Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5157–5168
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A general implicit iteration for finding fixed points of nonexpansive mappings

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Communicated by Y. H. Yao

Abstract

The aim of the paper is to construct an iterative method for finding the fixed points of nonexpansive mappings. We introduce a general implicit iterative scheme for finding an element of the set of fixed points of a nonexpansive mapping defined on a nonempty closed convex subset of a real Hilbert space. The strong convergence theorem for the proposed iterative scheme is proved under certain assumptions imposed on the sequence of parameters. Our results extend and improve the results given by Ke and Ma [Y. Ke, C. Ma, Fixed Point Theory Appl., **2015** (2015), 21 pages], Xu et al. [H. K. Xu, M. A. Alghamdi, N. Shahzad, Fixed Point Theory Appl., **2015** (2015), 12 pages], and many others.

Keywords: Metric projection mapping, nonexpansive mapping, variational inequality, viscosity method, implicit rules.
2010 MSC: 47H09, 47J25.

1. Introduction

A well-known iteration method for approximating fixed points of a nonexpansive mapping is the viscosity approximation method introduced by Moudafi [9] in 2000. Extensions of viscosity approximation method were obtained by Xu [11] in 2004. For arbitrary $x_1 \in H$, let $\{x_n\}$ be a sequence in H defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

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Received 2016-07-29

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and f and T are contractions and nonexpansive mappings from H onto itself, respectively. This method is called the explicit viscosity method for nonexpansive mappings. It is well-known that under certain assumptions imposed on the parameters, the sequence $\{x_n\}$ generated by (1.1) converges strongly to the fixed point x^* of T which solves the following variational inequality:

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \quad \text{for all } x \in C := Fix(T). \tag{1.2}$$

The implicit midpoint rules are the most powerful techniques for solving ordinary differential equations, (see [3, 5, 10] and the references therein). In 2014, Alghamdi et al. [2] introduced the implicit midpoint rule for nonexpansive mapping as follows: let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping. For arbitrary $x_1 \in H$, let $\{x_n\}$ be a sequence in H defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ for all $n \in \mathbb{N}$. They proved that under some suitable conditions imposed on sequence of parameters, the sequence $\{x_n\}$ generated by (1.3) converges weakly to some fixed point of T. By using the idea of contractions to regularize the implicit midpoint rule for nonexpansive mappings and to find the strong convergence results, in 2015, Xu et al. [12] introduced the following viscosity implicit midpoint rule: let C be a nonempty closed convex subset of a Hilbert space H and f, T are contraction and nonexpansive mappings from C onto itself, respectively. For arbitrary $x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \text{for all } n \in \mathbb{N}, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ for all $n \in \mathbb{N}$. They proved that the sequence generated by (1.4) converges strongly to the fixed point x^* of T, which solves the variational inequality (1.2). In the same year, Ke and Ma [7] have generalized the viscosity implicit midpoint rule of Xu et al. [12] in the following way:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\big(\beta_n x_n + (1-\beta_n)x_{n+1}\big), \quad \text{for all } n \in \mathbb{N}, \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some sequences in $(0,1)$ for all $n \in \mathbb{N}$ and proved the strong convergence of the proposed implicit rule (1.5). Also as we know, there are a large number of algorithms for solving the fixed point problem of nonexpansive mappings in the literature, see [14–18].

On the other hand, the projection methods have played an important role in Hilbert spaces, depending on their convergence analysis. By virtue of projections, in 2011, Ceng et al. [4] introduced implicit and explicit iterative schemes for finding the fixed points of a nonexpansive mapping T defined on a nonempty, closed and convex subset C of a real Hilbert space H as follows:

$$x_t = P_C[t\gamma V x_t + (I - t\mu F)Tx_t] \tag{1.6}$$

and

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)Tx_n], \quad \text{for all } n \in \mathbb{N}, \tag{1.7}$$

where $F : C \to H$ is k-Lipschitzian and η-strongly monotone operator with $k > 0, \eta > 0$, $V : C \to H$ is an L-Lipschitzian mapping with $L \geq 0$, $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$, $\{\alpha_n\} \subset (0,1)$ and $x_1 \in C$ an arbitrary initial point. They proved that the sequences generated by the iterative schemes (1.6) and (1.7) converge strongly to a fixed point x^* of T which solves the following variational inequality problem:

$$\langle (\mu F - \gamma V)x^*, x - x^*\rangle \geq 0, \quad \text{for all } x \in C := Fix(T).$$

In this paper, motivated by the work of Alghamdi et al. [2], Ceng et al. [4], Ke and Ma al [7] and Xu et al. [12], we propose a more general implicit iteration than (1.5) for finding fixed points of a nonexpansive mapping and prove the strong convergence of the sequence generated by the proposed iteration to fixed point of nonexpansive mapping. Our results extend and improve the results given by Ke and Ma [7], Xu et al. [12], and many others.

