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Common fixed point theorems in Menger PMT-spaces with applications

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Abstract

In this paper, we introduce the concept of Menger PMT-spaces. Further, we prove common fixed point theorems in a complete Menger probabilistic metric type space and, by using the main result, we give applications on the existence and uniqueness of a solution for a class of integral equations.

Keywords: Nonlinear probabilistic contractive mapping, complete probabilistic metric type space, Menger space, fixed point theorem, integral equation.
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1. Introduction and preliminaries

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by

$$\begin{aligned}\Delta^+ = \{F : \mathbb{R}\cup\{-\infty,+\infty\} \longrightarrow [0,1] : &F \text{ is left-continuous and non-decreasing on } \mathbb{R},\\ &F(0)=0 \text{ and } F(+\infty)=1\},\end{aligned}$$

and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^-F(+\infty) = 1\}$, where $l^-f(x)$ denotes the left limit of the function f at the point x and $l^-f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual

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point-wise ordering of functions, i.e., $F \leq G$, if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1 ([11])**.** A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *continuous t–norm*, if T satisfies the following conditions:

(t1) T is commutative and associative;

(t2) T is continuous;

(t3) $T(a,1) = a$, for all $a \in [0,1]$;

(t4) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $c \leq d$, and $a,b,c,d \in [0,1]$.

Two typical examples of continuous t–norm are $T(a,b) = ab$ and $T(a,b) = \min\{a,b\}$.

Now, the t–norm T are recursively defined by $T^1 = T$ and

$$T^n(x_1, \cdots, x_{n+1}) = T(T^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for each $n \geq 2$ and $x_i \in [0,1]$ for each $i \in \{1,2,\cdots,n+1\}$. The t-norm T is *of Hadžić type I*, if for any $\varepsilon \in]0,1[$, there exists $\delta \in]0,1[$ (which may depend on m) such that

$$T^m(1-\delta, \cdots, 1-\delta) > 1 - \varepsilon \tag{1.1}$$

for each $m \in \mathbb{N}$.

We assume that, in this paper, all the t–norms are of Hadžić type I.

Definition 1.2 ([11])**.** A mapping $S : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *continuous s–norm*, if S satisfies the following conditions:

(s1) S is associative and commutative;

(s2) S is continuous;

(s3) $S(a,0) = a$, for all $a \in [0,1]$;

(s4) $S(a,b) \leq S(c,d)$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

Two typical examples of continuous s–norm are $S(a,b) = \min\{a+b, 1\}$ and $S(a,b) = \max\{a,b\}$.

Definition 1.3. A *Menger probabilistic metric type space* (briefly, *Menger PMT-space*) is a triple $(X, \mathcal{F}, T)$, where X is a nonempty set, T is a continuous t–norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), then the following conditions hold:
for all $x,y,z \in X$,

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$, if and only if $x = y$;

(PM2) $F_{x,y}(t) = F_{y,x}(t)$;

(PM3) $F_{x,z}(K(t+s)) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in X$ and $t,s \geq 0$ for some constant $K \geq 1$.

Definition 1.4. A *Menger probabilistic normed type space* (briefly, Menger PNT-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t–norm, and μ is a mapping from X into D^+ such that the following conditions hold for all $x, y \in X$,

(PN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$, if and only if $x = 0$;

(PN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for $\alpha \neq 0$;

(PN3) $\mu_{x+y}(K(t+s)) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$ for some constant $K \geq 1$.

Probabilistic metric space, Probabilistic normed space and Menger probabilistic normed type spaces have been studied by some authors [1]-[7],[9], [10], [12], [13].

Remark 1.5. The space $L_p(0 < p < 1)$ of all real-valued functions $f(x)$ for all $x \in [0, 1]$ such that $\int_0^1 |f(x)|^p dx < \infty$ is a type metric space. Define

$$D(f, g) = \left(\int_0^1 |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}$$

for all $f, g \in L_p$. Then D is a metric type space with $K = 2^{\frac{1}{p}}$.

Example 1.6. Let M be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^p dx < \infty$, where $p > 0$ is a real number. Define

$$F_{f,g}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{t}{t + (\int_0^1 |f(x)-g(x)|^p dx)^{\frac{1}{p}}}, & \text{if } t > 0. \end{cases}$$

Then, by Remark 1.5, $(M, \mathcal{F}, T_p)$ is a PMT-space with $K = 2^{\frac{1}{p}}$.

