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Singularities of dual hypersurfaces of spacelike hypersurfaces in lightcone and Legendrian duality

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Abstract

The theory of the Legendrian singularity is applied for lightcones that are canonically embedded in the higher-dimensional lightcone and de Sitter space in the Minkowski space-time. The singularities of two classes of hypersurfaces that are dual to space-like hypersurface in the lightcone under Legendrian dualities are analyzed in detail.

Keywords: Singularity, Legendrian duality, light-cone frame.
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1. Introduction

It is well-known that the Minkowski space-time is the mathematical model of Einsteins Theory of relativity. Several geometric objects in the Minkowski space-time have been investigated from various perspectives and using differential geometry and physics [2–4, 8, 10]. In particular, submanifolds in the three types of pseudo-spheres (i.e., the hyperbolic space, the de Sitter space and the lightcone) in the Minkowski space-time have received recent attention. Izumiya introduced the mandala of Legendrian dualities between pseudo-spheres in the Minkowski space-time [4]. This framework of the theory of Legendrian duality is fundamentally useful to study space-like submanifolds in lightcones. The third author and Pei et al. have also performed significant research regarding submanifolds in the Minkowski space-time from the viewpoint of singularity theory [12–16]. In this paper, inspired by the study of Izumiya and the collaborative research of the second author and Izumiya et al. [5–7], we study the geometric properties of space-like hypersurfaces in lightcones. The second author et al. studied the curves in the unit 2-sphere and 3-sphere, considering

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Legendrian duality [5, 7], and investigated hypersurfaces in the unit n-sphere in the framework of the theory of Legendrian dualities between pseudo-spheres in the Minkowski $(n+2)$-space [6]. In fact, the core practices in their study are that they embed the unit sphere into the lightcone and de Sitter space and investigate the hypersurfaces in the unit sphere by using the singularity theory and the theory of Legendrian duality comprehensively. A natural question thus arises: what if this hypersurface exists in a lower-dimensional lightcone embedded in the de Sitter space or in the light-cone space? In fact, for the de Sitter space and the lightcone, naturally embedded lower-dimensional lightcones exist. If a space-like hypersurface resides in the lower-dimensional lightcone, then it certainly resides in both the higher-dimensional lightcone and the higher-dimensional de Sitter space through the embeddings. Moreover, we note that because the embeddings are the isometries, these two hypersurfaces have the same geometric structures via the isometries based on the spherical geometry. Based on the embeddings of the lightcone in the de Sitter space or the lightcone, we use the theory of Legendrian duality to obtain two dual hypersurfaces of space-like hypersurfaces in the lightcone. On the lightcone, there is a projection onto the canonically embedded hyperbolic space. We investigate the singular points of the dual hypersurfaces and the projection images of the singular value sets onto the hyperbolic space in the lightcone. An interesting consequence is that the critical value sets of these dual hypersurfaces have the same projections onto the hyperbolic space and are both equal to the hyperbolic focal set (or the hyperbolic evolute). In general, to study the singularity of the dual hypersurfaces of space-like hypersurfaces, we should first provide the properties of differential geometry on the hypersurface. However, the situation of the hypersurface in the lightcone is quite different from that of the hypersurface in other spaces because the metric on the lightcone is degenerate. For the space-like hypersurfaces $M = \boldsymbol{x}(U)$ in the lightcone, we define a map $G : U \to L_0^n$ by $G(u) = \boldsymbol{x}_L(u)$, which is called the *lightcone quasi-Gauss map* of $M = \boldsymbol{x}(U)$. Thereby, we can define the *lightcone quasi-Gauss-Kronecker curvature* of M at some point. We call G the lightcone quasi-Gauss map because $G(u) = \boldsymbol{x}_L(u)$ is light-like and belongs to the normal space of $\boldsymbol{x}(u)$, although $\boldsymbol{x}(u)$ and $\boldsymbol{x}_L(u)$ are not orthogonal. Applying the properties of differential geometry on the space-like hypersurface, the following study on space-like hypersurfaces in the lightcone can be smoothly conducted.

Our paper is organized as follows: Section 2 reviews basic definitions and characterizations of the Minkowski $(n+2)$-space and establishes the differential geometry of a space-like hypersurfaces in the lightcone. Several duality relationships are presented in Proposition 2.2; we define the light-cone dual hypersurface and sphere-cone dual hypersurface along a space-like hypersurface in the lightcone, and the hyperbolic evolutes are obtained from the critical value sets of the light-cone dual hypersurfaces of $M = \boldsymbol{x}(U)$. A singularity study is presented in Sections 3 and 4. First, in Section 3, we define the light-cone focal surface and the sphere-cone focal surface along the space-like hypersurface in the lightcone. Theorem 3.3 interprets the important relationships between the hyperbolic evolutes of a space-like hypersurface in the lightcone, the light-cone focal surface and the sphere-cone focal surface. We also define a family of light-cone height functions and a family of sphere-cone height functions along space-like hypersurfaces in the lightcone. The equivalent conditions on the singular sets of the sphere-cone height functions and the light-cone height function are given in Propositions 3.1 and 3.2, respectively. Then, in Section 4, we interpret the geometric meaning of the light-cone dual hypersurfaces of the submanifolds in L_0^n and the sphere-cone dual hypersurfaces of the submanifolds in L_+^n in the theory of Legendrian singularities; that is, the two classes of dual hypersurfaces can be the wave fronts of the Legendrian immersion. In Section 5, using the theory of contact from Montaldi [11], we consider the contact between hypersurfaces in the lightcone with parabolic $(n-1)$-hyperquadrics and parabolic n-hyperquadrics. Some equivalent relationships at singularities are shown clearly. In Section 6, we consider the surfaces in the 3-lightcone as a special case of the previous sections.

2. Preliminaries

Let $\mathbb{R}^{n+2}$ be an $(n+2)$-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}), \boldsymbol{y} = (y_1, y_2, \ldots, y_{n+2})$ in $\mathbb{R}^{n+2}$, their *pseudo scalar product* is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2}$.

Here, $(\mathbb{R}^{n+2}, \langle , \rangle)$ is called *Minkowski* $(n+2)$*-space*, which is denoted by $\mathbb{R}^{n+2}_1$. For any $n+1$ vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n+1} \in \mathbb{R}^{n+2}_1$, their *pseudo vector product* is defined by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \ldots \wedge \boldsymbol{x}_{n+1} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{n+2} \\ x_1^1 & x_1^2 & \cdots & x_1^{n+2} \\ x_2^1 & x_2^2 & \cdots & x_2^{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n+1}^1 & x_{n+1}^2 & \cdots & x_{n+1}^{n+2} \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}\}$ is the canonical basis of $\mathbb{R}^{n+2}_1$ and $\boldsymbol{x}_i = (x_i^1, x_i^2, \ldots, x_i^{n+2})$. A non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+2}_1$ is called spacelike, lightlike, or timelike if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. The norm of $\boldsymbol{x} \in \mathbb{R}^{n+2}_1$ is defined by $\| \boldsymbol{x} \| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

We define the *de Sitter* $(n+1)$*-space* by

$$S^{n+1}_1 = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

We define the $(n+1)$-dimensional *open light-cone at the origin* by

$$LC^{n+1}_* = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2}_1 \backslash \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We consider a submanifold in the de Sitter $(n+1)$-space defined by

$$L^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}) \in S^{n+1}_1 \mid x_2 = 1\},$$

and a submanifold in the lightcone defined by

$$H^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}) \in LC^{n+1}_* \mid x_2 = 1\},$$

we call L^n_+ the *spherical light-cone* and call H^n_+ *the lightlike hyperbolic sphere*. We also consider the n-dimensional open lightcone L^n_0 in LC^{n+1}_* defined by

$$L^n_0 = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}) \in LC^{n+1}_* \mid x_2 = 0\},$$

and the n-dimensional hyperbolic space H^n_0 defined by

$$H^n_0 = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2}_1 \mid x_2 = 0, -x_1^2 + x_3^2 + \cdots + x_5^2 = -1\}.$$

We have a canonical light-cone projection $\pi : LC^{n+1}_* \to H^n_+$ defined by

$$\pi(\boldsymbol{x}) = \widetilde{\boldsymbol{x}} = \left(\frac{x_1}{x_2}, 1, \frac{x_3}{x_2}, \ldots, \frac{x_{n+2}}{x_2}\right),$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+2})$.

Let $\boldsymbol{x} : U \longrightarrow L^n_0$ be an embedding from an open set $U \subset \mathbb{R}^{n-1}$. We identify $M = \boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$. Obviously, the tangent space T_pM is all spacelike (i.e., consists only spacelike vectors), so M is a spacelike hypersurface in $L^n_0 \subset \mathbb{R}^{n+2}_1$. In addition, the isometric mapping $\Phi : L^n_0 \to L^n_+$ is defined by $\Phi(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{e_2}$, $\boldsymbol{v} \in L^n_0$, and the isometric mapping $\Theta : H^n_+ \to H^n_0$ is given by $\Theta(\boldsymbol{v}) = \boldsymbol{v} - \boldsymbol{e_2}$, $\boldsymbol{v} \in H^n_+$. Hence, via the isometry Φ, we have a hypersurface $\overline{\boldsymbol{x}} : U \to L^n_+$ defined by $\overline{\boldsymbol{x}}(\boldsymbol{u}) = \Phi(\boldsymbol{x}(\boldsymbol{u})) = \boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{e}_2$, and we identify $\overline{M} = \overline{\boldsymbol{x}}(U)$ with U through the embedding $\overline{\boldsymbol{x}}$, so that $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ have the same geometric properties as spherical hypersurfaces. For any $p = \boldsymbol{x}(\boldsymbol{u})$, we can obtain a unique lightlike vector $\boldsymbol{x}_L(\boldsymbol{u})$ as

$$\boldsymbol{x}_L(\boldsymbol{u}) = \frac{-2}{\langle V, \boldsymbol{x}(\boldsymbol{u}) \rangle}\left(V - \frac{\langle V, V \rangle}{2\langle V, \boldsymbol{x}(\boldsymbol{u}) \rangle}\boldsymbol{x}(\boldsymbol{u})\right)$$

with V being an arbitrary vector field that satisfies the conditions $\langle V, \boldsymbol{x}(\boldsymbol{u})\rangle \neq 0$ and $\langle V, \boldsymbol{x}_{u_i}(\boldsymbol{u})\rangle = \langle V, \boldsymbol{e}_2\rangle = 0$.