2. Preliminaries

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A mapping $T : C \to H$ is called

(1) *monotone* if
$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \text{for all } x, y \in C;$$

(2) *η-strongly monotone* if there exists a positive real number η such that
$$\langle Tx - Ty, x - y\rangle \geq \eta \|x - y\|^2, \quad \text{for all } x, y \in C;$$

(3) *k-Lipschitzian* if there exists a constant $k \geq 0$ such that
$$\|Tx - Ty\| \leq k \|x - y\|, \quad \text{for all } x, y \in C;$$

(4) *nonexpansive* if
$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C.$$

Throughout this paper, the symbol $\mathbb{N}$ stands for the set of all natural numbers. Also, we denote by I the identity mapping of H.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point $P_C(x)$ of C such that
$$\|x - P_C(x)\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

The mapping P_C is called the metric projection [8] from H onto C. It is remarkable that the metric projection mapping P_C is nonexpansive from H onto C (see Agarwal et al. [1] for other properties of projection mappings).

The following lemmas will be needed to prove our main results.

Lemma 2.1 ([6]). *For the metric projection mapping P_C, the following properties hold:*

(i) $P_C(x) \in C$ *for all* $x \in H$;

(ii) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;

(iii) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$ *for all* $x \in H$ *and* $y \in C$;

(iv) $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2$ *for all* $x, y \in H$.

Lemma 2.2 ([13]). *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0, 1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator on C. Define the mapping $T_\lambda : C \to H$ by*
$$T_\lambda(x) = x - \lambda\mu Fx, \quad \text{for all } x \in C \text{ and } \lambda \in (0, 1).$$
Then T_λ is a contraction provided $0 < \mu < 2\frac{\eta}{k^2}$. More precisely, for $\mu \in (0, 2\frac{\eta}{k^2})$,
$$\|T_\lambda(x) - T_\lambda(y)\| \leq (1 - \lambda\tau)\|x - y\|, \quad \text{for all } x, y \in C,$$
where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1]$.

Lemma 2.3 ([4]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $V : C \to H$ be an L-Lipschitzian mapping with constant $L \geq 0$ and $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator with constants $k, \eta > 0$. Then for $0 \leq \gamma L < \mu\eta$,*
$$\langle x - y, (\mu F - \gamma V)x - (\mu F - \gamma V)y\rangle \geq (\mu\eta - \gamma L)\|x - y\|^2 \quad \text{for all } x, y \in C.$$
That is, $(\mu F - \gamma V)$ is strongly monotone with coefficient $(\mu\eta - \gamma L)$.

Lemma 2.4 ([1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C weakly converging to some $x^* \in C$ and the sequence $\{(I - T)x_n\}$ converges strongly to 0, then $x^* \in Fix(T)$.*

Lemma 2.5 ([12]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + b_n, \quad \text{for all } n \in \mathbb{N},$$

where $\{a_n\}$ is a sequences in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} a_n = \infty$, *and*

(b) *either* $\limsup_{n\to\infty} \frac{b_n}{a_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

In this section, we introduce a more general implicit iteration method than (1.5) for finding the fixed points of a nonexpansive mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping with $L \geq 0$. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that $0 < \mu < \frac{2\eta}{k^2}$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. For arbitrary $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by the following iterative algorithm:*

$$\begin{cases} x_1 \in C, \\ z_n = \gamma_n x_n + (1 - \gamma_n)Tx_n, \\ x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)T(\beta_n z_n + (1 - \beta_n)x_{n+1})] \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are some sequences with $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\gamma_n\} \subset [0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < \epsilon \leq \beta_n \leq \beta_{n+1} < 1$ *for all* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $\gamma_n \in [0, 1]$ *for all* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(T)$, which is also the unique solution of the variational inequality:

$$\langle (\mu F - \gamma V)x^*, x^* - x \rangle \leq 0, \quad \text{for all } x \in Fix(T). \tag{3.2}$$

Proof. Set $y_n = \beta_n z_n + (1 - \beta_n)x_{n+1}$ and $\theta_n = (1 - \beta_n)(1 - \alpha_n \tau)$ for all $n \in \mathbb{N}$. We now proceed with the following steps.

STEP 1. $\{x_n\}$ is bounded.

Let $q \in Fix(T)$. From (3.1), we have

$$\begin{aligned} \|z_n - q\| &\leq \gamma_n \|x_n - q\| + (1 - \gamma_n) \|Tx_n - q\| \\ &\leq \gamma_n \|x_n - q\| + (1 - \gamma_n) \|x_n - q\| = \|x_n - q\|, \end{aligned}$$

and

$$\begin{aligned} \|y_n - q\| &\leq \beta_n \|z_n - q\| + (1 - \beta_n) \|x_{n+1} - q\| \\ &\leq \beta_n \|x_n - q\| + (1 - \beta_n) \|x_{n+1} - q\|. \end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1}-q\| &\le \|\alpha_n\gamma V x_n + (I-\alpha_n\mu F)T(y_n) - q\| \\
&= \|\alpha_n(\gamma V x_n - \mu F q) + (I-\alpha_n\mu F)T(y_n) - (I-\alpha_n\mu F)q\| \\
&\le \alpha_n \|(\gamma V x_n - \mu F q)\| + (1-\alpha_n\tau)\|T(y_n)-q\| \\
&\le \alpha_n(\|\gamma V x_n - \gamma V q\| + \|\gamma V q - \mu F q\|) + (1-\alpha_n\tau)\|y_n - q\| \\
&\le \alpha_n\gamma L\|x_n-q\| + \alpha_n\|(\gamma V-\mu F)q\| + (1-\alpha_n\tau)\big[\beta_n\|x_n-q\| + (1-\beta_n)\|x_{n+1}-q\|\big] \\
&= \alpha_n\gamma L\|x_n-q\| + \alpha_n\|(\gamma V-\mu F)q\| + \beta_n(1-\alpha_n\tau)\|x_n-q\| \\
&\quad + (1-\beta_n)(1-\alpha_n\tau)\|x_{n+1}-q\| \\
&= \big(\alpha_n\gamma L + \beta_n(1-\alpha_n\tau)\big)\|x_n-q\| + \alpha_n\|(\gamma V-\mu F)q\| + \theta_n\|x_{n+1}-q\|,
\end{aligned}$$