Definition 1.7. Let $(X, \mathcal{F}, T)$ be a Menger PMT-space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X, if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that
$$F_{x_n,x}(\epsilon) > 1 - \lambda,$$
whenever $n \geq N$, which is denoted by $\lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}_n$ in X is called a *Cauchy sequence*, if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that
$$F_{x_n,x_m}(\epsilon) > 1 - \lambda,$$
whenever $n, m \geq N$.

(3) A Menger PMT-space $(X, \mathcal{F}, T)$ is said to be *complete*, if every Cauchy sequence in X is convergent to a point in X.

Definition 1.8. Let $(X, \mathcal{F}, T)$ be a Menger PMT-space. For any $p \in X$ and $\lambda > 0$, the strong $\lambda-$*neighborhood* of p is the set
$$N_p(\lambda) = \{q \in X : F_{p,q}(\lambda) > 1 - \lambda\},$$
and the strong neighborhood system for X is the union $\bigcup_{p\in V} \mathcal{N}_p$, where $\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}$.

The strong neighborhood system for X determines a Hausdorff topology for X.

Remark 1.9. In this paper, we assume that, if $(X, \mathcal{F}, T)$ is a PMT-space and $\{p_n\}$, $\{q_n\}$ are two sequences such that $p_n \to p$ and $q_n \to q$, then

$$\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t).$$

Remark 1.10. In certain situations, we assume the following:

Suppose that, for any $\mu \in]0,1[$, there exists $\lambda \in]0,1[$ (which does not depend on n) such that

$$T^{n-1}(1-\lambda, \cdots, 1-\lambda) > 1-\mu \tag{1.2}$$

for each $n \in \{1, 2, \cdots\}$.

Lemma 1.11. *Let $(X, \mathcal{F}, T)$ be a Menger PMT-space. If we define $E_{\lambda,F} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,F}(x, y) = \inf\{t > 0 : f_{x,y}(t) > 1-\lambda\}$$

for all $\lambda \in (0, 1)$ and $x, y \in X$, then we have the following:

(1) *For any $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,F}(x_1, x_k) \leq KE_{\lambda,F}(x_1, x_2) + K^2E_{\lambda,F}(x_2, x_3) + \cdots + K^{n-1}E_{\lambda,F}(x_{k-1}, x_k)$$

for any $x_1, \cdots, x_k \in X$.

(2) *For any sequence $\{x_n\}$ in X, $F_{x_n,x}(t) \longrightarrow 1$, if and only if $E_{\lambda,F}(x_n, x) \to 0$. Also, the sequence $\{x_n\}$ is a Cauchy sequence with respect to F, if and only if it is a Cauchy sequence with respect to $E_{\lambda,F}$.*

Proof. (1) For any $\mu \in (0, 1)$, we can find $\lambda \in (0, 1)$ such that

$$T^{n-1}(1-\lambda, \cdots, 1-\lambda) > 1-\mu.$$

By the triangular inequality, we have

$$\begin{aligned} &F_{x,x_n}(KE_{\lambda,F}(x_1, x_2) + \cdots + K^{n-1}E_{\lambda,F}(x_{n-1}, x_n) + Kn\delta) \\ &\quad\geq T^{n-1}(f_{x_1,x_2}(E_{\lambda,F}(x_1, x_2) + \delta), \cdots, F_{x_{n-1},x_n}(E_{\lambda,F}(x_{n-1}, x_n) + \delta)) \\ &\quad\geq T^{n-1}(1-\lambda, \cdots, 1-\lambda) > 1-\mu \end{aligned}$$

for any $\delta > 0$, which implies that

$$E_{\mu,F}(x_1, x_n) \leq Kf_{\lambda,F}(x_1, x_2) + K^2E_{\lambda,F}(x_2, x_3) + \cdots + K^{n-1}E_{\lambda,F}(x_{n-1}, x_n) + Kn\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu,F}(x_1, x_n) \leq KE_{\lambda,F}(x_1, x_2) + K^2E_{\lambda,F}(x_2, x_3) + \cdots + K^{n-1}E_{\lambda,F}(x_{n-1}, x_n).$$

(2) It follows that

$$F_{x_n,x}(\eta) > 1-\lambda \iff E_{\lambda,F}(x_n, x) < \eta$$

for any $\eta > 0$. This completes the proof. □

Remark 1.12. If (1.2) holds, then the λ in part (1) of Lemma 1.11 does not depend on k (see [8]).