We have $\langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{x}(\boldsymbol{u})\rangle = \langle \boldsymbol{x}_L(\boldsymbol{u}), \boldsymbol{x}_L(\boldsymbol{u})\rangle = 0$, $\langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{x}_L(\boldsymbol{u})\rangle = -2$, $\langle \boldsymbol{e}_2, \boldsymbol{e}_2\rangle = 1$, and $\langle \boldsymbol{e}_2, \boldsymbol{x}(\boldsymbol{u})\rangle = \langle \boldsymbol{e}_2, \boldsymbol{x}_L(\boldsymbol{u})\rangle = \langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{x}_{u_i}\rangle = \langle \boldsymbol{x}_L(\boldsymbol{u}), \boldsymbol{x}_{u_i}\rangle = 0$. The system $\{\boldsymbol{e}_2, \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{x}_L(\boldsymbol{u}), \boldsymbol{x}_{u_1}(\boldsymbol{u}), \dots, \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})\}$ is a basis of $T_p\mathbb{R}^{n+2}_1$. We define a map $G : U \longrightarrow L^n_0$ by $G(\boldsymbol{u}) = \boldsymbol{x}_L(\boldsymbol{u})$. We call it the *lightcone quasi-Gauss map* of the hypersurface $M = \boldsymbol{x}(U)$. We have a linear mapping provided by the derivation of the lightcone quasi-Gauss map at $p \in M$, $dG(\boldsymbol{u}) : T_pM \longrightarrow T_pM$. We call the linear transformation $S_p = dG(\boldsymbol{u})$ the *shape operator* of M at $p = \boldsymbol{x}(\boldsymbol{u})$. The eigenvalues of S_p denoted by $\{\kappa_i(p)\}_{i=1}^{n-1}$ are called the *principal curvatures* of M at p. The *lightcone quasi-Gauss-Kronecker curvature* of M at p is defined to be $K(p) = \det S_p$. A point p is called an *umbilic point* if all the principal curvatures coincide at p and thus we have $S_p = \kappa(p)\mathrm{id}_{T_pM}$ for some $\kappa(p) \in \mathbb{R}$. We say that M is *totally umbilic* if all the points on M are umbilic. Since $\boldsymbol{x}$ is a spacelike embedding, we have a *Riemannian metric* (or the *first fundamental form*) on M given by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(\boldsymbol{u}) = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \boldsymbol{x}_{u_j}(\boldsymbol{u})\rangle$ for any $\boldsymbol{u} \in U$. The second fundamental form on M is given by $h_{ij}(\boldsymbol{u}) = \langle \boldsymbol{x}_{Lu_i}(\boldsymbol{u}), \boldsymbol{x}_{u_j}(\boldsymbol{u})\rangle$ at any $\boldsymbol{u} \in U$, where $\boldsymbol{x}_{Lu_i}(\boldsymbol{u}) = \frac{\partial \boldsymbol{x}_L}{\partial u_i}(\boldsymbol{u})$. Under the above notations, we have the following Weingarten formula

$$G_{u_i} = \sum_{j=1}^{n-1} h_i^j \boldsymbol{x}_{u_j} (i = 1, \dots, n-1),$$

where $(h_i^j) = (h_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. This formula induces an explicit expression of the lightcone Gauss-Kronecker curvature in terms of the Riemannian metric and the second fundamental invariant given by $K = \det(h_{ij})/\det(g_{\alpha\beta})$. A point p is a parabolic point if $K(p) = 0$. A point p is a *flat point* if it is an umbilic point and $K(p) = 0$.

Each *hyperbolic evolute* of $M = \boldsymbol{x}(U)$ is defined to be

$$\boldsymbol{\varepsilon}^{\pm}_M = \bigcup_{i=1}^{n-1} \left\{ \pm \left(\frac{\sqrt{-\kappa_i(p)}}{2}\boldsymbol{x}(\boldsymbol{u}) + \frac{1}{2\sqrt{-\kappa_i(p)}}\boldsymbol{x}_L(\boldsymbol{u}) \right) \;\middle|\; p = \boldsymbol{x}(\boldsymbol{u}) \in M = \boldsymbol{x}(U) \right\}.$$

We now show the basic theorem in this paper which is the fundamental tool for the study of spacelike submanifolds in lightcone in Minkowski space. We define one-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_0dv_0 + \sum_{i=1}^n w_idv_i$, $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_0dw_0 + \sum_{i=1}^n v_idw_i$ on $\mathbb{R}^{n+2}_1 \times \mathbb{R}^{n+2}_1$ and consider the following four double fibrations with one-forms:

(i) (a) $H^{n+1}(-1) \times S^{n+1}_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\}$;
 (b) $\pi_{11} : \Delta_1 \longrightarrow H^{n+1}(-1), \pi_{12} : \Delta_1 \longrightarrow S^{n+1}_1$;
 (c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1, \theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_1$;

(ii) (a) $H^{n+1}(-1) \times LC^{n+1}_* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -1\}$;
 (b) $\pi_{21} : \Delta_2 \longrightarrow H^{n+1}(-1), \pi_{22} : \Delta_2 \longrightarrow LC^{n+1}_*$;
 (c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_2, \theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_2$;

(iii) (a) $LC^{n+1}_* \times S^{n+1}_1 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 1\}$;
 (b) $\pi_{31} : \Delta_3 \longrightarrow LC^{n+1}_*, \pi_{32} : \Delta_3 \longrightarrow S^{n+1}_1$;
 (c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_3, \theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_3$;

(iv) (a) $LC^{n+1}_* \times LC^{n+1}_* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -2\}$;
 (b) $\pi_{41} : \Delta_4 \longrightarrow LC^{n+1}_*, \pi_{42} : \Delta_4 \longrightarrow LC^{n+1}_*$;
 (c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_4, \theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_4$.

Here, $\pi_{i1}(v, w) = v$, $\pi_{i2}(v, w) = w$ are the canonical projections. Moreover, $\theta_{i1} = \langle dv, w\rangle \mid_{\triangle_i}$ and $\theta_{i2} = \langle v, dw\rangle \mid_{\triangle_i}$ are the restrictions of the one-forms $\langle dv, w\rangle$ and $\langle v, dw\rangle$ on $\triangle_i$. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over $\triangle_i$ which is denoted by K_i. The basic theorem in this paper is the following theorem:

Theorem 2.1. *Under the same notations as the previous paragraph, each $(\triangle_i; K_i)$ $(i = 1, 2, 3, 4)$ is a contact manifold and both of π_{ij} $(j = 1, 2)$ are Legendrian fibrations. Moreover, those contact manifolds are contact diffeomorphic to each other.*

The proof of this theorem can be found in [4]. In this paper, we will only consider (Δ_3, K_3) and (Δ_4, K_4). If we have an isotropic mapping $i : L \to \Delta_i$ (i.e., $i^*\theta_{i1} = 0$), we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i*-dual* to each other $(i = 3, 4)$. For detailed properties of Legendrian fibrations, see [1].

Now we define hypersurfaces in LC^{n+1}_* associated with the hypersurfaces in L^n_0 or L^n_+. Let $\boldsymbol{x} : U \longrightarrow L^n_0$ be a hypersurface. We define $LD_M : U \times \mathbb{R} \longrightarrow LC^{n+1}_*$ by

$$LD_M(\boldsymbol{u}, \eta) = \frac{\eta^2}{4}\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x_L}(\boldsymbol{u}) + \eta \boldsymbol{e}_2,$$

and we call LD_M the *light-cone dual hypersurface along* M. We also define $\overline{LD}_{\overline{M}} : U \times \mathbb{R} \longrightarrow LC^{n+1}_*$ by

$$\overline{LD}_{\overline{M}}(\boldsymbol{u}, \eta) = \frac{\eta^2}{2(\eta - 1)}\boldsymbol{x}(\boldsymbol{u}) + \frac{\eta - 1}{2}\boldsymbol{x_L}(\boldsymbol{u}) + \eta \boldsymbol{e}_2,$$

and we call $\overline{LD}_{\overline{M}}$ the *sphere-cone dual hypersurface along* $\overline{M}$. Then we have the following proposition.

Proposition 2.2. *Under the above notations, we have the following:*

(i) $\boldsymbol{x}$ *and* LD_M *are* Δ_4*-dual to each other.*
(ii) $\overline{\boldsymbol{x}}$ *and* $\overline{LD}_{\overline{M}}$ *are* Δ_3*-dual to each other.*

Proof.

(i) Consider the mapping $\mathcal{L}_4 : U \times \mathbb{R} \longrightarrow \Delta_4$ defined by $\mathcal{L}_4(\boldsymbol{u}, \eta) = (LD_M(\boldsymbol{u}, \eta), \boldsymbol{x}(\boldsymbol{u}))$. Then we have

$$\langle LD_M(\boldsymbol{u}, \eta), \boldsymbol{x}(\boldsymbol{u})\rangle = \langle \boldsymbol{x_L}(\boldsymbol{u}), \boldsymbol{x}(\boldsymbol{u})\rangle = -2.$$

Moreover, we have

$$\mathcal{L}_4^*\theta_{42} = \langle LD_M(\boldsymbol{u}, \eta), d\boldsymbol{x}(\boldsymbol{u})\rangle = \sum_{i=1}^{n-1}\langle LD_M(\boldsymbol{u}, \mu), \boldsymbol{x}_{u_i}\rangle du_i = 0.$$

Hence the assertion (i) holds.

(ii) Consider the mapping $\mathcal{L}_3 : U \times \mathbb{R} \longrightarrow \Delta_3$ defined by $\mathcal{L}_3(\boldsymbol{u}, \eta) = (\overline{LD}_{\overline{M}}(\boldsymbol{u}, \eta), \overline{\boldsymbol{x}}(\boldsymbol{u}))$. Then we have

$$\langle \overline{LD}_{\overline{M}}(\boldsymbol{u}, \eta), \overline{\boldsymbol{x}}(\boldsymbol{u})\rangle = \langle \frac{\eta - 1}{2}\boldsymbol{x_L}(\boldsymbol{u}) + \eta \boldsymbol{e}_2, \boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{e}_2\rangle = 1.$$

Moreover, we have

$$\mathcal{L}_3^*\theta_{32} = \langle \overline{LD}_{\overline{M}}(\boldsymbol{u}, \mu), d\overline{\boldsymbol{x}}(\boldsymbol{u})\rangle = \sum_{i=1}^{n-1}\langle \overline{LD}_{\overline{M}}(\boldsymbol{u}, \mu), \boldsymbol{x}_{u_i}\rangle du_i = 0.$$

The assertion (ii) is complete. □

3. The light-cone height functions and sphere-cone height functions of hypersurfaces

Let $\boldsymbol{x} : U \to L^n_0$ be a hypersurface in the L^n_0. Then we define two families of functions as follows:

$$H : U \times LC^{n+1}_* \to \mathbb{R}, \qquad H(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{v}\rangle + 2,$$
$$\overline{H} : U \times LC^{n+1}_* \longrightarrow \mathbb{R}, \qquad \overline{H}(\boldsymbol{u}, \overline{\boldsymbol{v}}) = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle - 1.$$

We call H a *light-cone height function* of M. For any fixed $\boldsymbol{v}_0 \in LC^{n+1}_*$, we denote $h_{v_0}(\boldsymbol{u}) = H(\boldsymbol{u}, \boldsymbol{v}_0)$. We also call $\overline{H}$ a *sphere-cone height function* of $\overline{M}$. For any fixed $\overline{\boldsymbol{v}}_0 \in LC^{n+1}_*$, we denote $\overline{h}_{\overline{v}_0}(\boldsymbol{u}) = \overline{H}(\boldsymbol{u}, \overline{\boldsymbol{v}}_0)$.

Proposition 3.1. *Let M be a hypersurface in L_0^n and H be the light-cone height function on M. For $p = \boldsymbol{x}(\boldsymbol{u}) \neq \boldsymbol{v}$, we have the following:*

(i) $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0,\ (i = 1, \ldots, n-1)$ *if and only if* $\boldsymbol{v} = LD_M(\boldsymbol{u}, \eta)$ *for some* $\eta \in \mathbb{R}\backslash\{\boldsymbol{0}\}$.

