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Young's inequality for multivariate functions

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Abstract

This paper presents a generalization of Young's inequality to the real functions of several variables. Moreover, the relevant facts about Young's inequality and its extension including improved proofs are provided in a review. The basic results are initiated by applying the integral method to a strictly increasing continuous function of one variable.

Keywords: Strictly increasing function, integral sum, Young's inequality.
2010 MSC: 26B20, 26D15.

1. Introduction

Studying the inequalities, we estimate the relationships between different types of means. Let us take two nonnegative numbers x and y, and two positive coefficients k and l satisfying $k + l = 1$. The basic inequality says that the geometric mean $x^k y^l$ is less than or equal to the arithmetic mean $kx + ly$. By putting $k = 1/p$, $l = 1/q$, $x = a^p$ and $y = b^q$, the geometric-arithmetic mean inequality can be expressed by

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{1.1}$$

By using this simple inequality, one can prove the important Hölder's inequality and Minkowski's inequality concerning with the norms of integrable functions. The inequality in equation (1.1) represents the discrete form of Young's inequality.

More general and interesting is the integral form of Young's inequality. It uses a bijective continuous function $f : [0, \infty) \to [0, \infty)$. Such a function is strictly increasing satisfying $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$, and its inverse function has the same properties. These functions have their place in a convex analysis because their antiderivative functions

$$F(t) = \int_0^t f(x)\, dx \tag{1.2}$$

are convex.

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2. Review of Young's inequality

In this review of the integral form of Young's inequality, we use a nonnegative strictly increasing unbounded continuous real function $f(x)$ on the unbounded closed interval $[0, \infty)$. We omit the usual assumption $f(0) = 0$ because it is not actually required.

The theorem that follows is the starting point of the mathematical story on Young's inequality. Its graphic interpretation is more than clear, but its analytical proof is slightly complicated.

Theorem 2.1. *Let $f(x)$ be a nonnegative strictly increasing unbounded continuous function on $[0, \infty)$, and let $g(y)$ be its inverse function. Then, the equality*

$$af(a) = \int_0^a f(x)\,dx + \int_{f(0)}^{f(a)} g(y)\,dy \tag{2.1}$$

holds for every number $a \geq 0$.

Proof. Given a nonnegative real number a, and a positive integer n, for each index $i = 0, 1, \ldots, n$ we take the points $x_i = x_i(n) = (a/n)i$ and $y_i = y_i(n) = f(x_i)$. Thus, $x_i - x_{i-1} = a/n$ and $g(y_i) = x_i$. The product $af(a)$ can be expressed as the sum,

$$\begin{aligned} af(a) &= \sum_{i=1}^{n} \frac{a}{n}\big[if(x_i) - (i-1)f(x_{i-1})\big] \\ &= \sum_{i=1}^{n} \frac{a}{n} f(x_{i-1}) + \sum_{i=1}^{n} \big[f(x_i) - f(x_{i-1})\big]\frac{a}{n}i \\ &= \sum_{i=1}^{n} \big(x_i - x_{i-1}\big)f(x_{i-1}) + \sum_{i=1}^{n}(y_i - y_{i-1})g(y_i), \end{aligned} \tag{2.2}$$

where the augend is the n-integral sum of f, and the addend is the n-integral sum of g. Letting $n \to \infty$, and respecting the bounds $x_0 = 0$, $x_n = a$, $y_0 = f(0)$, and $y_n = f(a)$, we obtain the equality in equation (2.1). □

The following corollary extends the equality in equation (2.1) to numbers $a \geq 0$ and $b \geq f(0)$.

Corollary 2.2. *Let $f(x)$ and $g(y)$ be functions as in Theorem 2.1. Then, the equality*

$$ab = \int_0^a f(x)\,dx + \int_{f(0)}^{b} g(y)\,dy - \int_a^{g(b)} \big[b - f(x)\big]dx \tag{2.3}$$

holds for every pair of numbers $a \geq 0$ and $b \geq f(0)$.

Proof. Equation (2.1) applied to the function $f(x)$ on the interval $[0, g(b)]$ yields

$$g(b)b = \int_0^{g(b)} f(x)\,dx + \int_{f(0)}^{b} g(y)\,dy. \tag{2.4}$$

Integral features provide that

$$ab - g(b)b = \int_{g(b)}^{a} f(x)\,dx - \int_a^{g(b)} \big[b - f(x)\big]dx. \tag{2.5}$$

The equality in equation (2.3) follows by summing equations (2.4) and (2.5). □

The integral of a nondecreasing continuous function f satisfies the inequality

$$f(a_1)(a_2 - a_1) \leq \int_{a_1}^{a_2} f(x)\,dx \leq f(a_2)(a_2 - a_1)$$

for every pair of numbers a_1 and a_2 of the domain of f. Putting $a_1 = a$ and $a_2 = g(b)$, and rearranging, we can bring out the inequality

$$0 \leq \int_a^{g(b)} \big[b - f(x)\big]\,dx \leq \big[g(b) - a\big]\big[b - f(a)\big]. \tag{2.6}$$

Applying the above estimation to equation (2.3), we obtain a generalization of equation (2.1) in the form of the double inequality as follows.

