Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5119–5135
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Convergence theorems for mixed type asymptotically nonexpansive mappings in the intermediate sense

Gurucharan S. Saluja[a], Mihai Postolache[b,c,*], Adrian Ghiura[c]

[a]Department of Mathematics, Govt. N. P. G. College of Science, Raipur-492010 (C.G.), India.

[b]China Medical University, No.91, Hsueh-Shih Road, Taichung, Taiwan.

[c]Department of Mathematics & Informatics, University "Politehnica" of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania.

Communicated by R. Saadati

Abstract

We provide a new two-step iteration scheme of mixed type for two asymptotically nonexpansive self mappings in the intermediate sense and two asymptotically nonexpansive non-self mappings in the intermediate sense and establish some strong and weak convergence theorems for mentioned scheme and mappings in uniformly convex Banach spaces. Our results extend and generalize the corresponding results of Chidume et al. [C. E. Chidume, E. U. Ofoedu, H. Zegeye, J. Math. Anal. Appl., **280** (2003), 364–374] and [C. E. Chidume, N. Shahzad, H. Zegeye, Numer. Funct. Anal. Optim., **25** (2004), 239–257], Guo et al. [W. Guo, W. Guo, Appl. Math. Lett., **24** (2011), 2181–2185] and [W. Guo, Y. J. Cho, W. Guo, Fixed Point Theory Appl., **2012** (2012), 15 pages], Saluja [G. S. Saluja, J. Indian Math. Soc. (N.S.), **81** (2014), 369–385], Schu [J. Schu, Bull. Austral. Math. Soc., **43** (1991), 153–159], Tan and Xu [K. K. Tan, H. K. Xu, J. Math. Anal. Appl., **178** (1993), 301–308], Wang [L. Wang, J. Math. Anal. Appl., **323** (2006), 550–557], Wei and Guo [S. I. Wei, W. Guo, Commun. Math. Res., **31** (2015), 149–160] and [S. Wei, W. Guo, J. Math. Study, **48** (2015), 256–264].

Keywords: Asymptotically nonexpansive self and non-self mapping in intermediate sense, new two-step iteration scheme of mixed type, common fixed point, uniformly convex Banach space, strong convergence, weak convergence.
2010 MSC: 47H09, 47H10, 65J15, 47J25.

1. Introduction and preliminaries

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a nonlinear mapping, then

*Corresponding author
Email addresses: saluja1963@gmail.com (Gurucharan S. Saluja), mihai@mathem.pub.ro (Mihai Postolache), adrianghiura25@gmail.com (Adrian Ghiura)

Received 2016-05-24

we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of four mappings S_1, S_2, T_1 and T_2 will be denoted by $F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)$.

A mapping $T\colon K \to K$ is said to be asymptotically nonexpansive in the intermediate sense [1] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T^n(x) - T^n(y)\| - \|x-y\|\Big) \le 0.$$

From the above definition, it follows that an asymptotically nonexpansive mapping must be an asymptotically nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example shows.

Example 1.1. Let $E = \mathbb{R}$ be a normed linear space and $K = [0, 1]$. For each $x \in K$, we define

$$T(x) = \begin{cases} kx, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0,\end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n|x-y| \le |x-y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\limsup_{n\to\infty}\{|T^n x - T^n y| - |x-y|\} = \limsup_{n\to\infty}\{k^n|x-y| - |x-y|\} \le 0,$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$, for all $x, y \in K$, $n \in \mathbb{N}$ and T is continuous. Hence T is an asymptotically nonexpansive mapping in the intermediate sense.

Example 1.2. Let $E = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|\lambda| < 1$. For each $x \in K$, consider

$$T(x) = \begin{cases} \lambda\, x \sin(1/x), & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

Clearly $F(T) = \{0\}$. Since

$$Tx = \lambda\, x \sin(1/x), \qquad T^2 x = \lambda^2\, x \sin(1/x) \sin\Big(\frac{1}{\lambda\, x\, \sin(1/x)}\Big), \ \ldots,$$

we obtain $\{T^n x\} \to 0$ uniformly on K as $n \to \infty$. Thus

$$\limsup_{n\to\infty}\Big\{\|T^n x - T^n y\| - \|x-y\| \vee 0\Big\} = 0$$

for all $x, y \in K$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense (ANI in short), but it is not a Lipschitz mapping. In fact, suppose that there exists $\lambda > 0$ such that $|Tx - Ty| \le \lambda|x-y|$ for all $x, y \in K$. If we take $x = \frac{2}{5\pi}$ and $y = \frac{2}{3\pi}$, then

$$|Tx - Ty| = \Big|\lambda\,\frac{2}{5\pi}\,\sin\Big(\frac{5\pi}{2}\Big) - \lambda\,\frac{2}{3\pi}\,\sin\Big(\frac{3\pi}{2}\Big)\Big| = \frac{16\lambda}{15\pi},$$

whereas

$$\lambda|x-y| = \lambda\Big|\frac{2}{5\pi} - \frac{2}{3\pi}\Big| = \frac{4\lambda}{15\pi},$$

and hence it is not an asymptotically nonexpansive mapping.

Definition 1.3. A subset K of a Banach space E is said to be a retract of E if there exists a continuous mapping $P\colon E \to K$ (called a retraction) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E.

If $P\colon E \to K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

In 2004, Chidume et al. [3] introduced the concept of non-self asymptotically nonexpansive mappings in the intermediate sense as follows.

Definition 1.4. Let K be a nonempty subset of a real Banach space E and let $P\colon E \to K$ be a nonexpansive retraction of E onto K. A non-self mapping $T\colon K \to E$ is said to be asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| - \|x-y\| \Big) \le 0.$$

For the sake of convenience, we restate the following concepts and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{ 1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\| \Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Let $\mathcal{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has:

(i) Gâteaux differentiable norm if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $\mathcal{S}$.