(ii) $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0,\ (i = 1, \ldots, n-1)$ *and* $\det Hess(h_v)(\boldsymbol{u}) = 0$ *if and only if* $\boldsymbol{v} = LD_M(\boldsymbol{u}, \eta)$, *and* $-\frac{\eta^2}{4}$ *is one of the non-zero principle curvatures* $\kappa_i(p)$ *of* M.

Proof.

(i) Since $\boldsymbol{v} \in LC_*^{n+1}$, there exist $\lambda, \mu, \xi_i, (i = 1, \ldots, n-1), \eta \in \mathbb{R}$ such that $\boldsymbol{v} = \lambda\boldsymbol{x}(\boldsymbol{u}) + \mu\boldsymbol{x_L}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i \boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_2$ with $-4\lambda\mu + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) + \eta^2 = 0$. The condition

$$h_v(\boldsymbol{u}) = \langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{v}\rangle + 2 = \langle \boldsymbol{x}(\boldsymbol{u}), \lambda\boldsymbol{x}(\boldsymbol{u}) + \mu\boldsymbol{x_L}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_2\rangle + 2 = -2\mu + 2 = 0$$

means that $\mu = 1$, so that $\boldsymbol{v} = \lambda\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x_L}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_2$ and $-4\lambda + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) + \eta^2 = 0$. Therefore, $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0$ if and only if

$$\partial h_v/\partial u_i(\boldsymbol{u}) = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \boldsymbol{v}\rangle = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \lambda\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x_L}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_2\rangle = \sum_{j=1}^{n-1} g_{ij}\xi_j = 0.$$

Since g_{ij} is positive definite, we have $\xi_j = 0$ $(j = 1, \ldots, n-1)$. Then we have $-4\lambda + \eta^2 = 0$, so that $\lambda = \frac{\eta^2}{4}$. Thus, we have $\boldsymbol{v} = \frac{\eta^2}{4}\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x_L}(\boldsymbol{u}) + \eta\boldsymbol{e}_2$. The converse direction also holds.

(ii) Suppose that $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0$. Then we have

$$\begin{aligned}
\text{Hess}\,(h_v)(\boldsymbol{u}) &= \big(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{v}\rangle\big)\\
&= \Big(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \frac{\eta^2}{4}\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x_L}(\boldsymbol{u}) + \eta\boldsymbol{e}_2\rangle\Big)\\
&= \frac{\eta^2}{4}\big(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{x}(\boldsymbol{u})\rangle\big) + \big(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{x_L}(\boldsymbol{u})\rangle\big)\\
&= -\frac{\eta^2}{4}\big(g_{ij}(\boldsymbol{u})\big) - \big(h_{ij}(\boldsymbol{u})\big).
\end{aligned}$$

Therefore, $\det(\mathrm{Hess(h_v)(u)}) = 0$ if and only if $-\det \text{Hess}\,(h_v)(\boldsymbol{u})\big(g_{ij}(\boldsymbol{u})\big)^{-1} = \det\big((h_i^j)(\boldsymbol{u}) - (-\frac{\eta^2}{4}I)\big) = 0$, so that $\det \text{Hess}\,(h_v)(\boldsymbol{u}){=}0$ if and only if $-\frac{\eta^2}{4}$ is one of the non-zero principle curvatures of M at p. □

Proposition 3.2. *Let $\overline{M}$ be a hypersurface in L_+^n and $\overline{H}$ the sphere-cone height function on $\overline{M}$. For $p = \boldsymbol{x}(\boldsymbol{u})$ and $\overline{p} = \overline{\boldsymbol{x}}(\boldsymbol{u}) \neq \overline{\boldsymbol{v}}$, we have the following:*

(i) $\overline{h}_{\overline{v}}(\boldsymbol{u}) = \partial\overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = 0, (i = 1, \ldots, n-1)$ *if and only if*

$$\overline{\boldsymbol{v}} = \overline{LD}_{\overline{M}}(\boldsymbol{u}, \eta) \textit{ for some } \eta \in \mathbb{R}\backslash\{\boldsymbol{0}\}.$$

(ii) $\overline{h}_{\overline{v}}(\boldsymbol{u}) = \partial\overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = 0, (i = 1, \ldots, n-1)$ *and* $\det(\mathrm{Hess}(\overline{\mathrm{h}}_{\overline{\mathrm{v}}})(\mathrm{u})) = 0$ *if and only if* $\overline{\boldsymbol{v}} = \overline{LD}_{\overline{M}}(\boldsymbol{u}, \eta)$, $-(\frac{\eta}{\eta-1})^2$ *is one the non-zero principle curvatures* $\kappa_i(p)$ *of* M.

Proof.

(i) Since $\overline{\boldsymbol{v}} \in LC_*^{n+1}$, there exist $\lambda, \mu, \xi_i, (i = 1, \ldots, n-1), \eta \in \mathbb{R}$ such that $\overline{\boldsymbol{v}} = \lambda\boldsymbol{x}(\boldsymbol{u}) + \mu\boldsymbol{x_L}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_2$ with $-4\lambda\mu + \sum_{i,j=1}^{n-1} \xi_i\xi_j g_{ij}(\boldsymbol{u}) + \eta^2 = 0$. The condition

$$\overline{h}_{\overline{v}}(\boldsymbol{u}) = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle - 1 = \langle \boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{e}_2, \lambda\boldsymbol{x}(\boldsymbol{u}) + \mu\boldsymbol{x_L}(\boldsymbol{u}) + \sum_{i=1}^{n-1} \xi_i\boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta\boldsymbol{e}_2\rangle - 1 = -2\mu + \eta - 1 = 0$$

implies $\mu = \frac{\eta-1}{2}$, so that $\overline{\boldsymbol{v}} = \lambda \boldsymbol{x}(\boldsymbol{u}) + \frac{\eta-1}{2}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) + \sum_{i=1}^{n-1}\xi_i \boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta \boldsymbol{e}_2$ and $-2\lambda(\eta-1) + \sum_{i,j=1}^{n-1}\xi_i\xi_j g_{ij}(\boldsymbol{u}) + \eta^2 = 0$. Therefore, $\overline{h}_{\overline{v}}(\boldsymbol{u}) = \partial \overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = 0$ if and only if

$$\partial \overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \lambda \boldsymbol{x}(\boldsymbol{u}) + \frac{\eta-1}{2}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) + \sum_{i=1}^{n-1}\xi_i \boldsymbol{x}_{u_i}(\boldsymbol{u}) + \eta \boldsymbol{e}_2\rangle = \sum_{j=1}^{n-1} g_{ij}\xi_j = 0.$$

Since g_{ij} is positive definite, we have $\xi_j = 0$ $(j = 1, \dots, n-1)$. Then we have $-2\lambda(\eta-1) + \eta^2 = 0$, so that $\lambda = \frac{\eta^2}{2(\eta-1}$. Thus, we have $\overline{\boldsymbol{v}} = \frac{\eta^2}{2(\eta-1)}\boldsymbol{x}(\boldsymbol{u}) + \frac{\eta-1}{2}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) + \eta \boldsymbol{e}_2$. The converse direction also holds.

(ii) Suppose that $\overline{h}_{\overline{v}}(\boldsymbol{u}) = \partial \overline{h}_{\overline{v}}/\partial u_i(\boldsymbol{u}) = 0$. Then we have

$$\begin{aligned}
\mathrm{Hess}(\overline{h}_{\overline{v}})(\boldsymbol{u}) &= (\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \overline{\boldsymbol{v}}\rangle)\\
&= (\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \frac{\eta^2}{2(\eta-1)}\boldsymbol{x}(\boldsymbol{u}) + \frac{\eta-1}{2}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) + \eta \boldsymbol{e}_2\rangle)\\
&= \frac{\eta^2}{2(\eta-1)}(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{x}(\boldsymbol{u})\rangle) + \frac{\eta-1}{2}(\langle \boldsymbol{x}_{u_iu_j}(\boldsymbol{u}), \boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u})\rangle)\\
&= -\frac{\eta^2}{2(\eta-1)}(g_{ij}(\boldsymbol{u})) - \frac{\eta-1}{2}(h_{ij}(\boldsymbol{u})).
\end{aligned}$$

It follows that $\det(\mathrm{Hess}(\overline{h}_{\overline{v}})(\boldsymbol{u})) = 0$ if and only if $\det(\mathrm{Hess}(\overline{h}_{\overline{v}})(\boldsymbol{u})\big(g_{ij}(\boldsymbol{u})\big)^{-1}/(-\frac{\eta-1}{2})) = \det\big((h_i^j(\boldsymbol{u})) - (-(\frac{\eta}{\eta-1})^2 I)\big) = 0$. Thus, $\det(\mathrm{Hess}(\overline{h}_{\overline{v}})(\boldsymbol{u})) = 0$ if and only if $-(\frac{\eta}{\eta-1})^2$ is one of the non-zero principle curvatures of M at p. □

Let $(\boldsymbol{u}, \eta)$ be a singular point of $LD_M(\boldsymbol{u}, \eta)$. By Proposition 3.1, we have $-\frac{\eta^2}{4} = \kappa_i(p)(1 \le i \le n-1)$, where $\kappa_i(p)$ is one of the non-zero principle curvatures of M at $p = \boldsymbol{x}(\boldsymbol{u})$. It follows that we have $\eta = \pm 2\sqrt{-\kappa_i(p)}$. Then the critical value sets of LD_M are given by

$$C(LD_M)^{\pm}(u) = \bigcup_{i=1}^{n-1}\Big\{ -\kappa_i(p)\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) \pm 2\sqrt{-\kappa_i(p)}\boldsymbol{e}_2 \mid \boldsymbol{u} \in U \Big\}.$$

Let $(\boldsymbol{u}, \eta)$ be a singular point of each one of $\overline{LD}_{\overline{M}}$. By Proposition 3.2, we have $-(\frac{\eta}{\eta-1})^2 = \kappa_i(p)(1 \le i \le n-1)$, where $\kappa_i(p)$ is one of the non-zero principle curvatures of M at $p = \boldsymbol{x}(\boldsymbol{u})$. It follows that $\eta = \frac{\pm\sqrt{-\kappa_i(p)}}{\pm\sqrt{-\kappa_i(p)}-1}$. Therefore the critical value sets of $\overline{LD}_{\overline{M}}$ are given by

$$C(\overline{LD}_{\overline{M}})^{\pm}(u) = \bigcup_{i=1}^{n-1}\left\{ \frac{-\kappa_i(p)}{2(\pm\sqrt{-\kappa_i(p)}-1)}\Big(\boldsymbol{x}(\boldsymbol{u}) - \frac{1}{\kappa_i(p)}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) \pm \frac{2}{\sqrt{-\kappa_i(p)}}\boldsymbol{e}_2\Big)\Big| \ \boldsymbol{u} \in U \right\}.$$

We respectively denote that

$$LF_M^{\pm} = \bigcup_{i=1}^{n-1}\Big\{ -\kappa_i(p)\boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) \pm 2\sqrt{-\kappa_i(p)}\boldsymbol{e}_2 \mid \boldsymbol{u} \in U \Big\},$$
$$LF_{\overline{M}}^{\pm} = \bigcup_{i=1}^{n-1}\left\{ \frac{-\kappa_i(p)}{2(\pm\sqrt{-\kappa_i(p)}-1)}\Big(\boldsymbol{x}(\boldsymbol{u}) - \frac{1}{\kappa_i(p)}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}) \pm \frac{2}{\sqrt{-\kappa_i(p)}}\boldsymbol{e}_2\Big)\Big| \ \boldsymbol{u} \in U \right\}.$$