which immediately gives that

$$(1-\theta_n)\|x_{n+1}-q\| \le [\alpha_n\gamma L + \beta_n(1-\alpha_n\tau)]\,\|x_n-q\| + \alpha_n\|(\gamma V-\mu F)q\|. \tag{3.3}$$

Since $\alpha_n, \beta_n \in (0,1)$, it follows that $(1-\theta_n) = 1-(1-\beta_n)(1-\alpha_n\tau) > 0$ for all $n \in \mathbb{N}$. Hence, from (3.3), we get

$$\begin{aligned}
\|x_{n+1}-q\| &\le \frac{\alpha_n\gamma L + \beta_n(1-\alpha_n\tau)}{1-\theta_n}\|x_n-q\| + \frac{\alpha_n}{1-\theta_n}\|(\gamma V-\mu F)q\| \\
&= \left(1-\frac{\alpha_n(\tau-\gamma L)}{1-\theta_n}\right)\|x_n-q\| + \frac{\alpha_n}{1-\theta_n}\|(\gamma V-\mu F)q\| \\
&= \left(1-\frac{\alpha_n(\tau-\gamma L)}{1-\theta_n}\right)\|x_n-q\| + \frac{\alpha_n(\tau-\gamma L)}{1-\theta_n}\left(\frac{1}{(\tau-\gamma L)}\|(\gamma V-\mu F)q\|\right).
\end{aligned}$$

Thus, we have

$$\|x_{n+1}-q\| \le \max\left\{\|x_n-q\|, \frac{1}{(\tau-\gamma L)}\|(\gamma V-\mu F)q\|\right\}, \quad \text{for all } n\in\mathbb{N}.$$

Hence $\{x_n\}$ is bounded and so the sequences $\{Vx_n\}, \{z_n\}, \{y_n\}, \{Tx_n\}, \{Ty_n\}$, and $\{FTx_n\}$ are bounded.

Step 2. $\|x_{n+1}-x_n\| \to 0$ as $n\to\infty$.

Choose M_1, M_2 and M_3 such that

$$M_1 \ge \sup_{n\ge1}\|\gamma V x_n - \mu F T y_n\|, \quad M_2 \ge \sup_{n\ge1}\|x_n - Tx_n\|, \quad \text{and} \quad M_3 \ge \sup_{n\ge1}\|z_n - x_n\|.$$

From (3.1), we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|\gamma_{n+1}x_{n+1} + (1-\gamma_{n+1})Tx_{n+1} - \gamma_n x_n - (1-\gamma_n)Tx_n\| \\
&= \|\gamma_{n+1}x_{n+1} - \gamma_{n+1}x_n + \gamma_{n+1}x_n - \gamma_n x_n + (1-\gamma_{n+1})Tx_{n+1} \\
&\quad -(1-\gamma_{n+1})Tx_n + (1-\gamma_{n+1})Tx_n - (1-\gamma_n)Tx_n\| \\
&\le \gamma_{n+1}\|x_{n+1}-x_n\| + (1-\gamma_{n+1})\|Tx_{n+1}-Tx_n\| + |\gamma_{n+1}-\gamma_n|\|x_n-Tx_n\| \\
&\le \gamma_{n+1}\|x_{n+1}-x_n\| + (1-\gamma_{n+1})\|x_{n+1}-x_n\| + |\gamma_{n+1}-\gamma_n|\|x_n-Tx_n\| \\
&= \|x_{n+1}-x_n\| + |\gamma_{n+1}-\gamma_n|\|x_n-Tx_n\|.
\end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned}
y_{n+1}-y_n &= \beta_{n+1}z_{n+1} + (1-\beta_{n+1})x_{n+2} - \beta_n z_n - (1-\beta_n)x_{n+1} \\
&= (1-\beta_{n+1})x_{n+2} - (1-\beta_{n+1})x_{n+1} + (1-\beta_{n+1})x_{n+1} + \beta_{n+1}z_{n+1} - \beta_n z_n - (1-\beta_n)x_{n+1} \\
&= (1-\beta_{n+1})(x_{n+2}-x_{n+1}) + \beta_{n+1}z_{n+1} - \beta_{n+1}x_{n+1} - \beta_n z_n + \beta_n x_{n+1} \\
&= (1-\beta_{n+1})(x_{n+2}-x_{n+1}) + \beta_n(x_{n+1}-z_n) + \beta_{n+1}(z_{n+1}-x_{n+1}) \\
&\quad + \beta_n(z_{n+1}-x_{n+1}) - \beta_n(z_{n+1}-x_{n+1}) \\
&= (1-\beta_{n+1})(x_{n+2}-x_{n+1}) + \beta_n(z_{n+1}-z_n) + (\beta_{n+1}-\beta_n)(z_{n+1}-x_{n+1}).
\end{aligned}$$