(ii) Fréchet differentiable norm [11] if for each x in $\mathcal{S}$, the above limit exists and is attained uniformly for y in $\mathcal{S}$ and in this case, it is also well-known that

$$\begin{aligned} \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 &\le \frac{1}{2}\|x+h\|^2 \\ &\le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|x\|) \end{aligned} \tag{1.1}$$

for all $x,\ h \in E$, where J is the Fréchet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $x \in E$, $\langle \cdot\,\cdot\rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\to 0} \frac{b(t)}{t} = 0$.

(iii) Opial condition [7] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \text{ for all } y \in E \text{ with } y \ne x.$$

Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \ne 2$ fails to satisfy Opial condition.

A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A mapping $T\colon K \to K$ is said to be completely continuous if $\{Tx_n\}$ has a convergent subsequence in K whenever $\{x_n\}$ is bounded in K.

A Banach space E has the Kadec-Klee property [10] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

In 2003, Chidume et al. [2] studied the following iteration process for non-self asymptotically nonexpansive mappings:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \ge 1, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and proved some strong and weak convergence theorems in the framework of uniformly convex Banach spaces.

In 2004, Chidume et al. [3] studied the following iteration scheme:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and K is a nonempty closed convex subset of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto K, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

In 2006, Wang [13] generalized the iteration process (1.2) as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{aligned}$$

where $T_1, T_2 \colon K \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings.

In 2012, Guo et al. [5] generalized the iteration process (1.3) as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $S_1, S_2 \colon K \to K$ are two asymptotically nonexpansive self mappings and $T_1, T_2 \colon K \to E$ are two asymptotically nonexpansive non-self-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings. Recently, Wei and Guo [15] studied the following.

Let E be a real Banach space, K a nonempty closed convex subset of E and $P \colon E \to K$ a nonexpansive retraction of E onto K. Let $S_1, S_2 \colon K \to K$ be two asymptotically nonexpansive self mappings and $T_1, T_2 \colon K \to E$ two asymptotically nonexpansive non-self-mappings. Then Wei and Guo [15] defined the new iteration scheme of mixed type with mean errors as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P(\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1}y_n + \gamma_n u_n), \\ y_n &= P(\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1}x_n + \gamma'_n u'_n), \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$, and proved some weak convergence theorems in the setting of real uniformly convex Banach spaces. If $\gamma_n = \gamma'_n = 0$, for all $n \geq 1$, then the iteration scheme (1.5) reduces to the scheme (1.4).

The purpose of this paper is to study iteration scheme (1.5) for the mixed type asymptotically nonexpansive mappings in the intermediate sense which is more general than the class of asymptotically nonexpansive mappings in uniformly convex Banach spaces and establish some strong and weak convergence theorems for mentioned scheme and mappings.

Next we state the following useful lemmas to prove our main results.

Lemma 1.5 ([12])**.** Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then

(i) $\lim_{n\to\infty} \alpha_n$ exists;

(ii) in particular, if $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.6 ([9])**.** Let E be a uniformly convex Banach space and $0 < \alpha \le t_n \le \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.7 ([10])**.** Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p,\, q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.

Lemma 1.8 ([10])**.** Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi\colon [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each Lipschitzian mapping $T\colon K \to K$ with the Lipschitz constant L,

$$\|tTx + (1-t)Ty - T(tx + (1-t)y)\| \le L\phi^{-1}\Big(\|x-y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all $x,\, y \in K$ and all $t \in [0,1]$.

2. Strong convergence theorems

In this section, we prove some strong convergence theorems of iteration scheme (1.5) for the mixed type asymptotically nonexpansive mappings in the intermediate sense in real uniformly convex Banach spaces. First, we shall need the following lemma.

Lemma 2.1. Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E. Let $S_1,\, S_2\colon K \to K$ be two asymptotically nonexpansive self-mappings in the intermediate sense and $T_1,\, T_2\colon K \to E$ two asymptotically nonexpansive non-self-mappings in the intermediate sense. Put

$$G_n = \max\Big\{0, \sup_{x,\, y\in K,\, n\ge 1} \Big(\|S_1^n x - S_1^n y\| - \|x-y\|\Big), \sup_{x,\, y\in K,\, n\ge 1} \Big(\|S_2^n x - S_2^n y\| - \|x-y\|\Big)\Big\}, \tag{2.1}$$

and

$$\begin{aligned} H_n = \max\Big\{0, &\sup_{x,\, y\in K,\, n\ge 1} \Big(\|T_1(PT_1)^{n-1}(x) - T_1(PT_1)^{n-1}(y)\| - \|x-y\|\Big), \\ &\sup_{x,\, y\in K,\, n\ge 1} \Big(\|T_2(PT_2)^{n-1}(x) - T_2(PT_2)^{n-1}(y)\Big)\Big\}, \end{aligned} \tag{2.2}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. Let $\{x_n\}$ be the sequence defined by (1.5), where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \ge 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Assume that $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \ne \emptyset$. Then $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} d(x_n, F)$ both exist for any $q \in F$.