We respectively call each one of $LF_M^{\pm}$ the *ligt-cone focal surface* of M, and each one of $LF_{\overline{M}}^{\pm}$ the *sphere-cone focal surface* of $\overline{M}$. Then the projections of these surfaces to H_+ are given as follows:

$$\pi\big(C(LD_M)^{\pm}\big) = \bigcup_{i=1}^{n-1}\left\{ \pm\Big(\frac{\sqrt{-\kappa_i(p)}}{2}\boldsymbol{x}(\boldsymbol{u}) + \frac{1}{2\sqrt{-\kappa_i(p)}}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u})\Big) + \boldsymbol{e}_2 \mid \boldsymbol{u} \in U \right\},$$

$$\pi\big(C(\overline{LD}_{\overline{M}})^{\pm}\big) = \bigcup_{i=1}^{n-1}\left\{ \pm\Big(\frac{\sqrt{-\kappa_i(p)}}{2}\boldsymbol{x}(\boldsymbol{u}) + \frac{1}{2\sqrt{-\kappa_i(p)}}\boldsymbol{x_L}(\boldsymbol{u})\Big) + \boldsymbol{e}_2 \mid \boldsymbol{u} \in U \right\}.$$

By definition, we have $\boldsymbol{\varepsilon}^{\pm}_M = \Theta \circ \pi\big(C(LD_M)^{\pm}\big)$, where $\boldsymbol{\varepsilon}^{\pm}_M$ is the hyperbolic evolute of $M = \boldsymbol{x}(U)$. This means that the hyperbolic evolutes are obtained from the critical value sets of the light-cone dual hypersurfaces of $M = \boldsymbol{x}(U)$. We define $\pi^* = \Theta \circ \pi : LC^{n+1}_* \longrightarrow H^n_0$. Then we have the following theorem.

Theorem 3.3. *Both of the projections of the critical value sets $C(LD_M)^{\pm}$ and $C(\overline{LD}_{\overline{M}})^{\pm}$ in the n-dimension hyperbolic space H^n_0 are the images of the hyperbolic evolutes of M, that is,*

$$\pi^*\big(C(LD_M)^{\pm}\big) = \pi^*\big(C(\overline{LD}_{\overline{M}})^{\pm}\big) = \boldsymbol{\varepsilon}^{\pm}_M.$$

4. The two classes of dual hypersurfaces as wave fronts

We now naturally interpret the light-cone dual hypersurfaces of the submanifolds in L^n_0 and the sphere-cone dual hypersurfaces of the submanifolds in L^n_+ as wave front sets in the theory of Legendrian singularities. Let $\overline{\pi} : PT^*(LC^{n+1}_*) \longrightarrow LC^{n+1}_*$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau : TPT^*(LC^{n+1}_*) \longrightarrow PT^*(LC^{n+1}_*)$ and the differential map $d\overline{\pi} : TPT^*(LC^{n+1}_*) \longrightarrow T(LC^{n+1}_*)$ of $\overline{\pi}$. For any $X \in TPT^*(LC^{n+1}_*)$, there exists an element $\alpha \in T^*(LC^{n+1}_*)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_{\boldsymbol{v}}(LC^{n+1}_*)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we have the canonical contact structure on $PT^*(LC^{n+1}_*)$ by

$$K = \big\{X \in TPT^*(LC^{n+1}_*) \mid \tau(X)(d\overline{\pi}(X))\big\} = 0.$$

On the other hand, we consider a point $\boldsymbol{v} = (v_1, v_2, \dots, v_{n+2}) \in LC^{n+1}_*$, then we have

$$v_1 = \pm\sqrt{v_2^2 + \cdots + v_{n+2}^2}.$$

So we adopt the coordinate system $(v_2, \dots, v_{n+2})$ of LC^{n+1}_*. For the local coordinate neighborhood $\big(U, (\pm\sqrt{v_2^2 + \cdots + v_{n+2}^2}, v_2, \dots, v_{n+2})\big)$ in LC^{n+1}_*, we have a trivialization $PT^*(LC^{n+1}_*) \equiv LC^{n+1}_* \times P(\mathbb{R}^n)^*$ and we call $\big((\pm\sqrt{v_2^2 + \ldots + v_{n+2}^2}, v_2, \dots, v_{n+2}), [\xi_2 : \cdots : \xi_{n+2}]\big)$ homogeneous coordinates of $PT^*(LC^{n+1}_*)$, where $[\xi_2 : \cdots : \xi_{n+2}]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^n)^*$. It is easy to show that $X \in K_{(\boldsymbol{v},[\xi])}$ if and only if $\sum_{i=2}^{n+2} \mu_i \xi_i = 0$, where $d\overline{\pi}(X) = \sum_{i=2}^{n+2} \mu_i(\partial/\partial v_i) \in T_{\boldsymbol{v}}LC^{n+1}_*$. An immersion $i : L \longrightarrow PT^*(LC^{n+1}_*)$ is said to be a *Legendrian immersion* if $\dim(L) = n$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\overline{\pi} \circ i$ is also called the *Legendrian map* and we call the set $W(i)$=image$\overline{\pi} \circ i$ the *wave front* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the map germ $\Delta^*F : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}^{k+1}, \boldsymbol{0})$ defined by $\Delta^*F = (F, \partial F/\partial u_1, \dots, \partial F/\partial u_k)$ is nonsingular. In this case, we have the following smooth $(n-1)$-dimensional smooth submanifold

$$\Sigma_*(F) = \left\{(\boldsymbol{u}, \boldsymbol{v}) \in (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \mid F(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{v}) = \cdots = \frac{\partial F}{\partial u_k}(\boldsymbol{u}, \boldsymbol{v}) = 0\right\} = (\Delta^*F)^{-1}(\boldsymbol{0}).$$

The map germ $\mathcal{L}_F : (\Sigma_*(F), \boldsymbol{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\mathcal{L}_F(\boldsymbol{u}, \boldsymbol{v}) = \left(\boldsymbol{v}, \left[\frac{\partial F}{\partial v_1}(\boldsymbol{u}, \boldsymbol{v}) : \cdots : \frac{\partial F}{\partial v_n}(\boldsymbol{u}, \boldsymbol{v})\right]\right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 18].

Proposition 4.1. *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the wave front of $\mathcal{L}_F$ is

$$W(\mathcal{L}_F) = \left\{ \boldsymbol{v} \in \mathbb{R}^n \mid \exists\ \boldsymbol{u} \in \mathbb{R}^k \text{ such that } F(\boldsymbol{u},\boldsymbol{v}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u},\boldsymbol{v}) = \cdots = \frac{\partial F}{\partial u_k}(\boldsymbol{u},\boldsymbol{v}) = 0 \right\}.$$

We claim here that we have a trivialization as follows:

$$\Phi : PT^*(LC_*^{n+1}) \equiv LC_*^{n+1} \times P(\mathbb{R}^n)^*, \Phi([\sum_{i=2}^{n+2} \xi_i dv_i]) = (v_1, v_2, \ldots, v_{n+2}), [\xi_2 : \cdots : \xi_{n+2}]).$$

By using the above coordinate system, we have the following proposition:

Proposition 4.2. *The light-cone height function $H : U \times LC_*^{n+1} \longrightarrow \mathbb{R}$ is a Morse family of the hypersurfaces around $(\boldsymbol{u},\boldsymbol{v}) \in \Sigma_*(H)$.*

Proof. Without loss of generality, we consider the future component LC_*^{n+1}. For any $\boldsymbol{v} = (v_1, v_2, \ldots, v_{n+2}) \in LC_*^{n+1}$, we have $v_2 = \sqrt{v_1^2 - v_3^2 \cdots - v_{n+2}^2}$. For $\boldsymbol{x}(\boldsymbol{u}) = \big(x_1(\boldsymbol{u}), 0, x_3(\boldsymbol{u}), \ldots, x_{n+2}(\boldsymbol{u})\big) \in L_0^n$, we get

$$H(\boldsymbol{u},\boldsymbol{v}) = -x_1(\boldsymbol{u})v_1 + x_3(\boldsymbol{u})v_3 + \cdots + x_{n+2}(\boldsymbol{u})v_{n+2} + 2.$$

We need to prove that the mapping

$$\triangle^* H = \left(H, \frac{\partial H}{\partial u_1}, \ldots, \frac{\partial H}{\partial u_{n-1}} \right)$$

is non-singular at any point on $(\Delta^* H)^{-1}(\boldsymbol{0})$. If $(\boldsymbol{u},\boldsymbol{v}) \in (\Delta^* H)^{-1}(\boldsymbol{0})$, then $\boldsymbol{v} = LD_M(\boldsymbol{u},\eta)$ by Proposition 3.1. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$A = \begin{pmatrix} \langle \boldsymbol{x}_{u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1}}, \boldsymbol{v} \rangle & -x_1 & x_3 & \cdots & x_{n+2} \\ \langle \boldsymbol{x}_{u_1 u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_1 u_{n-1}}, \boldsymbol{v} \rangle & -x_{1u_1} & x_{3u_1} & \cdots & x_{n+2u_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \langle \boldsymbol{x}_{u_{n-1} u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1} u_{n-1}}, \boldsymbol{v} \rangle & x_{1u_{n-1}} & x_{3u_{n-1}} & \cdots & x_{n+2u_{n-1}} \end{pmatrix}.$$

Since $\{\boldsymbol{x}, \boldsymbol{x}_{u_1}, \ldots, \boldsymbol{x}_{u_{n-1}}\}$ are linearly independent, $\operatorname{rank}(A) = n$. This completes the proof. □

Proposition 4.3. *The sphere-cone height function $\overline{H} : U \times LC_*^{n+1} \longrightarrow \mathbb{R}$ is a Morse family of the hypersurfaces around $(\boldsymbol{u},\overline{\boldsymbol{v}}) \in \Sigma_*(\overline{H})$.*

Proof. Without loss of generality, we consider the future component LC_*^{n+1}. For any $\overline{\boldsymbol{v}} = (v_1, v_2, \ldots, v_{n+2}) \in LC_*^{n+1}$, we have $v_1 = \sqrt{v_2^2 + \cdots + v_{n+2}^2}$. For $\overline{\boldsymbol{x}}(\boldsymbol{u}) = \big(x_1(\boldsymbol{u}), 1, x_3(\boldsymbol{u}), \ldots, x_{n+2}(\boldsymbol{u})\big) \in L_+^n$, we get

$$\overline{H}(\boldsymbol{u},\overline{\boldsymbol{v}}) = -x_1(\boldsymbol{u})\sqrt{v_2^2 + \cdots + v_{n+2}^2} + v_2 + x_3(\boldsymbol{u})v_3 + \cdots + x_{n+2}(\boldsymbol{u})v_{n+2} - 1.$$