Therefore,

$$\|y_{n+1}-y_n\| \leq (1-\beta_{n+1})\|x_{n+2}-x_{n+1}\| + \beta_n\|z_{n+1}-z_n\| + |\beta_{n+1}-\beta_n|\|z_{n+1}-x_{n+1}\|.$$

Using (3.4), we have

$$\begin{aligned}\|y_{n+1}-y_n\| \leq{}& (1-\beta_{n+1})\|x_{n+2}-x_{n+1}\| + \beta_n\|x_{n+1}-x_n\| \\ &+ \beta_n|\gamma_{n+1}-\gamma_n|\|x_n - Tx_n\| + |\beta_{n+1}-\beta_n|\|z_{n+1}-x_{n+1}\|.\end{aligned}$$

Again from (3.1), we have

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq \|\alpha_{n+1}\gamma Vx_{n+1} + (I-\alpha_{n+1}\mu F)Ty_{n+1} - \alpha_n\gamma Vx_n - (I-\alpha_n\mu F)Ty_n\| \\
&= \|\alpha_{n+1}\gamma Vx_{n+1} + (I-\alpha_{n+1}\mu F)Ty_{n+1} - \alpha_n\gamma Vx_n - (I-\alpha_n\mu F)Ty_n \\
&\quad -(I-\alpha_{n+1}\mu F)Ty_n + (I-\alpha_{n+1}\mu F)Ty_n\| \\
&\leq \|(I-\alpha_{n+1}\mu F)Ty_{n+1} - (I-\alpha_{n+1}\mu F)Ty_n\| + \|\alpha_{n+1}\gamma Vx_{n+1} \\
&\quad -\alpha_{n+1}\gamma Vx_n + \alpha_{n+1}\gamma Vx_n - \alpha_n\gamma Vx_n - (\alpha_{n+1}-\alpha_n)\mu FTy_n\| \\
&\leq (1-\alpha_{n+1}\tau)\,\|Ty_{n+1}-Ty_n\| + \alpha_{n+1}\gamma\|Vx_{n+1}-Vx_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\,\|\gamma Vx_n - \mu FTy_n\| \\
&\leq (1-\alpha_{n+1}\tau)\,\|y_{n+1}-y_n\| + \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|M_1 \\
&\leq (1-\alpha_{n+1}\tau)\big[(1-\beta_{n+1})\|x_{n+2}-x_{n+1}\| + \beta_n\|x_{n+1}-x_n\| \\
&\quad + \beta_n|\gamma_{n+1}-\gamma_n|\|x_n-Tx_n\| + |\beta_{n+1}-\beta_n|\|z_{n+1}-x_{n+1}\|] \\
&\quad + \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|M_1 \\
&= (1-\alpha_{n+1}\tau)(1-\beta_{n+1})\|x_{n+2}-x_{n+1}\| + |\alpha_{n+1}-\alpha_n|M_1 \\
&\quad + (1-\alpha_{n+1}\tau)\beta_n|\gamma_{n+1}-\gamma_n|\|x_n-Tx_n\| + (1-\alpha_{n+1}\tau)|\beta_{n+1}-\beta_n|\|z_{n+1}-x_{n+1}\| \\
&\quad + \big(\alpha_{n+1}\gamma L + (1-\alpha_{n+1}\tau)\beta_n\big)\|x_{n+1}-x_n\| \\
&\leq \theta_{n+1}\|x_{n+2}-x_{n+1}\| + |\alpha_{n+1}-\alpha_n|M_1 + (1-\alpha_{n+1}\tau)\beta_n|\gamma_{n+1}-\gamma_n|M_2 \\
&\quad + (1-\alpha_{n+1}\tau)|\beta_{n+1}-\beta_n|M_3 + \big(\alpha_{n+1}\gamma L + (1-\alpha_{n+1}\tau)\beta_n\big)\|x_{n+1}-x_n\| \\
&\leq \theta_{n+1}\|x_{n+2}-x_{n+1}\| + |\alpha_{n+1}-\alpha_n|M_1 + |\gamma_{n+1}-\gamma_n|M_2 + |\beta_{n+1}-\beta_n|M_3 \\
&\quad + \big(\alpha_{n+1}\gamma L + (1-\alpha_{n+1}\tau)\beta_n\big)\|x_{n+1}-x_n\|,\end{aligned}$$

which immediately gives that

$$\begin{aligned}(1-\theta_{n+1})\|x_{n+2}-x_{n+1}\| \leq{}& (\beta_n - \beta_n\alpha_{n+1}\tau + \alpha_{n+1}\gamma L)\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|M_1 \\ &+ |\gamma_{n+1}-\gamma_n|M_2 + |\beta_{n+1}-\beta_n|M_3,\end{aligned}$$