Proof. Let $q \in F$. From (1.5), (2.1) and (2.2), we have

$$\begin{aligned} \|y_n - q\| &= \|P(\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n) - P(q)\| \\ &\le \|\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n - q\| \\ &\le \alpha'_n \|S_2^n x_n - q\| + \beta'_n \|T_2(PT_2)^{n-1} x_n - q\| + \gamma'_n \|u'_n - q\| \\ &\le \alpha'_n [\|x_n - q\| + G_n] + \beta'_n [\|x_n - q\| + H_n] + \gamma'_n \|u'_n - q\| \\ &\le (\alpha'_n + \beta'_n)\|x_n - q\| + G_n + H_n + \gamma'_n \|u'_n - q\| \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&= (1-\gamma'_n)\|x_n - q\| + G_n + H_n + \gamma'_n\|u'_n - q\| \\
&\le \|x_n - q\| + A_n,
\end{aligned}
$$

where $A_n = G_n + H_n + \gamma'_n\|u'_n - q\|$. Since by assumptions $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} H_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$. Again from (1.5), (2.1) and (2.2), we have

$$
\begin{aligned}
\|x_{n+1} - q\| &= \|P(\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n) - P(q)\| \\
&\le \|\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n - q\| \\
&\le \alpha_n\|S_1^n x_n - q\| + \beta_n\|T_1(PT_1)^{n-1} y_n - q\| + \gamma_n\|u_n - q\| \\
&\le \alpha_n[\|x_n - q\| + G_n] + \beta_n[\|y_n - q\| + H_n] + \gamma_n\|u_n - q\| \\
&\le \alpha_n\|x_n - q\| + G_n + H_n + \beta_n\|y_n - q\| + \gamma_n\|u_n - q\|.
\end{aligned} \tag{2.4}
$$

By using equation (2.3) in (2.4), we obtain

$$
\begin{aligned}
\|x_{n+1} - q\| &\le \alpha_n\|x_n - q\| + \beta_n[\|x_n - q\| + A_n] \\
&\quad + G_n + H_n + \gamma_n\|u_n - q\| \\
&= (\alpha_n + \beta_n)\|x_n - q\| + \beta_n A_n + G_n + H_n + \gamma_n\|u_n - q\| \\
&= (1-\gamma_n)\|x_n - q\| + \beta_n A_n + G_n + H_n + \gamma_n\|u_n - q\| \\
&\le \|x_n - q\| + B_n,
\end{aligned} \tag{2.5}
$$

where $B_n = \beta_n A_n + G_n + H_n + \gamma_n\|u_n - q\|$. Since by hypothesis $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} H_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} A_n < \infty$, it follows that $\sum_{n=1}^{\infty} B_n < \infty$. For any $q \in F$, from (2.5), we obtain the following inequality

$$
d(x_{n+1}, F) \le d(x_n, F) + B_n. \tag{2.6}
$$

By applying Lemma 1.5 in (2.5) and (2.6), we have $\lim_{n\to\infty}\|x_n - q\|$ and $d(x_n, F)$ both exist. This completes the proof. □

Lemma 2.2. Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E. Let $S_1, S_2 \colon K \to K$ be two asymptotically nonexpansive self-mappings in the intermediate sense and $T_1, T_2 \colon K \to E$ two asymptotically nonexpansive non-self-mappings in the intermediate sense and G_n and H_n be taken as in Lemma 2.1. Assume that $F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.5), where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \ge 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. If the following conditions hold:

(i) $\{\beta_n\}$ and $\{\beta'_n\}$ are real sequences in $[\rho, 1-\rho]$ for all $n \ge 1$ and for some $\rho \in (0, 1)$,

(ii) $\|x - T_i(PT_i)^{n-1} y\| \le \|S_i^n x - T_i(PT_i)^{n-1} y\|$ for all $x, y \in K$ and $i = 1, 2$,

then $\lim_{n\to\infty}\|x_n - S_i x_n\| = \lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for $i = 1, 2$.

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$ and therefore $\{x_n\}$ is bounded. Thus, there exists a real number $r > 0$ such that $\{x_n\} \subseteq K' = \overline{B_r(0)} \cap K$, so that K' is a closed convex subset of K. Let $\lim_{n\to\infty}\|x_n - q\| = c$. Then $c > 0$ otherwise there is nothing to prove. Now (2.3) implies that

$$
\limsup_{n\to\infty}\|y_n - q\| \le c. \tag{2.7}
$$

Also, we have

$$
\|S_2^n x_n - q\| \le \|x_n - q\| + G_n, \quad \forall\, n \ge 1,
$$

$$\|T_2(PT_2)^{n-1}x_n - q\| \le \|x_n - q\| + H_n, \quad \forall\, n \ge 1,$$

and

$$\|S_1^n x_n - q\| \le \|x_n - q\| + G_n, \quad \forall\, n \ge 1.$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} \|S_2^n x_n - q\| &\le c,\\ \limsup_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - q\| &\le c,\end{aligned}$$

and

$$\limsup_{n\to\infty} \|S_1^n x_n - q\| \le c. \tag{2.8}$$

Next,

$$\|T_1(PT_1)^{n-1}y_n - q\| \le \|y_n - q\| + H_n,$$

gives by virtue of (2.6) that

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - q\| \le c.$$

Also, it follows from

$$\begin{aligned}c &= \lim_{n\to\infty} \|x_{n+1} - q\|\\ &= \lim_{n\to\infty} \|\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1}y_n + \gamma_n u_n - q\|\\ &= \lim_{n\to\infty} \|\alpha_n[S_1^n x_n - q + \frac{\gamma_n}{2\alpha_n}(u_n - q)]\\ &\quad + \beta_n[T_1(PT_1)^{n-1}y_n - q + \frac{\gamma_n}{2\beta_n}(u_n - q)]\|\\ &= \lim_{n\to\infty} \|\alpha_n[S_1^n x_n - q + \frac{\gamma_n}{2\alpha_n}(u_n - q)]\\ &\quad + (1-\alpha_n)[T_1(PT_1)^{n-1}y_n - q + \frac{\gamma_n}{2\beta_n}(u_n - q)]\|,\end{aligned}$$

and Lemma 1.6 that

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}y_n + \Big(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n}\Big)(u_n - q)\| = 0.$$