We need to prove the mapping

$$\triangle^* \overline{H} = \left(\overline{H}, \frac{\partial \overline{H}}{\partial u_1}, \ldots, \frac{\partial \overline{H}}{\partial u_{n-1}} \right)$$

is non-singular at any point on $(\Delta^* \overline{H})^{-1}(\boldsymbol{0})$. If $(\boldsymbol{u},\overline{\boldsymbol{v}}) \in (\Delta^* \overline{H})^{-1}(\boldsymbol{0})$, then $\overline{\boldsymbol{v}} = \overline{LD}_{\overline{M}}(\boldsymbol{u},\eta)$ by Proposition 3.2. The Jacobian matrix of $\Delta^* \overline{H}$ is given as follows:

$$A = \begin{pmatrix} \langle \overline{\boldsymbol{x}}_{u_1}, \boldsymbol{v} \rangle & \cdots & \langle \overline{\boldsymbol{x}}_{u_{n-1}}, \boldsymbol{v} \rangle & -\frac{v_2}{v_1}x_1 + 1 & -\frac{v_3}{v_1}x_1 + x_3 & \cdots & -\frac{v_{n+2}}{v_1}x_1 + x_{n+2} \\ \langle \overline{\boldsymbol{x}}_{u_1 u_1}, \boldsymbol{v} \rangle & \cdots & \langle \overline{\boldsymbol{x}}_{u_1 u_{n-1}}, \boldsymbol{v} \rangle & -\frac{v_2}{v_1}x_{1u_1} + 1 & -\frac{v_3}{v_1}x_{1u_1} + x_{3u_1} & \cdots & -\frac{v_{n+2}}{v_1}x_{1u_1} + x_{n+2u_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \langle \overline{\boldsymbol{x}}_{u_{n-1} u_1}, \boldsymbol{v} \rangle & \cdots & \langle \overline{\boldsymbol{x}}_{u_{n-1} u_{n-1}}, \boldsymbol{v} \rangle & -\frac{v_2}{v_1}x_{1u_{n-1}} + 1 & -\frac{v_3}{v_1}x_{1u_{n-1}} + x_{3u_{n-1}} & \cdots & -\frac{v_{n+2}}{v_1}x_{1u_{n-1}} + x_{n+2u_{n-1}} \end{pmatrix}.$$

We now prove that rank $A = n$. For $(x_1, 0, x_3, \ldots, x_{n+2}) = \boldsymbol{x}$ and $(\frac{v_1}{v_2}, 0, \frac{v_3}{v_2}, \ldots, \frac{v_{n+2}}{v_2}) = \frac{v}{v_2} - \boldsymbol{e}_2 = \frac{\eta}{2(\eta-1)}\boldsymbol{x} + \frac{\eta-1}{2\eta}\boldsymbol{x_L}$, we have

$$(-\frac{v_1}{v_2} + x_1, 0, \frac{v_3}{v_2} + x_3, \ldots, \frac{v_{n+2}}{v_2} + x_{n+2}) = \boldsymbol{x} - (\boldsymbol{v}/v_2 - \boldsymbol{e}_2) = \frac{\eta - 2}{2(\eta - 1)}\boldsymbol{x} - \frac{\eta - 1}{2\eta}\boldsymbol{x_L}.$$

Since $\{\frac{\eta-2}{2(\eta-1)}\boldsymbol{x} - \frac{\eta-1}{2\eta}\boldsymbol{x_L}, \boldsymbol{x}_{u_1}, \ldots, \boldsymbol{x}_{u_{n-1}}\}$ are linearly independent, rank$(A) = n$. This completes the proof. □

Here, we consider the Legendrian immersion

$$\mathcal{L}_4 : (\boldsymbol{u}, \eta) \longrightarrow \Delta_4,\ \mathcal{L}_4(\boldsymbol{u}, \eta) = (LD_M(\boldsymbol{u}, \eta), \boldsymbol{x}(\boldsymbol{u})).$$

We define the following:

$$\Psi : \Delta_4 \longrightarrow LC^{n+1}_* \times P(\mathbb{R}^n)^*, \Psi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, [v_1w_2 - v_2w_1 : \cdots : v_1w_{n+2} - v_{n+2}w_1]).$$

For the canonical contact form $\theta = \sum_{i=2}^{n+2} \xi_i dv_i$ on $PT^*(LC^{n+1}_*)$, we have

$$\begin{aligned}\Psi^*\theta &= (v_1w_2 - v_2w_1)dv_2 + \cdots + (v_1w_{n+2} - v_{n+2}w_1)dv_{n+2}|_{\Delta_4}\\ &= v_1(-w_1dv_1 + w_2dv_2 + \cdots + w_{n+2}dv_{n+2}) - w_1(-v_1dv_1 + v_2dv_2 + \cdots + v_{n+2}dv_{n+2})|_{\Delta_4}\\ &= v_1\langle \boldsymbol{w}, d\boldsymbol{v}\rangle|_{\Delta_4} = v_1\theta_{42}|_{\Delta_4}.\end{aligned}$$

Thus Ψ is a contact morphism.

Theorem 4.4. *For any hypersurface $\boldsymbol{x} : U \longrightarrow L^n_0$, the light-cone height function $H : U \times LC^{n+1}_* \longrightarrow \mathbb{R}$ is a generating family of the Legendrian immersion $\mathcal{L}_4$.*

Proof. Since H is a Morse family of hypersurfaces, we have a Legendrian immersion $\mathcal{L}_H : \Sigma_*(H) \longrightarrow PT^*(LC^{n+1}_*)$ defined by $\mathcal{L}_H(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, [\partial H/\partial v_2(\boldsymbol{u}, \boldsymbol{v}) : \cdots : \partial H/\partial v_{n+2}(\boldsymbol{u}, \boldsymbol{v})])$, where $\boldsymbol{v} = (v_1, \ldots, v_{n+2})$ and $\Sigma_*(H) = \{(\boldsymbol{u}, \boldsymbol{v}) \in U \times LC^{n+1}_* \mid \boldsymbol{u} \in U, \boldsymbol{v} = LD_M(\boldsymbol{u}, \eta), \eta \in \mathbb{R}\}$. We observe that H is a generating family of the Legendrian submanifold $\mathcal{L}_H(\Sigma_*(H))$ whose wave front is the image of LD_M. We have

$$\frac{\partial H}{\partial v_i}(\boldsymbol{u}, \boldsymbol{v}) = -\frac{l_i(\boldsymbol{u}, \eta)}{l_1(\boldsymbol{u}, \eta)}x_1(\boldsymbol{u}) + x_i(\boldsymbol{u}),\ (i = 2, \ldots, n + 2),$$

where $\boldsymbol{x}(\boldsymbol{u}) = \big(x_1(\boldsymbol{u}), 0, \ldots, x_{n+2}(\boldsymbol{u})\big)$ and $\boldsymbol{v} = LD_M(\boldsymbol{u}, \eta) = \big(l_1(\boldsymbol{u}, \eta), \ldots, l_{n+2}(\boldsymbol{u}, \eta)\big)$. It follows that

$$\mathcal{L}_H\big(\boldsymbol{u}, LD_M(\boldsymbol{u}, \eta)\big) = \big(LD_M(\boldsymbol{u}, \eta), [l_1(\boldsymbol{u}, \eta)x_2(\boldsymbol{u}) - l_2(\boldsymbol{u}, \eta)x_1(\boldsymbol{u}) : \cdots : l_1(\boldsymbol{u}, \eta)x_{n+2}(\boldsymbol{u}) - l_{n+2}(\boldsymbol{u}, \eta)x_1(\boldsymbol{u})]\big).$$

Therefore we have $\Psi \circ \mathcal{L}_4(\boldsymbol{u}, \eta) = \mathcal{L}_H(\boldsymbol{u}, \eta)$. This completes the proof. □

Similarly, we consider the Legendrian immersion $\mathcal{L}_3 : (\boldsymbol{u}, \eta) \longrightarrow \Delta_3$ defined by

$$\mathcal{L}_3(\boldsymbol{u}, \eta) = (\overline{LD}_{\overline{M}}(\boldsymbol{u}, \eta), \overline{\boldsymbol{x}}(\boldsymbol{u})).$$

Then we have the following theorem.

Theorem 4.5. *For any hypersurface $\overline{\boldsymbol{x}} : U \longrightarrow L^n_+$, the sphere-cone height function $\overline{H} : U \times LC^{n+1}_* \longrightarrow \mathbb{R}$ is a generating family of the Legendrian immersion $\mathcal{L}_3$.*

5. Contact with parabolic $(n-1)$-light-cone and parabolic n-hyperquadrics

Before we start to consider the contact between hypersurfaces in the light-cone with parabolic $(n-1)$-light-cone and parabolic n-hyperquadrics, we briefly review the theory of contact due to Montaldi [11]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with dim(X_1)=dim(X_2) and dim(Y_1)=dim(Y_2). We say that the contact of X_1 and Y_1 at y_1 is *the same type as the contact of* X_2 *and* Y_2 *at* y_2 if there is a diffeomorphism $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1, y_1) = K(X_2, Y_2, y_2)$. Of course, in the definition, $\mathbb{R}^n$ can be replaced by any manifold. Two function germs $f_i : (\mathbb{R}^n, a_i) \longrightarrow \mathbb{R}$ $(i = 1, 2)$ are called $\mathcal{K}$-equivalent if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \longrightarrow (\mathbb{R}^n, a_2)$, and a function germ $\lambda : (\mathbb{R}^n, a_1) \longrightarrow \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (f_2 \circ \Phi)$.

Theorem 5.1 (Montaldi [11])**.** *Let X_i, Y_i (for i=1,2) be submanifolds of $\mathbb{R}^n$ with $dimX_1$=$dimX_2$ and $dimY_1$=$dimY_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1, y_1) = K(X_2, Y_2, y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Returning to the light-cone dual hypersurface LD_M, we now consider the function $\mathfrak{h} : L_0^n \times LC_*^{n+1} \longrightarrow \mathbb{R}$ defined by $\mathfrak{h}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$ and the function $\mathfrak{g} : LC_*^{n+1} \times LC_*^{n+1} \longrightarrow \mathbb{R}$ defined by $\mathfrak{g}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$. For a given $\boldsymbol{v}_0 \in LC_*^{n+1}$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathfrak{h}(\boldsymbol{u}, \boldsymbol{v_0})$ and $\mathfrak{g}_{v_0}(\boldsymbol{u}) = \mathfrak{g}(\boldsymbol{u}, \boldsymbol{v_0})$, then we have $\mathfrak{h}_{v_0}^{-1}(0) = L_0^n \cap HP(\boldsymbol{v_0}, -2)$ and $\mathfrak{g}_{v_0}^{-1}(0) = LC_*^{n+1} \cap HP(\boldsymbol{v_0}, -2)$. For any $\boldsymbol{u}_0 \in U$, $\eta_0 \in \mathbb{R}$, we take the point $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \eta_0)$. Then we have

$$\mathfrak{g}_{v_0} \circ \boldsymbol{x}(\boldsymbol{u}_0) = \mathfrak{g} \circ (\boldsymbol{x} \times id_{LC_*^{n+1}})(\boldsymbol{u}_0, \boldsymbol{v}_0) = \mathfrak{h}_{v_0} \circ \boldsymbol{x}(\boldsymbol{u}_0) = \mathfrak{h} \circ (\boldsymbol{x} \times id_{LC_*^{n+1}})(\boldsymbol{u}_0, \boldsymbol{v}_0) = H(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0.$$