that is,

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq \frac{\beta_n - \beta_n\alpha_{n+1}\tau + \alpha_{n+1}\gamma L}{1-\theta_{n+1}}\|x_{n+1}-x_n\| \\
&\quad + \frac{1}{1-\theta_{n+1}}\big(M_1|\alpha_{n+1}-\alpha_n| + M_2|\gamma_{n+1}-\gamma_n| + M_3|\beta_{n+1}-\beta_n|\big) \\
&= \left(1 - \frac{\alpha_{n+1}(\tau-\gamma L) + (\beta_{n+1}-\beta_n)(1-\alpha_{n+1}\tau)}{1-\theta_{n+1}}\right)\|x_{n+1}-x_n\| \\
&\quad + \frac{1}{1-\theta_{n+1}}\big(M_1|\alpha_{n+1}-\alpha_n| + M_2|\gamma_{n+1}-\gamma_n| + M_3|\beta_{n+1}-\beta_n|\big).\end{aligned}$$

Let $M = \max\{M_1, M_2, M_3\}$. Note that $0 < \epsilon \leq \beta_n \leq \beta_{n+1} < 1$, we have

$$\beta_{n+1} + \theta_{n+1} = \beta_{n+1} + (1-\beta_{n+1})(1-\alpha_{n+1}\tau) = \big(1 - \alpha_{n+1}\tau(1-\beta_{n+1})\big) < 1.$$

Hence

$$0 < \epsilon \leq \beta_{n+1} < (1-\theta_{n+1}) < 1. \tag{3.5}$$

From (3.5), $1-\theta_{n+1} < 1$ implies $1 < \frac{1}{1-\theta_{n+1}}$. Therefore,

$$\begin{aligned}\alpha_{n+1}(\tau-\gamma L) &\leq \alpha_{n+1}(\tau-\gamma L) + (\beta_{n+1}-\beta_n)(1-\alpha_{n+1}\tau)\\ &\leq \frac{\alpha_{n+1}(\tau-\gamma L) + (\beta_{n+1}-\beta_n)(1-\alpha_{n+1}\tau)}{1-\theta_{n+1}}\end{aligned}$$

implies

$$-\alpha_{n+1}(\tau-\gamma L) \geq -\frac{\alpha_{n+1}(\tau-\gamma L) + (\beta_{n+1}-\beta_n)(1-\alpha_{n+1}\tau)}{1-\theta_{n+1}}.$$

Thus,

$$\|x_{n+2}-x_{n+1}\| \leq \big[1-\alpha_{n+1}(\tau-\gamma L)\big]\,\|x_{n+1}-x_n\| + \frac{M}{\epsilon}\big(|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n| + |\beta_{n+1}-\beta_n|\big).$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, and $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty$, therefore by Lemma 2.5, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Step 3. $\|x_n - Tx_n\| \to 0$ as $n\to\infty$.

Note that

$$\begin{aligned}\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\| + \|x_{n+1}-Ty_n\| + \|Ty_n-Tx_n\|\\ &\leq \|x_n-x_{n+1}\| + \|x_{n+1}-Ty_n\| + \|y_n-x_n\|\\ &= \|x_n-x_{n+1}\| + \|P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu F)Ty_n] - P_C[Ty_n]\| + \|y_n-x_n\|\\ &\leq \|x_n-x_{n+1}\| + \alpha_n\|\gamma Vx_n - \mu FTy_n\| + \|y_n-x_n\|\\ &\leq \|x_n-x_{n+1}\| + \alpha_n M_1 + \|\beta_n z_n + (1-\beta_n)x_{n+1} - x_n\|\\ &\leq \|x_n-x_{n+1}\| + \alpha_n M_1 + \beta_n\|z_n-x_n\| + (1-\beta_n)\,\|x_{n+1}-x_n\|\\ &= (2-\beta_n)\,\|x_n-x_{n+1}\| + \alpha_n M_1 + \beta_n\,\|\gamma_n x_n + (1-\gamma_n)Tx_n - x_n\|\\ &= (2-\beta_n)\,\|x_n-x_{n+1}\| + \alpha_n M_1 + \beta_n(1-\gamma_n)\|x_n-Tx_n\|,\end{aligned}$$

which implies that

$$\big[1-\beta_n(1-\gamma_n)\big]\,\|x_n-Tx_n\| \leq (2-\beta_n)\|x_n-x_{n+1}\| + \alpha_n M_1.$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, we get

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0.$$

Moreover, from (3.1), we have

$$\begin{aligned}\|x_n-Ty_n\| &\leq \|x_n-x_{n+1}\| + \|x_{n+1}-Ty_n\|\\ &\leq \|x_n-x_{n+1}\| + \alpha_n\|\gamma Vx_n - \mu FTy_n\|\\ &\leq \|x_n-x_{n+1}\| + \alpha_n M_1\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{3.6}$$

Step 4. $\limsup_{n\to\infty}\langle(\mu F-\gamma V)x^*, x^*-x_n\rangle \leq 0$, where $x^* = P_{Fix(T)}\big(I-(\mu F-\gamma V)\big)x^*$.