Corollary 2.3. *Let $f(x)$ and $g(y)$ be functions as in Theorem 2.1. Then, the double inequality*

$$ab \leq \int_0^a f(x)\,dx + \int_{f(0)}^b g(y)\,dy \leq af(a) + bg(b) - f(a)g(b) \tag{2.7}$$

holds for every pair of numbers $a \geq 0$ and $b \geq f(0)$.

If $f(a) = b$, the double inequality in equation (2.7) goes into the equality in equation (2.1). The middle area of equation (2.7) which is expressed by the sum of two integrals can be seen in Figure 1.

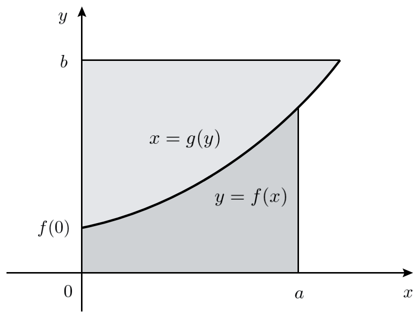

Figure 1: Graphic presentation of the middle area of equation (2.7).

The left-hand side of the inequality in equation (2.7) is known as Young's inequality. Using the function $f(x) = x^{p-1}$ in this inequality, we get the discrete inequality in equation (1.1).

Young proved the left-hand side of equation (2.7) by using the additional assumption that the function f is differentiable, see [14]. First analytic proofs of Young's inequality without the assumption of differentiability appeared in 1970s, see [1, 4], and [8]. A more general approach to Young's inequality and its consequences can be found in the books [11, pages 239–246] and [7, pages 14–20]. Some interesting details on Young's inequality can be seen in the papers [3, 6, 9]. Generalizations of the discrete and integral form, as well as a functional approach to Young's inequality, can be found in [10]. New refinements of Young's inequality were obtained in [12].

The proof of the equality in equation (2.1) similar to the above was made in [13] by using the lower and upper Riemann integral sums in the context of ε-notation. The bounds of the right term of Young's inequality were discussed in [2] and [5]. A convenient proof of Young's inequality was given in [13] by utilizing the convexity of the antiderivative function F of equation (1.2). The whole double inequality in equation (2.7) was also proved in [13] by applying the mean value theorem.

3. Main results

In order to extend Young's inequality to functions of two variables, we use a continuous function $f : [0,\infty)^2 \to [0,\infty)$ whose partial functions of one variable coincide with the functions in Section 2. Accordingly, we consider a nonnegative continuous real function $f(x,y)$ such that its partial function $f_x(y) = f(x,y)$ is strictly increasing and unbounded on $[0,\infty)$ for each fixed $x \in [0,\infty)$.

At the same time, we involve the corresponding two-variable function g determined by the rule, $g(x,z) = y$ if $z = f(x,y)$. Equivalently, if the functions $g_x(z)$ are inverses of the functions $f_x(y)$, then $g(x,z) = g_x(z)$.

Theorem 3.1. *Let $f(x,y)$ be a nonnegative continuous function on $A=[0,\infty)^2$ such that its partial function $f_x(y)=f(x,y)$ is strictly increasing and unbounded on $[0,\infty)$ for each $x\in[0,\infty)$. Let $g_x(z)$ be inverses of $f_x(y)$, and let $g(x,z)=g_x(z)$ be the corresponding function on the domain $B=\{(x,z): x\geq 0,\ z\geq f(x,0)\}$. Then, the integral equality*

$$b\int_0^a f(x,b)\,dx=\int_0^a dx\int_0^b f(x,y)\,dy+\int_0^a dx\int_{f(x,0)}^{f(x,b)} g(x,z)\,dz \tag{3.1}$$

holds for every pair of numbers $a\geq 0$ and $b\geq 0$.