We also have

$$\frac{\partial(\mathfrak{g}_{v_0} \circ \boldsymbol{x})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial(\mathfrak{h}_{v_0} \circ \boldsymbol{x})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial H}{\partial u_i}(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0$$

for $i = 1, \ldots, n-1$. This means that the $(n-1)$-hyperquadrics $\mathfrak{h}_{v_0}^{-1}(0) = L_0^n \cap HP(\boldsymbol{v_0}, -2)$ is tangent to $M = \boldsymbol{x}(U)$ at $p_0 = \boldsymbol{x}(\boldsymbol{u}_0)$. In this case, we call it the *light-cone tangent parabolic $(n-1)$-hyperquadrics* of M at p_0, which is denoted by $TPL_0^{n-1}(\boldsymbol{x}, \boldsymbol{u}_0)$. The n-hyperquadric $\mathfrak{g}_{v_0}^{-1}(0) = LC_*^{n+1} \cap HP(\boldsymbol{v_0}, -2)$ is also tangent to M at p_0. In this case, we call it the *light-cone tangent parabolic n-hyperquadric* of M at p_0, which is denoted by $TPLC_*^n(\boldsymbol{x}, \boldsymbol{u}_0)$. For the sphere-cone dual surfaces $\overline{LD}_{\overline{M}}$, we consider a function $\overline{\mathfrak{h}} : L_+^n \times LC_*^{n+1} \longrightarrow \mathbb{R}$ defined by $\overline{\mathfrak{h}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$ and a function $\overline{\mathfrak{g}} : S_1^{n+1} \times LC_*^{n+1} \longrightarrow \mathbb{R}$ defined by $\overline{\mathfrak{g}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$. For a given $\boldsymbol{v}_0 \in LC_*^{n+1}$, we denote that $\overline{\mathfrak{h}}_{v_0}(\boldsymbol{u}) = \overline{\mathfrak{h}}(\boldsymbol{u}, \boldsymbol{v_0})$ and $\overline{\mathfrak{g}}_{v_0}(\boldsymbol{u}) = \overline{\mathfrak{g}}(\boldsymbol{u}, \boldsymbol{v_0})$. Then we have $\overline{\mathfrak{h}}_{v_0}^{-1}(0) = L_+^n \cap HP(\boldsymbol{v_0}, 1)$ and $\overline{\mathfrak{g}}_{v_0}^{-1}(0) = S_1^{n+1} \cap HP(\boldsymbol{v_0}, 1)$. For any $\boldsymbol{u}_0 \in U$ and the points $\overline{\boldsymbol{v}}_0 = \overline{LD}_{\overline{M}}(\boldsymbol{u}_0, \eta_0)$, we have

$$\overline{\mathfrak{g}}_{\overline{\boldsymbol{v}}_0} \circ \overline{\boldsymbol{x}}(\boldsymbol{u}_0) = \overline{\mathfrak{g}} \circ (\overline{\boldsymbol{x}} \times id_{LC_*^{n+1}})(\boldsymbol{u}_0, \overline{\boldsymbol{v}}_0) = \overline{\mathfrak{h}}_{\overline{v}_0} \circ \overline{\boldsymbol{x}}(\boldsymbol{u}_0) = \overline{\mathfrak{h}} \circ (\overline{\boldsymbol{x}} \times id_{LC_*^{n+1}})(\boldsymbol{u}_0, \overline{\boldsymbol{v}}_0) = \overline{H}(\boldsymbol{u}_0, \overline{\boldsymbol{v}}_0) = 0.$$

We also have

$$\frac{\partial(\overline{\mathfrak{g}}_{\overline{v}_0} \circ \overline{\boldsymbol{x}})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial(\overline{\mathfrak{h}}_{\overline{\boldsymbol{v}}_0} \circ \overline{\boldsymbol{x}})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial \overline{H}}{\partial u_i}(\boldsymbol{u}_0, \overline{\boldsymbol{v}}_0) = 0$$

for $i = 1, \cdots, n-1$. It follows that each one of the $(n-1)$-hyperquadric $\overline{\mathfrak{h}}_{\overline{v}_0}^{-1}(0) = L_+^n \cap HP(\overline{\boldsymbol{v}}_0, 1)$ is tangent to $\overline{M}$ at $\overline{p}_0 = \overline{\boldsymbol{x}}(\boldsymbol{u}_0)$. In this case, we call each one the *de-Sitter tangent parabolic $(n-1)$-hyperquadric* of $\overline{M}$ at $\overline{p}_0$, which are denoted by $TPL_+^{n-1}(\boldsymbol{x}, \boldsymbol{u}_0)$. Also we have each of the n-hyperquadric $\overline{\mathfrak{g}}_{\overline{\boldsymbol{v}}_0}^{-1}(0) = S_1^{n+1} \cap HP(\overline{\boldsymbol{v}}_{\boldsymbol{0}}, 1)$ is tangent to $\overline{M}$ at $\overline{p}_0$. In this case, we call each one the *de-Sitter tangent parabolic n-hyperquadric* of $\overline{M}$ at $\overline{p}_0$, which are denoted by $TPS_1^n(\overline{\boldsymbol{x}}, \boldsymbol{u}_0)$.

Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (L_0^n, p_i)$ $(i = 1, 2)$ be hypersurface germs. For $\boldsymbol{v}_i = LD_{M_i}(\boldsymbol{u}_i, \eta_i)$, we denote $h_{i,v_i} : (U, \boldsymbol{u}_i) \longrightarrow (\mathbb{R}, 0)$ by $h_{i,v_i}(\boldsymbol{u}_i) = H(\boldsymbol{u}_i, \boldsymbol{v}_i)$. Then we have $h_{i,v_i}(\boldsymbol{u}) = (\mathfrak{h}_{i,v_i} \circ \boldsymbol{x}_i)(\boldsymbol{u}) = (\mathfrak{g}_{i,v_i} \circ \boldsymbol{x}_i)(\boldsymbol{u})$. For $\overline{\boldsymbol{v}}_i = \overline{LD}_{\overline{M}_i}(\boldsymbol{u}_i, \eta_i)$, we denote $\overline{h}_{i,\overline{v}_i} : (U, \boldsymbol{u}_i) \longrightarrow (\mathbb{R}, 0)$ by $\overline{h}_{i,\overline{v}_i}(\boldsymbol{u}_i) = \overline{H}(\boldsymbol{u}_i, \overline{\boldsymbol{v}}_i)$. Then we have $\overline{h}_{i,\overline{v}_i}(\boldsymbol{u}) = (\overline{\mathfrak{h}}_{i,\overline{v}_i} \circ \overline{\boldsymbol{x}}_i)(\boldsymbol{u}) = (\overline{\mathfrak{g}}_{i,\overline{v}_i} \circ \overline{\boldsymbol{x}}_i)(\boldsymbol{u})$. By Theorem 5.1, we have the following proposition.

Proposition 5.2. *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (L_0^n, p_i)$ $(i = 1, 2)$ be hypersurface germs. For $\boldsymbol{v}_i = LD_{M_i}(\boldsymbol{u}_i, \eta_i)$, the following conditions are equivalent:*

(i) $K(\boldsymbol{x}_1(U), TPL_0^{n-1}(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPL_0^{n-1}(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(ii) $K(\boldsymbol{x}_1(U), TPLC_*^n(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPLC_*^n(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(iii) h_{1,v_1} *and* h_{2,v_2} *are* $\mathcal{K}$*-equivalent.*

Moreover, for $\overline{\boldsymbol{v}}_i = \overline{LD}_{\overline{M}_i}(\boldsymbol{u}_i, \eta_i)$*, the following conditions are equivalent:*

(iv) $K(\boldsymbol{x}_1(U), TPL_+^{n-1}(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1) = K(\boldsymbol{x}_2(U), TPL_+^{n-1}(\boldsymbol{x}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2)$.

(v) $K(\boldsymbol{x}_1(U), TPS_1^n(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1) = K(\boldsymbol{x}_2(U), TPS_1^n(\boldsymbol{x}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2)$.

(vi) $\overline{h}_{1,\overline{v}_1}$ *and* $\overline{h}_{2,\overline{v}_2}$ *are* $\mathcal{K}$*-equivalent.*

On the other hand, we return to the review on the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we say that $\mathcal{L}_F(\Sigma_*(F))$ and $\mathcal{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, z) \longrightarrow (PT^*\mathbb{R}^n, z')$ such that H preserves fibers of π and that $H(\mathcal{L}_F(\Sigma_*(F))) = \mathcal{L}_G(\Sigma_*(G))$, where $z = \mathcal{L}_F(0), z' = \mathcal{L}_G(0)$. By using the Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs by the ordinary way (see, [1, Part III]). We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(\boldsymbol{q}, \boldsymbol{x}) = (\psi_1(\boldsymbol{q}, \boldsymbol{x}), \psi_2(\boldsymbol{x}))$ for $(\boldsymbol{q}, \boldsymbol{x}) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here, $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. We say that F is an *infinitesimally $\mathcal{K}$-versal deformation of* $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \dots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

The main result in the theory of Legendrian singularities ([1, §20.8] and [18, Theorem 2]) is the following:

Proposition 5.3 (Arnol'd, Zakalyukin)**.** *Let* $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ *be Morse families and we denote the corresponding Legendrian immersion germs by* $\mathcal{L}_F, \mathcal{L}_G$*. Then*

(i) $\mathcal{L}_F$ *and* $\mathcal{L}_G$ *are Legendrian equivalent if and only if* F *and* G *are* $\mathcal{P}$*-*$\mathcal{K}$*-equivalent.*
(ii) $\mathcal{L}_F$ *is Legendrian stable if and only if* F *is* $\mathcal{K}$*-versal deformation of* f*.*

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation [18, page 27]. For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define *the local ring of* f by $Q_r(f) = \mathcal{E}_n/(f^*(\mathfrak{M}_p)\mathcal{E}_n + \mathfrak{M}_n^{r+1})$. We have the following classification result of Legendrian stable germs (cf. [7, Proposition A.4]) which is the key for the purpose in this section.

Proposition 5.4. *Let* $F, G : (\mathbb{R}^n \times \mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ *be Morse families. Suppose that Legendrian immersion germs* $\mathcal{L}_F$ *and* $\mathcal{L}_G$ *are Legendrian stable, then the following conditions are equivalent.*

(i) $W(\mathcal{L}_F)$ *and* $W(\mathcal{L}_G)$ *are diffeomorphic as set germs.*

(ii) $\mathcal{L}_F$ *and* $\mathcal{L}_G$ *are Legendrian equivalent.*

(iii) $Q_{n+1}(f)$ *and* $Q_{n+1}(g)$ *are isomorphic as* $\mathbb{R}$*-algebras, where* $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ *and* $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$*.*

Let $Q_{n+1}(\boldsymbol{x}, u_0)$ be the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q_{n+1}(\boldsymbol{x}, \boldsymbol{u}_0) = C^\infty_{u_0}(U)/(\langle h_{v_0} \rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}_{n-1}^{n+2}),$$

and $Q_{n+1}(\overline{\boldsymbol{x}}, u_0)$ be the local ring of the function germ $\overline{h}_{\overline{v}_0} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q_{n+1}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0) = C^\infty_{u_0}(U)/(\langle \overline{h}_{\overline{v}_0} \rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}^{n+2}_{n-1}),$$

where $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \eta_0)$, $\overline{\boldsymbol{v}}_0 = \overline{LD}_{\overline{M}}(\boldsymbol{u}_0, \eta_0)$, and $C^\infty_{u_0}(U)$ is the local ring of function germ at $\boldsymbol{u}_0$ with the unique maximal ideal $\mathfrak{M}_{n-1}$.