Let us take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\mu F-\gamma V)x^*, x^*-x_n\rangle = \lim_{k\to\infty}\langle(\mu F-\gamma V)x^*, x^*-x_{n_k}\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup q \in H$. Without loss of generality, we may assume that $x_{n_k} \rightharpoonup q$. From Step 3 and Lemma 2.4, we have $q \in Fix(T)$. This together with the property of metric projection implies that

$$\begin{aligned}\limsup_{n\to\infty} \langle (\mu F - \gamma V)x^*, x^* - x_n\rangle &= \lim_{k\to\infty} \langle (\mu F - \gamma V)x^*, x^* - x_{n_k}\rangle \\ &= \langle (\mu F - \gamma V)x^*, x^* - q\rangle \\ &\le 0.\end{aligned}$$

Step 5. $x_n \to x^*$ as $n \to \infty$.

Set

$$A_n = \|x_n - x^*\|,\ B_n = \|y_n - x^*\|, \quad \phi_n = (1-\beta_n)(1-\alpha_n\tau)\big(1-\alpha_n(\tau-\gamma L)\big),$$

and

$$L_n = \alpha_n^2\|\gamma V x_n - \mu F x^*\|^2 + 2\alpha_n \langle (\gamma V - \mu F)x^*, \quad (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\rangle$$

for all $n \in \mathbb{N}$. Then from (3.1), we have

$$\begin{aligned}A_{n+1}^2 &= \|x_{n+1} - x^*\|^2 \\ &\le \|\alpha_n\gamma V x_n + (I-\alpha_n\mu F)Ty_n - x^*\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu F x^*) + (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\|^2 \\ &\le \alpha_n^2\|\gamma V x_n - \mu F x^*\|^2 + (1-\alpha_n\tau)^2\|Ty_n - x^*\|^2 \\ &\quad + 2\alpha_n \langle \gamma V x_n - \mu F x^*, (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\rangle \\ &= \alpha_n^2\|\gamma V x_n - \mu F x^*\|^2 + (1-\alpha_n\tau)^2\|Ty_n - x^*\|^2 \\ &\quad + 2\alpha_n \langle \gamma V x_n - \gamma V x^*, (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\rangle \\ &\quad + 2\alpha_n \langle \gamma V x^* - \mu F x^*, (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\rangle \\ &= (1-\alpha_n\tau)^2\|Ty_n - x^*\|^2 + 2\alpha_n \langle \gamma V x_n - \gamma V x^*, (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\rangle + L_n \\ &\le (1-\alpha_n\tau)^2\|y_n - x^*\|^2 + 2\alpha_n(1-\alpha_n\tau)\gamma\|V x_n - V x^*\|\|Ty_n - x^*\| + L_n \\ &\le (1-\alpha_n\tau)^2\|y_n - x^*\|^2 + 2\alpha_n(1-\alpha_n\tau)\gamma L\|x_n - x^*\|\|y_n - x^*\| + L_n \\ &= (1-\alpha_n\tau)^2 B_n^2 + 2\alpha_n(1-\alpha_n\tau)\gamma L A_n B_n + L_n.\end{aligned}$$

It turns out that

$$(1-\alpha_n\tau)^2 B_n^2 + 2\alpha_n(1-\alpha_n\tau)\gamma L A_n B_n + (L_n - A_{n+1}^2) \ge 0.$$

Solving above quadratic inequality for B_n, we get

$$B_n \ge \frac{-\alpha_n\gamma L A_n + \sqrt{\alpha_n^2\gamma^2 L^2 A_n^2 - (L_n - A_{n+1}^2)}}{(1-\alpha_n\tau)}.$$

Note that

$$\begin{aligned}B_n &= \|y_n - x^*\| \\ &\le \beta_n\|z_n - x^*\| + (1-\beta_n)\|x_{n+1} - x^*\| \\ &= \beta_n\|\gamma_n x_n + (1-\gamma_n)Tx_n - x^*\| + (1-\beta_n)A_{n+1} \\ &\le \beta_n[\gamma_n\|x_n - x^*\| + (1-\gamma_n)\|Tx_n - x^*\|] + (1-\beta_n)A_{n+1} \\ &\le \beta_n\|x_n - x^*\| + (1-\beta_n)A_{n+1} \\ &= \beta_n A_n + (1-\beta_n)A_{n+1}.\end{aligned}$$

Therefore, we have

$$\beta_n A_n + (1-\beta_n)A_{n+1} \ge \frac{-\alpha_n\gamma L A_n + \sqrt{\alpha_n^2\gamma^2 L^2 A_n^2 - (L_n - A_{n+1}^2)}}{(1-\alpha_n\tau)},$$

or

$$(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)A_n + (1-\beta_n)(1-\alpha_n\tau)A_{n+1} \geq \sqrt{\alpha_n^2\gamma^2L^2A_n^2 - L_n + A_{n+1}^2},$$

or

$$(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)A_n + \theta_n A_{n+1} \geq \sqrt{\alpha_n^2\gamma^2L^2A_n^2 - L_n + A_{n+1}^2},$$

or

$$\begin{aligned}\alpha_n^2\gamma^2L^2A_n^2 - L_n + A_{n+1}^2 &\leq (\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)^2A_n^2 + \theta_n^2A_{n+1}^2 + 2(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\theta_n A_nA_{n+1}\\ &\leq (\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)^2A_n^2 + \theta_n^2A_{n+1}^2 + (\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\theta_n[A_n^2 + A_{n+1}^2],\end{aligned}$$