Since $\lim_{n\to\infty} \|\Big(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n}\Big)(u_n - q)\| = 0$, we obtain that

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{2.9}$$

By condition (ii), it follows that

$$\|x_n - T_1(PT_1)^{n-1}y_n\| \le \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|,$$

and so, from (2.9), we have

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{2.10}$$

Now

$$\begin{aligned}\|x_{n+1} - q\| &= \|\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1}y_n + \gamma_n u_n - q\|\\ &= \|\alpha_n(S_1^n x_n - q) + \beta_n(T_1(PT_1)^{n-1}y_n - S_1^n x_n) + \gamma_n(u_n - S_1^n x_n)\|\\ &\le \alpha_n\|S_1^n x_n - q\| + \beta_n\|T_1(PT_1)^{n-1}y_n - S_1^n x_n\| + \gamma_n\|u_n - S_1^n x_n\|\|,\end{aligned}$$

yields that

$$c \le \liminf_{n\to\infty} \|S_1^n x_n - q\|,$$

so that (2.8) gives

$$\lim_{n\to\infty} \|S_1^n x_n - q\| = c.$$

On the other hand,

$$\begin{aligned}\|S_1^n x_n - q\| &\le \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\| + \|T_1(PT_1)^{n-1} y_n - q\| \\ &\le \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\| + \|y_n - q\| + H_n,\end{aligned}$$

so, we have

$$c \le \liminf_{n\to\infty} \|y_n - q\|. \tag{2.11}$$

By using (2.7) and (2.11), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = c.$$

Thus

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - q\| \\ &= \lim_{n\to\infty} \|\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n - q\| \\ &= \lim_{n\to\infty} \|\alpha'_n [S_2^n x_n - q + \frac{\gamma'_n}{2\alpha'_n}(u'_n - q)] \\ &\quad + \beta'_n [T_2(PT_2)^{n-1} x_n - q + \frac{\gamma'_n}{2\beta'_n}(u'_n - q)]\| \\ &= \lim_{n\to\infty} \|\alpha'_n [S_2^n x_n - q + \frac{\gamma'_n}{2\alpha'_n}(u'_n - q)] \\ &\quad + (1-\alpha'_n)[T_2(PT_2)^{n-1} x_n - q + \frac{\gamma'_n}{2\beta'_n}(u'_n - q)]\|, \end{aligned}$$

and Lemma 1.6 implies that

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1} x_n + \Big(\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n}\Big)(u'_n - q)\| = 0.$$

Since $\lim_{n\to\infty} \|\Big(\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n}\Big)(u'_n - q)\| = 0$, we obtain that

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1} x_n\| = 0. \tag{2.12}$$

By condition (ii), it follows that

$$\|x_n - T_2(PT_2)^{n-1} x_n\| \le \|S_2^n x_n - T_2(PT_2)^{n-1} x_n\|,$$

and so, from (2.12), we have

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1} x_n\| = 0. \tag{2.13}$$

Since $S_2^n x_n = P(S_2^n x_n)$ and $P\colon E \to K$ is a nonexpansive retraction of E onto K, we have

$$\begin{aligned}\|y_n - S_2^n x_n\| &= \|\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n - S_2^n x_n\| \\ &= \|(1-\beta'_n-\gamma'_n) S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n - S_2^n x_n\|\end{aligned}$$

$$\leq \beta_n' \|S_2^n x_n - T_2(PT_2)^{n-1} x_n\| + \gamma_n' \|u_n' - S_2^n x_n\|,$$

and so

$$\lim_{n\to\infty} \|y_n - S_2^n x_n\| = 0. \tag{2.14}$$

Again, we have

$$\|y_n - x_n\| \leq \|y_n - S_2^n x_n\| + \|S_2^n x_n - T_2(PT_2)^{n-1} x_n\| + \|T_2(PT_2)^{n-1} x_n - x_n\|.$$

Thus, it follows from (2.12), (2.13) and (2.14) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.15}$$

Since $\|x_n - T_1(PT_1)^{n-1} x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\|$ by condition (ii) and

$$\begin{aligned}\|S_1^n x_n - T_1(PT_1)^{n-1} x_n\| &\leq \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\| \\ &\quad + \|T_1(PT_1)^{n-1} y_n - T_1(PT_1)^{n-1} x_n\| \\ &\leq \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\| + \|y_n - x_n\| + H_n.\end{aligned}$$

By using (2.9), (2.15) and $H_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1} x_n\| = 0, \tag{2.16}$$

and

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1} x_n\| = 0. \tag{2.17}$$

It follows from

$$\begin{aligned}\|x_{n+1} - S_1^n x_n\| &= \|P(\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n) - P(S_1^n x_n)\| \\ &\leq \|\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n - S_1^n x_n\| \\ &\leq \beta_n \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\| + \gamma_n \|u_n - S_1^n x_n\|,\end{aligned}$$

and (2.9) that

$$\lim_{n\to\infty} \|x_{n+1} - S_1^n x_n\| = 0. \tag{2.18}$$

In addition, we have

$$\|x_{n+1} - T_1(PT_1)^{n-1} y_n\| \leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\|.$$

By using (2.9) and (2.18), we have

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1} y_n\| = 0. \tag{2.19}$$

Now, by using (2.16), (2.17) and the inequality

$$\|S_1^n x_n - x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1} x_n\| + \|T_1(PT_1)^{n-1} x_n - x_n\|,$$

we have $\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0$. It follows from (2.13) and the inequality

$$\|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1} x_n\|,$$

that

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| = 0. \tag{2.20}$$

Since

$$\|x_{n+1} - T_2(PT_2)^{n-1}y_n\| \le \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + \|x_n - y_n\| + H_n,$$

from (2.15), (2.18), (2.20) and $H_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - T_2(PT_2)^{n-1}y_n\| = 0. \tag{2.21}$$

