \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^{t_1}(t_1-s)^{\beta_i-1}h_i(s,x(s))ds\right|\\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}v(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}\right|
\end{aligned}$$

$$\leq \sum_{i=1}^{m} \frac{\|q_i\|\Omega_i(r)}{\Gamma(\beta_i)} \left\{ \int_1^{t_1} [(t_2-s)^{\beta_i-1} - (t_1-s)^{\beta_i-1}]ds + \int_{t_1}^{t_2} (t_2-s)^{\beta-1} ds \right\}$$
$$+ \frac{\|p_1\|\psi(r) + \|p_2\|\varphi_0 r(T-1)}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1} \right] \frac{1}{s} ds \right.$$
$$\left. + \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right|.$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that $\Omega_F : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F is upper semicontinuous. It is known [15, Proposition 1.2] that Ω_F will be upper semicontinuous if we prove that it has a closed graph, since Ω_F is already shown to be completely continuous. Thus we will prove that Ω_F has a closed graph. The proof is similar to that of Claim III of Theorem 2.5, and thus is omitted.

Finally, we show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \notin \Omega_F(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega_F(x)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$|x(t)| \leq \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} \|q_i\|\Omega_i(\|x\|) + \|p_1\|\psi(\|x\|)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \|p_2\|\varphi_0\|x\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].$$

Consequently, we get

$$\frac{\left(1 - \|p_2\|\varphi_0 \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]\right)\|x\|}{\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} \|q_i\|\Omega_i(\|x\|) + \|p_1\|\psi(\|x\|)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}} \leq 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega_F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.4), we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

Example 2.8. Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential equation

$$\begin{cases} D^{1/2}\left(x(t) - \sum_{i=1}^{4} I^{(2i+1/2}h_i(t,x(t))\right) \in F(t,x(t),Kx(t)), \quad t \in [1,e], \\ x(1) = 0, \end{cases} \tag{2.4}$$

where

$$h_i(t,x(t)) = \left(\frac{1}{i+\sqrt{3}\log t}\right)\left(\frac{x(t)}{25+i}\right).$$

(i) Consider the multi-valued map $F : [1,e] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned} x \to F(t,x,Kx) = \Bigg[&\frac{1}{5+t^2}\left(\frac{x^2}{1+|x|}+1\right) + \frac{1+\cos^2 t}{4}\int_1^t \frac{e^{1-st}}{5+st}x(s)ds \\ &, \frac{1}{8+3t^4}\left(\frac{x^2}{2+3|x|}+1\right) + \frac{1}{4+2e^{3t}}\int_1^t \frac{e^{1-st}}{5+st}x(s)ds \Bigg]. \end{aligned} \tag{2.5}$$

Here $\alpha = 1/2$, $\beta_1 = 3/2$, $\beta_2 = 5/2$, $\beta_3 = 7/2$, $\beta_4 = 9/2$, $m = 4$, $T = e$. With the given data, we find that $\varphi_0 = 1/6$, $|h_i(t,x)| \le (1/(i+\sqrt{3}\log t))(|x|/(25+i))$, $i = 1,2,3,4$ which satisfies (H_2) with $q_i(t) = 1/(i+\sqrt{3}\log t)$ and $\Omega_i(|x|) = |x|/(25+i)$, $i = 1,2,3,4$. Since $\int_0^1 u^{-1/2}e^{-u}du = \sqrt{\pi}\mathrm{erf}(1)$, where $\mathrm{erf}(\cdot)$ is the Gauss error function, we have $\gamma = 4.06015694$. For $f \in F$, we have

$$\begin{aligned}|f| &\le \max\left(\frac{1}{5+t^2}\left(\frac{x^2}{1+|x|}+1\right)+\frac{1+\cos^2 t}{4}K|x|, \frac{1}{8+3t^4}\left(\frac{x^2}{2+3|x|}+1\right)+\frac{1}{4+2e^{3t}}K|x|\right)\\ &\le \frac{1}{5+t^2}(|x|+1)+\frac{1+\cos^2 t}{4}|Kx|.\end{aligned}$$

Thus

$$\|F(t,x,y)\| \le \frac{1}{5+t^2}(|x|+1)+\frac{1+\cos^2 t}{4}|y|,$$

for all $(t,x,y) \in [1,e]\times\mathbb{R}^2$ with $p_1(t) = 1/(5+t^2)$, $p_2(t) = (1+\cos^2 t)/4$ and $\psi(|x|) = |x|+1$. Thus, the condition (H_1) is satisfied. Also we have $\|p_1\| = 1/6$, $\|p_2\| = 1/2$, $\|q_i\| = 1/i$, $i = 1,2,3,4$. We can find that there exists a positive $M_0 > 0.5527464752$ satisfying (H_3).

Thus all conditions of Theorem 2.7 are satisfied. Therefore, by the conclusion of Theorem 2.7, the problem (2.4) with the $F(t,x,Kx)$ is given by (2.5) has at least one solution on $[1,e]$.