Theorem 5.5. *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (L^n_0, p_i)$ $(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.*

(i) *The lightcone hypersurface germs $LD_{M_1}(U \times \mathbb{R})$ and $LD_{M_2}(U \times \mathbb{R})$ are diffeomorphic.*

(ii) *Legendrian immersion germs $\mathcal{L}^1_4$ and $\mathcal{L}^2_4$ are Legendrian equivalent.*

(iii) *The lightcone height functions germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.*

(iv) *h_{1,v_1} and h_{2,v_2} are $\mathcal{K}$-equivalent.*

(v) $K(\boldsymbol{x}_1(U), TPL^{n-1}_0(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPL^{n-1}_0(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(vi) $K(\boldsymbol{x}_1(U), TPLC^n_*(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPLC^n_*(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(vii) *Local rings $Q_{n+1}(\boldsymbol{x}_1, \boldsymbol{u}_1)$ and $Q_{n+1}(\boldsymbol{x}_2, \boldsymbol{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.*

Proof. By Proposition 5.3 and Proposition 5.4, the conditions (i) ∼(iii) and (vii) are equivalent. By definition, the condition (iii) implies the condition (iv). By Proposition 5.3, H_i is a $\mathcal{K}$-versal deformation of h_{i,v_i}. We can apply the uniqueness result of $\mathcal{K}$-versal deformations (cf., [9]), so that the condition (iv) implies the condition (iii). By Theorem 5.1, the conditions (iv) ∼ (vi) are equivalent. This completes the proof. □

Theorem 5.6. *Let $\overline{\boldsymbol{x}}_i : (U, \boldsymbol{u}_i) \longrightarrow (L_+, p_i)$ $(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.*

(i) *The lightcone hypersurface germs $\overline{LD}_{\overline{M}_1}(U \times \mathbb{R})$ and $\overline{LD}_{\overline{M}_2}(U \times \mathbb{R})$ are diffeomorphic.*

(ii) *Legendrian immersion germs $\mathcal{L}^1_3$ and $\mathcal{L}^2_3$ are Legendrian equivalent.*

(iii) *The lightcone height functions germs $\overline{H}_1$ and $\overline{H}_2$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.*

(iv) *$\overline{h}_{1,\overline{v}_1}$ and $\overline{h}_{2,\overline{v}_2}$ are $\mathcal{K}$-equivalent.*

(v) $K(\overline{\boldsymbol{x}}_1(U), TPL^{n-1}_+(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1) = K(\overline{\boldsymbol{x}}_2(U), TPL^{n-1}_+(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2)$.

(vi) $K(\overline{\boldsymbol{x}}_1(U), TPS^n_1(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1) = K(\overline{\boldsymbol{x}}_2(U), TPS^n_1(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2)$.

(vii) *Local rings $Q_{n+1}(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1)$ and $Q_{n+1}(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2)$ are isomorphic as $\mathbb{R}$-algebras.*

The proof is similar to the proof of the above theorem, so that we omit it.

Lemma 5.7. *Let $\boldsymbol{x} : U \longrightarrow L^n_0$ be a hypersurface germ such that the corresponding Legendrian immersion germs $\mathcal{L}_4$ and $\mathcal{L}_3$ are Legendrian stable. Then at the singular point $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \pm 2\sqrt{-\kappa_i(p_0)})(1 \leq i \leq n-1)$ of LD_M and the singular points $\overline{\boldsymbol{v}}_0 = \overline{LD}_{\overline{M}}(\boldsymbol{u}_0, \frac{\pm\sqrt{-\kappa_i(p_0)}}{\pm\sqrt{-\kappa_i(p_0)}-1})$ of $\overline{LD}_{\overline{M}}$, we have the following equivalent assertions.*

(i) *The lightcone hypersurface germs $LD_M(U \times \mathbb{R})$ and $\overline{LD}_{\overline{M}}(U \times \mathbb{R})$ are diffeomorphic.*

(ii) *Legendrian immersion germs $\mathcal{L}_4$ and $\mathcal{L}_3$ are Legendrian equivalent.*

(iii) *The lightcone height functions germs H and $\overline{H}$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.*

(iv) h_{v_0} *and* $\overline{h}_{\overline{v}_0}$ *are* $\mathcal{K}$*-equivalent.*

(v) $K(\boldsymbol{x}(U), TPL_0^{n-1}(\boldsymbol{x}, \boldsymbol{u}_0), \boldsymbol{v}_0) = K(\overline{\boldsymbol{x}}(U), TPL_+^{n-1}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0), \overline{\boldsymbol{v}}_0)$.

(vi) $K(\boldsymbol{x}(U), TPLC_*^n(\boldsymbol{x}, \boldsymbol{u}_0), \boldsymbol{v}_0) = K(\overline{\boldsymbol{x}}(U), TPS_1^n(\overline{\boldsymbol{x}}, \boldsymbol{u}_0), \overline{\boldsymbol{v}}_0)$.

(vii) *Local rings* $Q_{n+1}(\boldsymbol{x}, \boldsymbol{u}_0)$ *and* $Q_{n+1}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0)$ *are isomorphic as* $\mathbb{R}$*-algebras.*

Proof. By definition, we have

$$h_{v_0}(\boldsymbol{u}) = \langle \boldsymbol{x}(\boldsymbol{u}), -\kappa_i(p_0)\boldsymbol{x}(\boldsymbol{u}_0) + \boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}_0) \pm 2\sqrt{-\kappa_i(p_0)}\boldsymbol{e}_2\rangle + 2,$$

so that

$$\frac{h_{v_0}(\boldsymbol{u})}{\pm 2\sqrt{-\kappa_i(p_0)}} = \Big\langle \boldsymbol{x}(\boldsymbol{u}), \pm\Big(\frac{\sqrt{-\kappa_i(p_0)}}{2}\boldsymbol{x}(\boldsymbol{u}_0) + \frac{1}{2\sqrt{-\kappa_i(p_0)}}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}_0)\Big) + \boldsymbol{e}_2\Big\rangle \pm \frac{1}{\sqrt{-\kappa_i(p_0)}}.$$

We also have

$$\overline{h}_{\overline{v}_0}(\boldsymbol{u}) = \Big\langle \boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{e}_2, \frac{-\kappa_i(p_0)}{2(\pm\sqrt{-\kappa_i(p_0)} - 1)}\Big(\boldsymbol{x}(\boldsymbol{u}_0) - \frac{1}{\kappa_i(p_0)}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}_0) \pm \frac{2}{\sqrt{-\kappa_i(p_0)}}\boldsymbol{e}_2\Big)\Big\rangle - 1,$$

and

$$\begin{aligned}\frac{(\pm\sqrt{-\kappa_i(p_0)} - 1)\overline{h}_{\overline{v}_0}(\boldsymbol{u})}{\pm\sqrt{-\kappa_i(p_0)}} &= \Big\langle \boldsymbol{x}(\boldsymbol{u}) + \boldsymbol{e}_2, \pm\Big(\frac{\sqrt{-\kappa_i(p_0)}}{2}\boldsymbol{x}(\boldsymbol{u}_0) + \frac{1}{2\sqrt{-\kappa_i(p_0)}}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}_0)\Big) + \boldsymbol{e}_2\Big\rangle \\ &\quad \mp \frac{(\pm\sqrt{-\kappa_i(p_0)} - 1)}{\sqrt{-\kappa_i(p_0)}} \\ &= \Big\langle \boldsymbol{x}(\boldsymbol{u}), \pm\Big(\frac{\sqrt{-\kappa_i(p_0)}}{2}\boldsymbol{x}(\boldsymbol{u}_0) + \frac{1}{2\sqrt{-\kappa_i(p_0)}}\boldsymbol{x}_{\boldsymbol{L}}(\boldsymbol{u}_0)\Big) + \boldsymbol{e}_2\Big\rangle \pm \frac{1}{\sqrt{-\kappa_i(p_0)}}.\end{aligned}$$

Therefore, we have

$$h_{v_0} = 2(\pm\sqrt{-\kappa_i(p_0)} - 1)\overline{h}_{\overline{v}_0}.$$

This means that the assertion (iv) holds. By the uniqueness of the $\mathcal{K}$-versal deformation, we have the assertion (iii). By Proposition 5.3, we have the assertion (ii). By Proposition 5.4, we have the assertions (i) and (vii). On the other hand, for $\mathfrak{g}_{v_0} \circ \boldsymbol{x} = \mathfrak{h}_{v_0} \circ \boldsymbol{x} = h_{v_0}$ and $\overline{\mathfrak{g}}_{\overline{v}_0} \circ \overline{\boldsymbol{x}} = \overline{\mathfrak{h}}_{\overline{v}_0} \circ \overline{\boldsymbol{x}} = \overline{h}_{\overline{v}_0}$, by Theorem 5.1, we have the assertions (v) and (vi). This completes the proof. □

By Lemma 5.7, we have our main result as the following theorem.

Theorem 5.8. *Let* $\boldsymbol{x}_i : (U, \boldsymbol{u}_i) \longrightarrow (L_0^n, p_i)$ $(i = 1, 2)$ *be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. At the singular points* $\boldsymbol{v}_j = LD_M(\boldsymbol{u}_0, \pm 2\sqrt{-\kappa_j(p)})$ $(1 \leq j \leq n-1)$ *of* LD_M*, and the singular points* $\overline{\boldsymbol{v}}_j = \overline{LD}_{\overline{M}}(\boldsymbol{u}_0, \frac{\pm\sqrt{-\kappa_j(p_0)}}{\pm\sqrt{-\kappa_j(p)} - 1})$ *of* $\overline{LD}_{\overline{M}}$*, the conditions* (i) $\sim$ (vii) *in Theorem* 5.5 *and the conditions* (i) $\sim$ (vii) *in Theorem* 5.6 *are all equivalent.*

6. Surfaces in the 3-lightcone

In this section, we stick to the case $n = 3$. We consider the surfaces in the 3-lightcone as a special case of the previous sections. First, we consider the generic properties of spacelike submanifolds in the open lightcone L_0^3. We consider the space of embeddings $\mathrm{Emb}(U, L_0^3)$ with Whitney C^∞-topology. We also consider the function $\mathcal{H} : L_0^3 \times LC_*^{n+1} \longrightarrow \mathbb{R}$ which is given by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v}\rangle + 2$. We claim that $\mathfrak{h}_v$ is a submersion for any $\boldsymbol{v} \in LC_*^{n+1}$, where $\mathfrak{h}_v(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{x} \in \mathrm{Emb}(U, L_0^3)$, we have $H = \mathcal{H} \circ (\boldsymbol{x} \times id_{LC_*^{n+1}})$. We have the k-jet extension

$$j_1^k \overline{H} : U \times LC_*^{n+1} \longrightarrow J^k(U, \mathbb{R}),$$

defined by $j_1^k H(\boldsymbol{u}, \boldsymbol{v}) = j^k h_v(\boldsymbol{u})$. We consider the trivialization $J^k(U, \mathbb{R}) = U \times \mathbb{R} \times J^k(2,1)$. For any submanifold $Q \subset J^k(2,1)$, we denote $\tilde{Q} = U \times 0 \times Q$. Then we have the following proposition as a corollary of [17, Lemma 6].