Since T_i for $i = 1, 2$ is uniformly continuous, P is nonexpansive retraction, it follows from (2.21) that

$$\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\| = \|T_i[(PT_i)(PT)^{n-2})y_{n-1}] - T_i(Px_n)\| \to 0 \quad \text{as} \quad n \to \infty \tag{2.22}$$

for $i = 1, 2$. Moreover, we have

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\| + \|x_n - y_n\|.$$

By using (2.10), (2.15) and (2.19), we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.23}$$

In addition, we have

$$\begin{aligned}\|x_n - T_1 x_n\| &\le \|x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_{n-1}\| \\ &\quad + \|T_1(PT_1)^{n-1}y_{n-1} - T_1 x_n\| \\ &\le \|x_n - T_1(PT_1)^{n-1}x_n\| + \|x_n - y_{n-1}\| + H_n \\ &\quad + \|T_1(PT_1)^{n-1}y_{n-1} - T_1 x_n\|.\end{aligned}$$

Thus, it follows from (2.17), (2.22), (2.23) and $H_n \to 0$ as $n \to \infty$, that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0.$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0.$$

Finally, we have

$$\begin{aligned}\|x_n - S_1 x_n\| &\le \|x_n - T_1(PT_1)^{n-1}x_n\| + \|S_1 x_n - T_1(PT_1)^{n-1}x_n\| \\ &\le \|x_n - T_1(PT_1)^{n-1}x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| \quad (\text{by condition (ii)}).\end{aligned}$$

Thus, it follows from (2.16) and (2.17) that

$$\lim_{n\to\infty} \|x_n - S_1 x_n\| = 0.$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - S_2 x_n\| = 0.$$

This completes the proof. □

Theorem 2.3. Under the assumptions of Lemma 2.2, if one of S_1, S_2, T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.

Proof. Without loss of generality we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 2.1, there exists a subsequence $\{S_1x_{n_k}\}$ of $\{S_1x_n\}$ such that $\{S_1x_{n_k}\}$ converges strongly to some q_1. Moreover, by definition of complete continuity and from Lemma 2.2, we have

$$\lim_{k\to\infty}\|x_{n_k}-S_1x_{n_k}\|=\lim_{k\to\infty}\|x_{n_k}-S_2x_{n_k}\|=0,$$

and

$$\lim_{k\to\infty}\|x_{n_k}-T_1x_{n_k}\|=\lim_{k\to\infty}\|x_{n_k}-T_2x_{n_k}\|=0,$$

which implies that

$$\|x_{n_k}-q_1\|\le\|x_{n_k}-S_1x_{n_k}\|+\|S_1x_{n_k}-q_1\|\to 0,$$

as $k\to\infty$ and so $x_{n_k}\to q_1\in K$. Thus, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\|q_1-S_iq_1\|=\lim_{k\to\infty}\|x_{n_k}-S_ix_{n_k}\|=0,$$

and

$$\|q_1-T_iq_1\|=\lim_{k\to\infty}\|x_{n_k}-T_ix_{n_k}\|=0$$

for $i=1,2$. Thus it follows that $q_1\in F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. Again, since $\lim_{n\to\infty}\|x_n-q_1\|$ exists by Lemma 2.1, we have $\lim_{n\to\infty}\|x_n-q_1\|=0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings S_1, S_2, T_1 and T_2. This completes the proof. □

For our next result, we need the following definition.

A mapping $T\colon K\to K$ is said to be semi-compact if for any bounded sequence $\{x_n\}$ in K such that $\|x_n-Tx_n\|\to 0$ as $n\to\infty$, then there exists a subsequence $\{x_{n_r}\}\subset\{x_n\}$ such that $x_{n_r}\to x^*\in K$ strongly as $r\to\infty$.

Theorem 2.4. Under the assumptions of Lemma 2.2, if one of S_1, S_2, T_1 and T_2 is semi-compact, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.

Proof. Since we know that from Lemma 2.2, $\lim_{n\to\infty}\|x_n-S_ix_n\|=\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ for $i=1,2$ and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $q_*\in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have $\|q_*-S_iq_*\|=\lim_{j\to\infty}\|x_{n_j}-S_ix_{n_j}\|=0$ and $\|q_*-T_iq_*\|=\lim_{j\to\infty}\|x_{n_j}-T_ix_{n_j}\|=0$ for $i=1,2$. Thus it follows that $q_*\in F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. Since $\lim_{n\to\infty}\|x_n-q_*\|$ exists by Lemma 2.1, we have $\lim_{n\to\infty}\|x_n-q_*\|=0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 2.5. Under the assumptions of Lemma 2.2, if there exists a continuous function $f\colon[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(t)>0$ for all $t\in(0,\infty)$ such that

$$f(d(x,F))\le a_1\|x-S_1x\|+a_2\|x-S_2x\|+a_3\|x-T_1x\|+a_4\|x-T_2x\|$$

for all $x\in K$, where $F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$ and a_1, a_2, a_3, a_4 are nonnegative real numbers such that $a_1+a_2+a_3+a_4=1$, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.