2.2. The Lipschitz case

Here we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler.

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [27]).

Definition 2.9. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called:

(**a**) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \le \gamma d(x,y) \text{ for each } x,y \in X;$$

(**b**) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.10 ([13]). *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then* $\mathrm{Fix}(N) \neq \emptyset$.

Theorem 2.11. *Assume that:*

(B_1) $F : J\times\mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x,y) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x,y \in \mathbb{R}$;

(B_2) $H_d(F(t,x,y),F(t,\bar{x},\bar{y})) \le m(t)(|x-\bar{x}|+|y-\bar{y}|)$ *for almost all* $t \in J$ *and* $x,\bar{x},y,\bar{y} \in \mathbb{R}$ *with* $m \in C(J,\mathbb{R}^+)$ *and* $d(0,F(t,0,0)) \le m(t)$ *for almost all* $t \in J$.

Then the boundary value problem (1.1) *has at least one solution on J if*

$$\|m\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(1+\varphi_0(T-1)) < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J,\mathbb{R})$ by the assumption (B_1), so F has a measurable selection (see [11, Theorem III.6]). Now we show that the operator Ω_F, defined by (2.1) satisfies the assumptions of Lemma 2.10. To show that $\Omega_F(x) \in \mathcal{P}_{cl}((CJ,\mathbb{R}))$ for each $x \in C(J,\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C(J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$u_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x_n(t)).$$

As F has compact values, we pass to a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$, we have

$$v_n(t) \to v(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)).$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\delta < 1$ $(\delta := \|m\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(1+\varphi_0(T-1)))$ such that

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \leq \delta\|x - \bar{x}\| \quad \text{for each} \quad x, \bar{x} \in C^1(J,\mathbb{R}).$$

Let $x, \bar{x} \in C^1(J,\mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t), Kx(t))$ such that, for each $t \in J$,

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)).$$

By (B_2), we have

$$H_d(F(t, x, Kx), F(t, \bar{x}, K\bar{x})) \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|).$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|), \quad t \in J.$$

Define $V : J \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|)\}.$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t), K\bar{x}(t))$ is measurable ([11, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, \bar{x}(t), K\bar{x}(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|)$.

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_2(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)).$$

Thus,

$$\begin{aligned}|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |v_1(s) - v_2(s)|\frac{ds}{s} \\ &\leq \|m\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(1+\varphi_0(T-1))\|x - \bar{x}\|.\end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \|m\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(1+\varphi_0(T-1))\|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\Omega_F(x),\Omega_F(\bar{x})) \leq \delta\|x-\bar{x}\| \leq \|m\|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}(1+\varphi_0(T-1))\|x-\bar{x}\|.$$

Since Ω_F is a contraction, it follows by Lemma 2.10 that Ω_F has a fixed point x which is a solution of (1.1). This completes the proof. □

Example 2.12. Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential equation

$$\begin{cases} D^{1/2}\left(x(t)-\displaystyle\sum_{i=1}^{3} I^{(2i+1)/2}h_i(t,x(t))\right) \in F(t,x(t),Kx(t)), \quad t\in[1,e], \\ x(1)=0, \end{cases} \tag{2.6}$$

where

$$h_1(t,x)=\frac{\log t}{4}\frac{|x|}{1+|x|},\quad h_2(t,x)=\frac{\tan^{-1}|x|}{5(1+\log t)},\quad h_3(t,x)=\frac{2e^{-t}}{3}\sin|x|.$$

Let $F:[1,e]\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x\to F(t,x,Kx)=\left[0,\frac{|x|}{(\sqrt{2}+\log t)^2(3+|x|)}+\frac{1}{(\sqrt{2}+\log t)^2}\sin\left|\int_0^t e^{-\sqrt{t-s}}x(s)ds\right|+\frac{1}{9}\right]. \tag{2.7}$$

Then we have

$$\sup\{|x| : x\in F(t,x,Kx)\} \leq \frac{2}{(\sqrt{2}+\log t)^2}+\frac{1}{9},$$

and

$$H_d(F(t,x,Kx),F(t,\bar{x},K\bar{x})) \leq \frac{1}{(\sqrt{2}+\log t)^2}\left(|x-\bar{x}|+|Kx-K\bar{x}|\right).$$

Let $m(t)=1/((\sqrt{2}+\log t)^2)$. Then we have $H_d(F(t,x,Kx),F(t,\bar{x},K\bar{x}))\leq m(t)|x-\bar{x}|$ with $d(0,F(t,0,0))=1/9\leq m(t)$ and $\|m\|=1/2$. Further

$$\|m\|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}(1+\varphi_0(T-1))=0.8065487605<1.$$

Thus all the conditions of Theorem 2.11 are satisfied. Therefore, by the conclusion of Theorem 2.11, the problem (2.6) with $F(t,x,Kx)$ given by (2.7) has at least one solution on $[1,e]$.

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