Proposition 6.1. *Let Q be a submanifold of $J^k(2,1)$. Then the set*

$$T_Q = \{\boldsymbol{x} \in \mathrm{Emb}(U, L_0^3) \mid j_1^k H \textit{ is transversal to } \tilde{Q}\},$$

is a residual subset of $\mathrm{Emb}(U, L_0^3)$. *If Q is a closed set, then T_Q is open.*

By the previous arguments and Appendix of [7], we have the following theorem.

Theorem 6.2. *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}(U, L_0^3)$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the corresponding Legendrian immersion germ $\mathcal{L}_4$ at any point is Legendrian stable.*

If we consider $\overline{\mathcal{H}} : L_+^3 \times LC_*^4 \longrightarrow \mathbb{R}$ defined by $\overline{\mathcal{H}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$ instead of $\mathcal{H} : L_0^3 \times LC_*^4 \longrightarrow \mathbb{R}$, we can show that the corresponding Legendrian immersion germ $\mathcal{L}_3$ at any point is Legendrian stable for a generic hypersurface $\overline{\boldsymbol{x}} : U \longrightarrow L_+^3$.

We now classify the singularities of the light-cone dual hypersurfaces. Here, we only consider the case for $M = \boldsymbol{x}(U)$ in L_0^3. By Theorem 5.5, a $\mathcal{K}$-invariant for the height function h_v is an invariant for the diffeomorphism class of the singularities of the lightcone duals of a hypersurface in L_0^3. Let $\boldsymbol{x} : U \longrightarrow L_0^3$ be an embedding from an open set $U \subset \mathbb{R}^2$, we define the $\mathcal{K}$-codimension (or Tyurina number) of the function germ h_{v_0} by

$$H\text{-ord}(\boldsymbol{x}, u_0) = \dim C_{u_0}^\infty / \langle h_{v_0}, \partial h_{v_0}/\partial u_i \rangle_{C_{u_0}^\infty}.$$

We call it the *order of contact* of M with parabolic $(n-1)$-hyperquadrics and parabolic n-hyperquadrics. We also define the corank of the function germ h_{v_0} by

$$H\text{-corank}(\boldsymbol{x}, u_0) = 2 - \mathrm{rank}(\mathrm{Hess}(h_{v_0})(u_0)).$$

By Theorem 4.4, Theorem 6.2 and Proposition 5.3, the light-cone height function H is a $\mathcal{K}$-versal deformation of h_{v_0} at each point $(\boldsymbol{u}_0, \boldsymbol{v}_0) \in U \times LC_*^4$. Therefore we can apply the classification of $\mathcal{K}$-versal deformations of function germs up to 4-parameters [1]. Suppose that the lightcone height function H is a $\mathcal{K}$-versal deformation of h_{v_0} at each point $(\boldsymbol{u}_0, \boldsymbol{v}_0) \in U \times LC_*^4$. Then it is P-$\mathcal{K}$-equivalent to one of the following germs:

$$\begin{aligned}
&(A_k) \quad F(u_1, u_2, \boldsymbol{\lambda}) = u_1^{k+1} \pm u_2^2 + \lambda_1 + \lambda_2 u_1 + \cdots + \lambda_k u_1^{k-1}, (1 \le k \le 4),\\
&(D_4^+) \quad F(u_1, u_2, \boldsymbol{\lambda}) = u_1^3 + u_2^3 + \lambda_1 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 u_1 u_2,\\
&(D_4^-) \quad F(u_1, u_2, \boldsymbol{\lambda}) = u_1^3 - u_1 u_2^2 + \lambda_1 + \lambda_2 u_1 + \lambda_3 u_2 + \lambda_4 (u_1^2 + u_2^2).
\end{aligned}$$

For any $F(u_1, u_2, \boldsymbol{\lambda})$, we have

$$W(\mathcal{L}_F) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^4 \mid \exists \boldsymbol{u} \in \mathbb{R}^2 \text{ such that } F(\boldsymbol{u}, \boldsymbol{\lambda}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{\lambda}) = \frac{\partial F}{\partial u_2}(\boldsymbol{u}, \boldsymbol{\lambda}) = 0 \right\}.$$

Let $f_i : (N_i, x_i) \longrightarrow (P_i, y_i) (i = 1, 2)$ be C^∞ map germs. We say that f_1 and f_2 are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi : (N_1, x_1) \longrightarrow (N_2, x_2)$ and $\psi : (P_1, y_1) \longrightarrow (P_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$. Then we have the following theorem.

Theorem 6.3. *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{sp}(U, L_0^3)$ such that for any $\boldsymbol{x} \in \mathcal{O}$, we have the following classifications*:

(a) *If H-corank$(\boldsymbol{x}, u_0) = 1$, then there are two distinct principle curvatures κ_1 and κ_2. In this case H-ord$(\boldsymbol{x}, u_0) \leq 4$ and we have the following*:

(A_1) *If H-ord$(\boldsymbol{x}, u_0) = 1$, then each one of LD_M is $\mathcal{A}$-equivalent to*

$$f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0).$$

(A_2) *If H-ord$(\boldsymbol{x}, u_0) = 2$, then each one of LD_M is $\mathcal{A}$-equivalent to*

$$f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3).$$

The image of f is diffeomorphic to $C \times \mathbb{R}^2$.

(A_3) *If H-ord$(\boldsymbol{x}, u_0) = 3$, then each one of LD_M is $\mathcal{A}$-equivalent to*

$$f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3).$$

The image of f is diffeomorphic to $SW \times \mathbb{R}$.

(A_4) *If H-ord$(\boldsymbol{x}, u_0) = 4$, then each one of LD_M is $\mathcal{A}$-equivalent to*

$$f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_2, u_3).$$

The image of f is diffeomorphic to BF.

(b) *If H-corank$(\boldsymbol{x}, u_0) = 2$ and the principle curvature $\kappa \neq 0$, then $\boldsymbol{u}_0$ is a non-flat umbilic point. In this case, we have H-ord$(\boldsymbol{x}, u_0) = 4$ and the following two cases*:

(D_4^+) *Each one of LD_M is $\mathcal{A}$-equivalent to*

$$f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3).$$

(D_4^-) *Each one of LD_M is $\mathcal{A}$-equivalent to*

$$f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3).$$

Here, $C = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$ is the *ordinary cusp*, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called a *swallowtail* and $BF = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$ is called a *butterfly*.

Proof. By Theorem 6.2, there exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{sp}(U, L_0^3)$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the corresponding Legendrian immersion germs $\mathcal{L}_4$ at any point are Legendrian stable. Therefore, the height function H is P-$\mathcal{K}$-equivalent to one of the germs of (A_k) $(k = 1, 2, 3, 4)$ and $D_4^{\pm}$. If we consider the germ $F(u_1, u_2, \boldsymbol{\lambda}) = u_1^3 \pm u_2^2 + \lambda_1 + \lambda_2 u_1$, then we have

$$W(\mathcal{L}_F) = \{(2u_1^3, -3u_1^2, \lambda_3, \lambda_4) \mid (u_1, \lambda_3, \lambda_4) \in \mathbb{R}^3\},$$

so that the corresponding Legendrian map germ is (A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$. Suppose that H is $\mathcal{P}$-$\mathcal{K}$-equivalent to F of type (A_2). By Propositions 5.3 and 5.4, LD_M is $\mathcal{A}$-equivalent to (A_2). Of course, the image of f is $C \times \mathbb{R}^2$. Moreover, the $\mathcal{K}$-codimension of $f(u_1, u_2) = u_1^3 \pm u_2^2$ is 2, so that we have $H - \mathrm{ord}(\boldsymbol{x}, u_0) = 2$. The proof of the other assertions are similar to this case. Therefore, we omit it. $\square$

By Lemma 5.7, the sphere-cone dual surface $\overline{LD}_{\overline{M}}$ of $\overline{\boldsymbol{x}} : U \longrightarrow L_+^3$ is locally diffeomorphic to the light-cone dual surface LD_M. Therefore, we obtain exactly the same assertions as the above theorem for the sphere-cone dual surface $\overline{LD}_{\overline{M}}$.

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References

[1] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of differentiable maps*, Vol. I, The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds, Monographs in Mathematics, Birkhäuser Boston, Inc., Boston, MA, (1985). 2, 4, 5, 6

[2] A. C. Asperti, M. Dajczer, *Conformally flat Riemannian manifolds as hypersurfaces of the light cone*, Canad. Math. Bull., **32** (1989), 281–285. 1

[3] S. J. Brodsky, H. C. Pauli, S. S. Pinsky, *Quantum chromodynamics and other field theories on the light cone*, Phys. Rep., **301** (1998), 299–486.

[4] S. Izumiya, *Lengendrian dualities and spacelike hypersurfaces in the lightcone*, Mosc. Math. J., **9** (2009), 325–357. 1, 2

[5] S. Izumiya, Y. Jiang, D. H. Pei, *Lightcone dualities for curves in the sphere*, Q. J. Math., **64** (2013), 221–234. 1

[6] S. Izumiya, Y. Jiang, D. H. Pei, *Lightcone dualities for hypersurfaces in the sphere*, Math. Nachr., **287** (2014), 1687–1700. 1

[7] S. Izumiya, Y. Jiang, T. Sato, *Lightcone dualities for curves in the lightcone unit 3-sphere*, J. Math. Phys., **54** (2013), 15 pages. 1, 5, 6

[8] M. Kasedou, *Spacelike submanifolds of codimension two in de Sitter space*, J. Geom. Phys., **60** (2010), 31–42. 1

[9] J. Martinet, *Singularities of smooth functions and maps*, Translated from the French by Carl P. Simon, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge-New York, (1982). 5

[10] R. R. Metsaev, C. B. Thorn, A. A. Tseytlin, *Light-cone superstring in AdS space-time*, Nuclear Phys. B, **596** (2001), 151–184. 1

[11] J. A. Montaldi, *On contact between submanifolds*, Michigan Math. J., **33** (1986), 191–195. 1, 5, 5.1

[12] Z. G. Wang, D. H. Pei, *Singularities of ruled null surfaces of the principal normal indicatrix to a null Cartan curve in de Sitter 3-space*, Phys. Lett. B, **689** (2010), 101–106. 1

[13] Z. G. Wang, D. H. Pei, *Null Darboux developable and pseudo-spherical Darboux image of null Cartan curve in Minkowski 3-space*, Hokkaido Math. J., **40** (2011), 219–240.

[14] Z. G. Wang, D. H. Pei, L. Chen, *Geometry of 1-lightlike submanifolds in antide Sitter n-space*, Proc. Roy. Soc. Edinburgh Sect. A, **143** (2013), 1089–1113.

[15] Z. G. Wang, D. H. Pei, X. M. Fan, *Singularities of null developable of timelike curve that lies on nullcone in semi-Euclidean 3-space with index 2*, Topology Appl., **160** (2013), 189–198.

[16] Z. G. Wang, D. H. Pei, L. L. Kong, *Gaussian surfaces and nullcone dual surfaces of null curves in a three-dimensional nullcone with index 2*, J. Geom. Phys., **73** (2013), 166–186. 1

[17] G. Wassermann, *Stability of caustics*, Math. Ann., **216** (1975), 43–50. 6

[18] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Funct. Anal. Appl., **10** (1976), 23–31. 4, 5, 5