Proof. From Lemma 2.2, we know that $\lim_{n\to\infty}\|x_n-S_ix_n\|=\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ for $i=1,2$, we have $\lim_{n\to\infty}f(d(x_n,F))=0$. Since $f\colon[0,\infty)\to[0,\infty)$ is a nondecreasing function satisfying $f(0)=0$ and $f(t)>0$ for all $t\in(0,\infty)$ and $\lim_{n\to\infty}d(x_n,F)$ exists by Lemma 2.1, we have $\lim_{n\to\infty}d(x_n,F)=0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Indeed, from (2.5), we have

$$\|x_{n+1}-q\|\le\|x_n-q\|+B_n$$

for each $n \geq 1$ with $\sum_{n=1}^{\infty} B_n < \infty$ and $q \in F$. For any m, n, $m > n \geq 1$, we have

$$\begin{aligned}
\|x_m - q\| &\leq \|x_{m-1} - q\| + B_{m-1} \\
&\leq \|x_{m-2} - q\| + B_{m-1} + B_{m-2} \\
&\ \vdots \\
&\leq \|x_n - q\| + \sum_{i=n}^{m-1} B_i \\
&\leq \|x_n - q\| + \sum_{i=n}^{\infty} B_i.
\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=n}^{\infty} B_i < \infty$, for any given $\varepsilon > 0$ there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4}, \quad \sum_{i=n_0}^{\infty} B_i < \frac{\varepsilon}{4}.$$

Therefore, there exists $q_1 \in F$ such that

$$\|x_{n_0} - q_1\| < \frac{\varepsilon}{4}, \quad \sum_{i=n_0}^{\infty} B_i < \frac{\varepsilon}{4}.$$

Thus, for all $m, n \geq n_0$, we get from the above inequality that

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - q_1\| + \|x_n - q_1\| \\
&\leq \|x_{n_0} - q_1\| + \sum_{i=n_0}^{\infty} B_i \\
&\quad + \|x_{n_0} - q_1\| + \sum_{i=n_0}^{\infty} B_i \\
&= 2\Big(\|x_{n_0} - q_1\| + \sum_{i=n_0}^{\infty} B_i\Big) \\
&< 2\Big(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\Big) = \varepsilon.
\end{aligned}$$

Thus, it follows that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset of E, the sequence $\{x_n\}$ converges strongly to some $q' \in K$. It is easy to prove that $F(S_1)$, $F(S_2)$, $F(T_1)$ and $F(T_2)$ are all closed and so F is a closed subset of K. Since $\lim_{n\to\infty} d(x_n, F) = 0$, we have $q' \in F$. Thus, the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings S_1, S_2, T_1 and T_2. This completes the proof. □

3. Weak convergence theorems

In this section, we prove some weak convergence theorems of iteration scheme (1.5) for the mixed type asymptotically nonexpansive mappings in the intermediate sense in real uniformly convex Banach spaces.

Lemma 3.1. Under the assumptions of Lemma 2.1, for all $p_1, p_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$$

exists for all $t \in [0, 1]$, where $\{x_n\}$ is the sequence defined by (1.5).

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - z\|$ exists for all $z \in F$ and therefore $\{x_n\}$ is bounded. By letting

$$a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$$

for all $t \in [0,1]$, we conclude that $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - p_2\|$ exists by Lemma 2.1. It, therefore, remains to prove Lemma 3.1 for $t \in (0,1)$. For all $x \in K$, we define the mapping $W_n\colon K \to K$ by:

$$U_n(x) = P(\alpha'_n S_2^n x + \beta'_n T_2(PT_2)^{n-1}x + \gamma'_n u'_n),$$

and

$$W_n(x) = P(\alpha_n S_1^n x + \beta_n T_1(PT_1)^{n-1}U_n(x) + \gamma_n u_n).$$

Then it follows that $x_{n+1} = W_n x_n$, $W_n p = p$ for all $p \in F$. Now from (2.3) and (2.5) of Lemma 2.1, we see that

$$\|U_n(x) - U_n(y)\| \le \|x - y\| + A_n,$$

and

$$\|W_n(x) - W_n(y)\| \le \|x - y\| + B_n, \tag{3.1}$$

with $\sum_{n=1}^{\infty} B_n < \infty$. By setting

$$Q_{n,m} = W_{n+m-1}W_{n+m-2}\dots W_n, \quad m \ge 1, \tag{3.2}$$

and

$$b_{n,m} = \|Q_{n,m}(tx_n + (1-t)p_1) - (tQ_{n,m}x_n + (1-t)Q_{n,m}p_2)\|.$$

From (3.1) and (3.2), we have

$$\begin{aligned}
\|Q_{n,m}(x) - Q_{n,m}(y)\| &= \|W_{n+m-1}W_{n+m-2}\dots W_n(x) - W_{n+m-1}W_{n+m-2}\dots W_n(y)\| \\
&\le \|W_{n+m-2}\dots W_n(x) - W_{n+m-2}\dots W_n(y)\| + B_{n+m-1} \\
&\le \|W_{n+m-3}\dots W_n(x) - W_{n+m-3}\dots W_n(y)\| \\
&\quad + B_{n+m-1} + B_{n+m-2} \\
&\ \ \vdots \\
&\le \|x - y\| + \sum_{j=n}^{n+m-1} B_j
\end{aligned}$$

for all $x, y \in K$, $Q_{n,m}x_n = x_{n+m}$ and $Q_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\
&\le b_{n,m} + \|Q_{n,m}(tx_n + (1-t)p_1) - p_2\| \\
&\le b_{n,m} + a_n(t) + \sum_{j=n}^{n+m-1} B_j.
\end{aligned} \tag{3.3}$$

By using [4, Theorem 2.3], we have

$$\begin{aligned}
b_{n,m} &\le \varphi^{-1}(\|x_n - u\| - \|Q_{n,m}x_n - Q_{n,m}u\|) \\
&\le \varphi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - Q_{n,m}u\|) \\
&\le \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|Q_{n,m}u - u\|)),
\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, i.e., $b_{n,m} \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} B_n = 0$, therefore from (3.3), we have

$$\limsup_{n\to\infty} a_n(t) \le \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is, $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. This completes the proof. $\square$

Lemma 3.2. Under the assumptions of Lemma 2.1, if E has a Fréchet differentiable norm, then for all $p_1, p_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the following limit exists

$$\lim_{n\to\infty} \langle x_n, J(p_1 - p_2)\rangle,$$

where $\{x_n\}$ is the sequence defined by (1.5), if $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle l_1 - l_2, J(p_1 - p_2)\rangle = 0$ for all $p_1, p_2 \in F$ and $l_1, l_2 \in W_w(\{x_n\})$.