which gives that

$$\begin{aligned}&[1-\theta_n^2 - \theta_n(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)]A_{n+1}^2\\ &\quad\leq [(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)^2 + (\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\theta_n - \alpha_n^2\gamma^2L^2]A_n^2 + L_n,\end{aligned}$$

or

$$[1-\theta_n(1-\alpha_n(\tau-\gamma L))]A_{n+1}^2 \leq [(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)^2 + (\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\theta_n - \alpha_n^2\gamma^2L^2]A_n^2 + L_n,$$

or

$$(1-\phi_n)A_{n+1}^2 \leq \big[(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\big(1-\alpha_n(\tau-\gamma L)\big) - \alpha_n^2\gamma^2L^2\big]A_n^2 + L_n.$$

It follows that

$$A_{n+1}^2 \leq \frac{(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\big(1-\alpha_n(\tau-\gamma L)\big) - \alpha_n^2\gamma^2L^2}{1-\phi_n}A_n^2 + \frac{L_n}{1-\phi_n}. \tag{3.7}$$

Let

$$u_n = \frac{1}{\alpha_n}\left\{1 - \frac{(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\big(1-\alpha_n(\tau-\gamma L)\big) - \alpha_n^2\gamma^2L^2}{1-\phi_n}\right\}.$$

Then

$$u_n = \frac{(\tau-\gamma L)\big(2-\alpha_n(\tau-\gamma L)\big) - \alpha_n^2\gamma^2L^2}{1-\phi_n}.$$

Since $\{\beta_n\}$ satisfies $0 < \epsilon \leq \beta_n \leq \beta_{n+1} < 1$ for all $n \geq 1$, it follows that $\lim_{n\to\infty}\beta_n$ exists. Assume that

$$\lim_{n\to\infty}\beta_n = \beta^* > 0.$$

Then

$$\lim_{n\to\infty} u_n = \frac{2(\tau-\gamma L)}{\beta^*} > 0.$$

Let σ_1 satisfies

$$0 < \sigma_1 < \frac{2(\tau-\gamma L)}{\beta^*}.$$

Then there exists a positive integer N_1 large enough such that $u_n > \sigma_1$ for all $n \geq N_1$, and hence

$$\frac{(\beta_n - \beta_n\alpha_n\tau + \alpha_n\gamma L)\big(1-\alpha_n(\tau-\gamma L)\big) - \alpha_n^2\gamma^2L^2}{1-\phi_n} \leq 1 - \sigma_1\alpha_n$$

for all $n \geq N_1$. Therefore, from (3.7), we have, for all $n \geq N_1$,

$$A_{n+1}^2 \leq (1-\sigma_1\alpha_n)A_n^2 + \frac{L_n}{1-\phi_n}. \tag{3.8}$$

Note that

$$\begin{aligned}L_n &= \alpha_n[\alpha_n\|\gamma Vx_n - \mu Fx^*\|^2 + 2\langle(\gamma V-\mu F)x^*, (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*\rangle] \\ &= \alpha_n[\alpha_n\|\gamma Vx_n - \mu Fx^*\|^2 + 2\langle(\gamma V-\mu F)x^*, (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x_n\rangle \\ &\quad + 2\langle(\gamma V-\mu F)x^*, (I-\alpha_n\mu F)x_n - (I-\alpha_n\mu F)x^*\rangle] \\ &\le \alpha_n[\alpha_n\|\gamma Vx_n - \mu Fx^*\|^2 + 2(1-\alpha_n\tau)\|(\gamma V-\mu F)x^*\|\|Ty_n - x_n\| \\ &\quad + 2\langle(\gamma V-\mu F)x^*, x_n - x^* + \alpha_n\mu(Fx^* - Fx_n)\rangle] \\ &\le \alpha_n[\alpha_n\|\gamma Vx_n - \mu Fx^*\|^2 + 2(1-\alpha_n\tau)\|(\gamma V-\mu F)x^*\|\|Ty_n - x_n\| \\ &\quad + 2\langle(\gamma V-\mu F)x^*, x_n - x^*\rangle + 2\alpha_n\mu k\|(\gamma V-\mu F)x^*\|\|x_n - x^*\|].\end{aligned}$$

Thus, by $\lim_{n\to\infty}\alpha_n = 0$, (3.6), and Step 4, we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{L_n}{\sigma_1\alpha_n(1-\phi_n)} &\le \limsup_{n\to\infty}\frac{1}{\sigma_1(1-\phi_n)}\Big[\alpha_n\|\gamma Vx_n - \mu Fx^*\|^2 \\ &\quad + 2(1-\alpha_n\tau)\|(\gamma V-\mu F)x^*\|\|Ty_n - x_n\| \\ &\quad + 2\langle(\gamma V-\mu F)x^*, x_n - x^*\rangle + 2\alpha_n\mu k\|(\gamma V-\mu F)x^*\|\|x_n - x^*\|\Big] \\ &\le 0.\end{aligned} \tag{3.9}$$

From (3.8), (3.9), and Lemma 2.5, we have $x_n \to x^*$ as $n\to\infty$. □

Remark 3.2. Several implicit rules can be deduced from our implicit rule (3.1) as follows:

- For $\gamma_n = 1$ for all $n\in\mathbb{N}$, our algorithm (3.1) reduces to

$$\begin{cases}x_1 \in C,\\ x_{n+1} = P_C\big[\alpha_n\gamma Vx_n + (I-\alpha_n\mu F)T\big(\beta_n x_n + (1-\beta_n)x_{n+1}\big)\big].\end{cases} \tag{3.10}$$

- For $\gamma_n = 0$ for all $n\in\mathbb{N}$, our algorithm (3.1) reduces to

$$\begin{cases}x_1 \in C,\\ x_{n+1} = P_C\big[\alpha_n\gamma Vx_n + (I-\alpha_n\mu F)T\big(\beta_n Tx_n + (1-\beta_n)x_{n+1}\big)\big].\end{cases} \tag{3.11}$$

- For $\gamma_n = 1$ and $\beta_n = \frac{1}{2}$ for all $n\in\mathbb{N}$, our algorithm (3.1) reduces to

$$\begin{cases}x_1 \in C,\\ x_{n+1} = P_C\big[\alpha_n\gamma Vx_n + (I-\alpha_n\mu F)T\big(\tfrac{x_n+x_{n+1}}{2}\big)\big].\end{cases} \tag{3.12}$$

- For $C = H$, $\mu F = I$, $V = f$, a contraction mapping with coefficient $\theta\in(0,1)$, $\gamma = 1$ and $L = \theta$ with $0<\theta<\tau = \mu\big(\eta - \mu\frac{1}{2}k^2\big)$, our algorithm (3.1) reduces to

$$\begin{cases}x_1 \in C,\\ z_n = \gamma_n x_n + (1-\gamma_n)T(x_n),\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\big(\beta_n z_n + (1-\beta_n)x_{n+1}\big).\end{cases}$$

- For $C = H$, $\mu F = I$, $V = f$, a contraction mapping with coefficient $\theta\in(0,1)$, $\gamma = 1$ and $L = \theta$ with $0<\theta<\tau = \mu\big(\eta - \frac{1}{2}\mu k^2\big)$, take $\gamma_n = 1$ for all $n\in\mathbb{N}$. Then, our algorithm (3.1) reduces to

$$\begin{cases}x_1 \in C,\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\big(\beta_n x_n + (1-\beta_n)x_{n+1}\big).\end{cases} \tag{3.13}$$

  The algorithm (3.13) is studied by Ke and Ma [7].

- For $C = H$, $\mu F = I$, $V = f$, a contraction mapping with coefficient $\theta \in (0, 1)$, $\gamma = 1$ and $L = \theta$ with $0 < \theta < \tau = \mu\big(\eta - \frac{1}{2}\mu k^2\big)$, take $\gamma_n = 1$ and $\beta_n = \frac{1}{2}$ for all $n \in \mathbb{N}$. Then, our algorithm (3.1) reduces to

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\big(\frac{x_n + x_{n+1}}{2}\big). \end{cases} \tag{3.14}$$

The algorithm (3.14) is studied by Xu et al. [12].

Remark 3.3. Iterative algorithms (3.10) and (3.12) improve and extend the algorithms (3.13) and (3.14), respectively, in the following ways:

(a) The self-mapping $f : C \to C$ is extended to non-self-mapping $V : C \to H$.
(b) The contraction coefficient $k \in (0, 1)$ is extended to Lipschitzian constant $L \in [0, \infty)$.

In particular, we derive the following interesting result.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping with $L \geq 0$. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that $0 < \mu < \frac{2\eta}{k^2}$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. For arbitrary $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by* (3.11) *and $\{\alpha_n\}$ and $\{\beta_n\}$ are some sequences in $(0, 1)$ satisfying the conditions* (i) *and* (ii) *of Theorem* 3.1. *Then the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(T)$, which is also the unique solution of the variational inequality* (3.2).

Proof. The proof follows from Theorem 3.1 by taking $\gamma_n = 0$ for all $n \geq 1$. □

We now present the result of Ke and Ma [7, Theorem 3.1] as a corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and $f : C \to C$ be a θ-contraction mapping. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. For arbitrary $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by* (3.13) *and $\{\alpha_n\}$ and $\{\beta_n\}$ are some sequences in* $(0, 1)$ *satisfying the conditions* (i) *and* (ii) *of Theorem* 3.1. *Then the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(T)$, which is also the unique solution of the variational inequality:*

$$\langle (I - f)x^*, x^* - x\rangle \leq 0, \quad \textit{for all } x \in Fix(T).$$

Proof. The proof follows from Theorem 3.1 by taking $C = H$, $\mu F = I$ and $V = f$, a contraction mapping with coefficient $\theta \in (0, 1)$, $\gamma = 1$ and $L = \theta$ with $0 < \theta < \tau = \mu\big(\eta - \frac{1}{2}\mu k^2\big)$, and $\gamma_n = 1$ for all $n \in \mathbb{N}$. □

Acknowledgment

The authors would like to thank the editor and referees for useful comments and suggestions.

The third author is supported by the Council of Scientific and Industrial Research (CSIR), New Delhi, India, through grant 09/013(0584)/2015-EMR-I.

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