Proof. Given $a\geq 0$ and $b\geq 0$, and a positive integer n, for each index $i=0,1,\ldots,n$ and $j=0,1,\ldots,n$ we take the points on the coordinate axes as follows

$$x_i=x_i(n)=\frac{a}{n}i,\ \ y_j=y_j(n)=\frac{b}{n}j,\ \ z_{ij}=z_{ij}(n)=f(x_i,y_j).$$

Thus, we have that

$$x_i-x_{i-1}=\frac{a}{n},\ \ y_j-y_{j-1}=\frac{b}{n},\ \ y_j=g(x_i,z_{ij}).$$

Generalizing the calculation of equation (2.2) to the two variables function $f(x,y)$, we obtain

$$\begin{aligned}
b\sum_{i=1}^{n}(x_i-x_{i-1})f(x_i,b)&=\sum_{i=1}^{n}(x_i-x_{i-1})\frac{b}{n}\sum_{j=1}^{n}\big[jf(x_i,y_j)-(j-1)f(x_i,y_{j-1})\big]\\
&=\sum_{i,j=1}^{n}(x_i-x_{i-1})\frac{b}{n}f(x_i,y_{j-1})+\sum_{i,j=1}^{n}(x_i-x_{i-1})\big[f(x_i,y_j)-f(x_i,y_{j-1})\big]\frac{b}{n}j\\
&=\sum_{i,j=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})f(x_i,y_{j-1})+\sum_{i,j=1}^{n}(x_i-x_{i-1})(z_{ij}-z_{i\,j-1})g(x_i,z_{ij}).
\end{aligned}\tag{3.2}$$

Respecting the given bounds

$$x_0=0,\ x_n=a,\ \ y_0=0,\ y_n=b,\ \ z_{i0}=f(x_i,0),\ z_{in}=f(x_i,b)$$

the equality in equation (3.1) follows by sending n to infinity. □

If $f(x,0)=f(a,0)$ and $f(x,b)=f(a,b)$ for every $x\in[0,a]$, then equation (3.1) takes the form

$$abf(a,b)=\int_0^a dx\int_0^b f(x,y)\,dy+\int_0^a dx\int_{f(a,0)}^{f(a,b)} g(x,z)\,dz,$$

pointing the visual similarity with equation (2.1).

Introducing the function bound $b(x)$ instead of the number bound b, we get the following generalization of Theorem 3.1.

Corollary 3.2. *Let $f(x,y)$ and $g(x,z)$ be functions as in Theorem* 3.1. *Then, the integral equality*

$$\int_0^a b(x)f(x,b(x))\,dx=\int_0^a dx\int_0^{b(x)} f(x,y)\,dy+\int_0^a dx\int_{f(x,0)}^{f(x,b(x))} g(x,z)\,dz \tag{3.3}$$

holds for every pair of a number $a\geq 0$ and a nonnegative continuous function $b(x)$ on the interval $[0,a]$.

Proof. Equation (3.3) is within a reach of the reflection moment (sending n to infinity) applied to equation (3.2) including the points

$$x_i=\frac{a}{n}i,\ \ y_{ij}=\frac{b(x_i)}{n}j,\ \ z_{ij}=f(x_i,y_{ij}),$$

as well as the bounds

$$x_0=0,\ x_n=a,\ \ y_{i0}=0,\ y_{in}=b(x_i),\ \ z_{i0}=f(x_i,0),\ z_{in}=f(x_i,b(x_i)).$$

□

Applying the equality in equation (3.3) to the function $f(x,y)$ on the curve trapeze $\{(x,y) : 0 \le x \le a, 0 \le y \le g(x,c)\}$, and using the integral properties, we obtain the next corollary.

Corollary 3.3. *Let $f(x,y)$ and $g(x,z)$ be functions as in Theorem 3.1. Then, the equality*

$$abc = \int_0^a dx \int_0^b f(x,y)\,dy + \int_0^a dx \int_{f(x,0)}^c g(x,z)\,dz - \int_0^a dx \int_b^{g(x,c)} \big[c - f(x,y)\big]\,dy \tag{3.4}$$

holds for every triple of numbers $a \ge 0$, $b \ge 0$, and $c \ge f(x,0)$ for all $x \in [0,a]$.

The integral estimation in equation (2.6) adapted to the function $f_x(y)$ on the interval $[b, g_x(c)]$ takes the form

$$0 \le \int_b^{g(x,c)} \big[c - f(x,y)\big]\,dy \le \big[g(x,c) - b\big]\big[c - f(x,b)\big],$$

which integrated over the interval $[0,a]$ yields

$$0 \le \int_0^a dx \int_b^{g(x,c)} \big[c - f(x,y)\big]\,dy \le \int_0^a \big[g(x,c) - b\big]\big[c - f(x,b)\big]\,dx.$$

Putting the above estimation and equation (3.4) together, we get the following double inequality as an extension of equation (2.7) to functions of two variables.

Corollary 3.4. *Let $f(x,y)$ and $g(x,z)$ be functions as in Theorem 3.1. Then, the double inequality*

$$abc \le \int_0^a dx \int_0^b f(x,y)\,dy + \int_0^a dx \int_{f(x,0)}^c g(x,z)\,dz \le \int_0^a \big[bf(x,b) + cg(x,c) - f(x,b)g(x,c)\big]\,dx \tag{3.5}$$

holds for every triple of numbers $a \ge 0$, $b \ge 0$, and $c \ge f(x,0)$ for all $x \in [0,a]$.