Proof. Suppose that $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in inequality (1.1). Then, we get

$$\begin{aligned} t\,\langle x_n - p_1,\, J(p_1 - p_2)\rangle + \frac{1}{2}\,\|p_1 - p_2\|^2 &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq t\,\langle x_n - p_1,\, J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 + b(t\|x_n - p_1\|). \end{aligned}$$

Since $\sup_{n\geq 1} \|x_n - p_1\| \leq M'$ for some $M' > 0$, we have

$$\begin{aligned} t\,\limsup_{n\to\infty} \langle x_n - p_1,\, J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 &\leq \frac{1}{2}\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq t\,\liminf_{n\to\infty} \langle x_n - p_1,\, J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 + b(tM'). \end{aligned}$$

That is,

$$\limsup_{n\to\infty} \langle x_n - p_1,\, J(p_1 - p_2)\rangle \leq \liminf_{n\to\infty} \langle x_n - p_1,\, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'}M'.$$

If $t \to 0$, then $\lim_{n\to\infty} \langle x_n - p_1,\, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F$; in particular, $\langle l_1 - l_2,\, J(p_1 - p_2)\rangle = 0$ for all $l_1, l_2 \in W_w(\{x_n\})$. □

Theorem 3.3. Under the assumptions of Lemma 2.2, if E has Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by (1.5) converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.

Proof. By Lemma 3.2, we obtain $\langle l_1 - l_2,\, J(p_1 - p_2)\rangle = 0$ for all $l_1, l_2 \in W_w(\{x_n\})$. Therefore, $\|q^* - p^*\|^2 = \langle q^* - p^*,\, J(q^* - p^*)\rangle = 0$ implies $q^* = p^*$. Consequently, $\{x_n\}$ converges weakly to a common fixed point in $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. This completes the proof. □

Theorem 3.4. Under the assumptions of Lemma 2.2, if the dual space E^* of E has the Kadec-Klee (KK) property and the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by (1.5) converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.

Proof. By Lemma 2.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $q_1 \in K$. By Lemma 2.2, we have

$$\lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0, \quad \text{and} \quad \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$$

for $i = 1, 2$. Since by hypothesis the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2$ are demiclosed at zero, therefore $S_i q_1 = q_1$ and $T_i q_1 = q_1$ for $i = 1, 2$, which means $q_1 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Now, we show that $\{x_n\}$ converges weakly to q_1. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q_2 \in K$. By the same method as above, we have $q_2 \in F$ and $q_1, q_2 \in W_w(\{x_n\})$. By Lemma 3.1, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0, 1]$ and so $q_1 = q_2$ by Lemma 1.7. Thus, the sequence $\{x_n\}$ converges weakly to $q_1 \in F$. This completes the proof. □

Theorem 3.5. Under the assumptions of Lemma 2.2, if E satisfies Opial condition and the mappings $I-S_i$ and $I-T_i$ for $i=1,2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by (1.5) converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2.

Proof. Let $q_* \in F$, from Lemma 2.1 the sequence $\{\|x_n - q_*\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $x^* \in K$. From Lemma 2.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i=1,2$. Since the mappings $I-S_i$ and $I-T_i$ for $i=1,2$ are demiclosed at zero, therefore $S_i x^* = x^*$ and $T_i x^* = x^*$ for $i=1,2$, which means $x^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to x^*. Suppose by the contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $y^* \in K$ and $x^* \neq y^*$. Then by the same method as given above, we can also prove that $y^* \in F$. From Lemma 2.1 the limits $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|x_n - y^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{n_k\to\infty} \|x_{n_k} - x^*\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - y^*\| \\
&= \lim_{n\to\infty} \|x_n - y^*\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - y^*\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - x^*\| \\
&= \lim_{n\to\infty} \|x_n - x^*\|,
\end{aligned}$$

which is a contradiction, so $x^* = y^*$. Thus, $\{x_n\}$ converges weakly to a common fixed point of the mappings S_1, S_2, T_1 and T_2. This completes the proof. □

Example 3.6. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K=[-1,1]$. Define two mappings S, $T\colon K \to K$, respectively, by the formulas

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0), \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

Then both S and T are asymptotically nonexpansive mappings with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschitzian mappings with $L = \sup_{n\geq 1}\{k_n\}$ and both are uniformly continuous on [-1,1] and hence they are asymptotically nonexpansive mappings in the intermediate sense. Also the unique common fixed point of S and T, that is, $F = F(S) \cap F(T) = \{0\}$. Furthermore, S and T satisfy the condition (ii) in Lemma 2.2 (see, [5, Example 3.1]).

Now, we give some more examples by taking two mappings, $T_1 = T_2 = T$ and $S_1 = S_2 = S$ as follows.

Example 3.7. Let $E = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$, $|\lambda| < 1$ and P be the identity mapping. For each $x \in K$, define the mappings S, $T\colon K \to K$ by the formulas

$$T(x) = \begin{cases} \lambda\, x\, \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$S(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\{T^n x\} \to 0$ uniformly on K as $n \to \infty$. Thus

$$\limsup_{n\to\infty} \Big\{ \|T^n x - T^n y\| - \|x - y\| \vee 0 \Big\} = 0$$

for all $x, y \in K$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense (ANI in short), but it is not a Lipschitz mapping and S is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschitzian with $L = \sup_{n\geq 1}\{k_n\}$. Also, $F(T) = \{0\}$ is the unique fixed point of T and $F(S) = \{0\}$ is the unique fixed point of S, that is, $F = F(S) \cap F(T) = \{0\}$ is the unique common fixed point of S and T.

Example 3.8. Let $E = \mathbb{R}$, $K = [0, 1]$, and P be the identity mapping. For each $x \in K$, define the mappings $S, T \colon K \to K$ by the formulas

$$T(x) = \begin{cases} k\,x, & \text{if } 0 \leq x \leq 1/2, \\ \frac{k}{2k-1}(k - x), & \text{if } 1/2 \leq x \leq k, \\ 0, & \text{if } k \leq x \leq 1, \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $1/2 < k < 1$. Then $F(T) = \{0\}$ and $\{T^n x\}$ converges uniformly to 0 on K. Obviously, T is uniformly continuous. It follows that T is an asymptotically nonexpansive mapping in the intermediate sense (ANI in short). Further, T is uniformly Lipschitzian. Indeed, if $0 \leq x \leq 1/2$ and $1/2 \leq y \leq k$, then $T^n x = k^n x$ and $T^n y = \frac{k^n}{2k-1}(k - y)$. Therefore, we see that

$$\begin{aligned} |T^n x - T^n y| &= \Big| k^n x - \frac{k^n}{2} + \frac{k^n}{2} - \frac{k^n}{2k-1}(k - y) \Big| \\ &= \Big| k^n \Big(x - \frac{1}{2} \Big) + \frac{k^n}{2k-1} \Big[\Big(k - \frac{1}{2} \Big) - (k - y) \Big] \Big| \\ &\leq k^n \Big| x - \frac{1}{2} \Big| + \frac{k^n}{2k-1} \Big| y - \frac{1}{2} \Big| \\ &\leq \frac{k^n}{2k-1} |x - y| \\ &\leq \frac{k}{2k-1} |x - y|. \end{aligned}$$

Hence T is uniformly $\frac{k}{2k-1}$-Lipschitzian and S is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschitzian with $L = \sup_{n\geq 1}\{k_n\}$. Also, $F(S) = \{0\}$ is the unique fixed point of S. Thus, $F = F(S) \cap F(T) = \{0\}$ is the unique common fixed point of S and T.

4. Concluding remarks

In this paper, we study the mixed type iteration scheme for two asymptotically nonexpansive self-mappings in the intermediate sense and two asymptotically nonexpansive non-self-mappings in the intermediate sense and establish some strong convergence theorems by using some completely continuous and semi-compactness conditions and some weak convergence theorems by using the following conditions:

(i) the space E has a Fréchet differentiable norm;

(ii) dual space E^* of E has the Kadec-Klee (KK) property;

(iii) the space E satisfies Opial condition.

Our results extend and generalize the corresponding results of Chidume et al. [2, 3], Guo et al. [5, 6], Saluja [8], Schu [9], Tan and Xu [12], Wang [13], Wei and Guo [14, 15].

Acknowledgment

The authors are grateful to the anonymous referee for his/her careful reading and useful suggestions that helped to improve the manuscript.

References

[1] R. Bruck, T. Kuczumow, S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math., **65** (1993), 169–179. 1
[2] C. E. Chidume, E. U. Ofoedu, H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **280** (2003), 364–374. 1, 4
[3] C. E. Chidume, N. Shahzad, H. Zegeye, *Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense*, Numer. Funct. Anal. Optim., **25** (2004), 239–257. 1, 1, 4
[4] J. García Falset, W. Kaczor, T. Kuczumow, S. Reich, *Weak convergence theorems for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal., **43** (2001), 377–401. 3
[5] W. Guo, Y. J. Cho, W. Guo, *Convergence theorems for mixed type asymptotically nonexpansive mappings*, Fixed Point Theory Appl., **2012** (2012), 15 pages. 1, 3.6, 4
[6] W. Guo, W. Guo, *Weak convergence theorems for asymptotically nonexpansive nonself-mappings*, Appl. Math. Lett., **24** (2011), 2181–2185. 4
[7] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597. 1
[8] G. S. Saluja, *Convergence theorems for two asymptotically nonexpansive non-self mappings in uniformly convex Banach spaces*, J. Indian Math. Soc. (N.S.), **81** (2014), 369–385. 4
[9] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43** (1991), 153–159. 1.6, 4
[10] K. Sitthikul, S. Saejung, *Convergence theorems for a finite family of nonexpansive and asymptotically nonexpansive mappings*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., **48** (2009), 139–152. 1, 1.7, 1.8
[11] W. Takahashi, G. E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Math. Japon., **48** (1998), 1–9. 1
[12] K. K. Tan, H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., **178** (1993), 301–308. 1.5, 4
[13] L. Wang, *Strong and weak convergence theorems for common fixed point of nonself asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **323** (2006), 550–557. 1, 4
[14] S. I. Wei, W. Guo, *Strong convergence theorems for mixed type asymptotically nonexpansive mappings*, Commun. Math. Res., **31** (2015), 149–160. 4
[15] S. Wei, W. Guo, *Weak convergence theorems for mixed type asymptotically nonexpansive mappings*, J. Math. Study, **48** (2015), 256–264. 1, 4