The middle volume of equation (3.5) which is expressed by the sum of two double integrals can be seen in Figure 2.

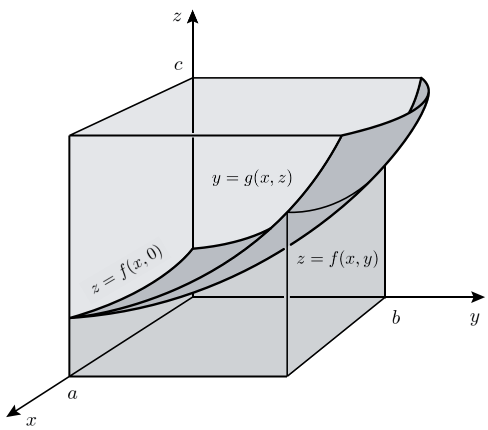

Figure 2: Graphic presentation of the middle volume of equation (3.5).

4. Generalizations

We generalize Theorem 3.1 and Corollary 3.4 to functions of several variables. The following is a generalization of Theorem 3.1.

Theorem 4.1. *Let $f(x_1,\ldots,x_n)$ be a nonnegative continuous n-variable function on $A=[0,\infty)^n$ such that its partial function $f_{x_1\ldots x_{n-1}}(x_n)=f(x_1,\ldots,x_n)$ is strictly increasing and unbounded on $[0,\infty)$ for each $(x_1,\ldots,x_{n-1})\in[0,\infty)^{n-1}$. Let $g_{x_1\ldots x_{n-1}}(x_{n+1})$ be inverse of $f_{x_1\ldots x_{n-1}}(x_n)$, and let*

$$g(x_1,\ldots,x_{n-1},x_{n+1})=g_{x_1\ldots x_{n-1}}(x_{n+1})$$

be the corresponding function on the domain

$$B=\big\{\big(x_1,\ldots,x_{n-1},x_{n+1}\big):(x_1,\ldots,x_{n-1})\in[0,\infty)^{n-1},x_{n+1}\geq f(x_1,\ldots,x_{n-1},0)\big\}.$$

Then, the integral equality

$$\begin{aligned}
&a_n\int_0^{a_1}dx_1\ldots\int_0^{a_{n-1}}f(x_1,\ldots,x_{n-1},a_n)\,dx_{n-1}\\
&=\int_0^{a_1}dx_1\ldots\int_0^{a_{n-1}}dx_{n-1}\int_0^{a_n}f(x_1,\ldots,x_{n-1},x_n)\,dx_n\\
&\quad+\int_0^{a_1}dx_1\ldots\int_0^{a_{n-1}}dx_{n-1}\int_{f(x_1,\ldots,x_{n-1},0)}^{f(x_1,\ldots,x_{n-1},a_n)}g(x_1,\ldots,x_{n-1},x_{n+1})\,dx_{n+1}
\end{aligned}$$

holds for every n-tuple of numbers $a_1\geq 0,\ \ldots,\ a_n\geq 0$.

It remains to carry out a generalization of Corollary 3.4.

Corollary 4.2. *Let $f(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_{n-1},x_{n+1})$ be functions as in Theorem 4.1. Then, the double inequality*

$$\begin{aligned}
a_1\ldots a_{n+1}&\leq\int_0^{a_1}dx_1\ldots\int_0^{a_{n-1}}dx_{n-1}\int_0^{a_n}f(x_1,\ldots,x_{n-1},x_n)\,dx_n\\
&\quad+\int_0^{a_1}dx_1\ldots\int_0^{a_{n-1}}dx_{n-1}\int_{f(x_1,\ldots,x_{n-1},0)}^{a_{n+1}}g(x_1,\ldots,x_{n-1},x_{n+1})\,dx_{n+1}\\
&\leq\int_0^{a_1}dx_1\ldots\int_0^{a_{n-1}}\big[a_nf(x_1,\ldots,x_{n-1},a_n)+a_{n+1}g(x_1,\ldots,x_{n-1},a_{n+1})\\
&\quad-f(x_1,\ldots,x_{n-1},a_n)\,g(x_1,\ldots,x_{n-1},a_{n+1})\big]\,dx_{n-1}
\end{aligned}$$

holds for every $(n+1)$-tuple of numbers $a_1\geq 0,\ \ldots,\ a_{n-1}\geq 0,\ a_n\geq 0$, and $a_{n+1}\geq f(x_1,\ldots,x_{n-1},0)$ for all $(x_1,\ldots,x_{n-1})\in[0,a_1]\times\ldots\times[0,a_{n-1}]$.

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