2. Common fixed point theorems

Now, we are in a position to prove some fixed point theorems in complete Menger PMT-spaces. We have more general results which improve Theorem 2.3 in [8] (we do not need to assume $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for any $t > 0$).

Definition 2.1. Let f and g be two mappings from a Menger PMT-space $(X, \mathcal{F}, T)$ into itself. The mappings f and g are said to be *weakly commuting*, if

$$F_{fgx,gfx}(t) \geq F_{fx,gx}(t)$$

for all $x \in X$ and $t > 0$.

For the remainder of the paper, let Φ be the set of all onto and strictly increasing functions

$$\phi : [0, \infty) \longrightarrow [0, \infty),$$

which satisfy $\lim_{n\to\infty} \phi^n(t) = 0$ for an $t > 0$, where $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$.

Remark 2.2. First, notice that, if $\phi \in \Phi$, then $\phi(t) < t$ for any $t > 0$. To see this, suppose that there exists $t_0 > 0$ with $t_0 \leq \phi(t_0)$. Then, since ϕ is nondecreasing, we have $t_0 \leq \phi^n(t_0)$ for each $n \in \{1, 2, \cdots\}$, which is a contradiction. Note also that $\phi(0) = 0$.

Lemma 2.3 ([8]). *Suppose that a Menger PMT-space $(X, \mathcal{F}, T)$ satisfies the following condition:*

$$F_{x,y}(t) = C$$

for all $t > 0$. Then we have $C = \varepsilon_0(t)$ and $x = y$.

Theorem 2.4. *Let $(X, \mathcal{F}, T)$ be a complete Menger PMT-space and f, g be weakly commuting self-mappings of X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$*;*

(b) f *or* g *is continuous;*

(c) $F_{fx,fy}(\phi(t)) \geq F_{gx,gy,}(t)$*, where* $\phi \in \Phi$.

Then we have the following:

(1) *If* (1.1) *holds and there exists* $x_0 \in X$ *such that*

$$E_F(gx_0, fx_0) = \sup\{E_{\gamma,F}(gx_0, fx_0) : \gamma \in (0, 1)\} < \infty,$$

then f and g have a unique common fixed point.

(2) *If* (1.2) *holds, then f and g have a unique common fixed point.*

Proof. (1) Choose $x_0 \in X$ with $E_F(gx_0, fx_0) < \infty$ and, next, choose $x_1 \in X$ with $fx_0 = gx_1$. Iteratively, choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$. Now, we have

$$F_{fx_n,fx_{n+1}}(\phi^{n+1}(t)) \geq F_{gx_n,gx_{n+1}}(\phi^n(t)) = F_{fx_{n-1},fx_n}(\phi^n(t)) \geq \cdots \geq F_{gx_0,gx_1}(t).$$

Note (see Lemma 1.9. of [8]) that, for any $\lambda \in (0, 1)$,

$$\begin{aligned} E_{\lambda,F}(fx_n, fx_{n+1}) &= \inf\{\phi^{n+1}(t) > 0 : F_{fx_n,fx_{n+1}}(\phi^{n+1}(t)) > 1 - \lambda\} \\ &\leq \inf\{\phi^{n+1}(t) > 0 : F_{gx_0,fx_0}(t) > 1 - \lambda\} \\ &\leq \phi^{n+1}(\inf\{t > 0 : F_{gx_0,fx_0}(t) > 1 - \lambda\}) \\ &= \phi^{n+1}(E_{\lambda,F}(gx_0, fx_0)) \\ &\leq \phi^{n+1}(E_F(gx_0, fx_0)), \end{aligned}$$

and so

$$E_{\lambda,F}(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_F(gx_0, fx_0))$$

for all $\lambda \in (0,1)$, which implies that

$$E_F(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_F(gx_0, fx_0)).$$

Let $\epsilon > 0$ and choose $n \in \{1, 2, \cdots\}$ so that

$$E_F(fx_n, fx_{n+1}) < \frac{\epsilon - \phi(\epsilon)}{K}.$$

Thus, for any $\lambda \in (0,1)$, there exists $\mu \in (0,1)$ such that

$$\begin{aligned} E_{\lambda,F}(fx_n, fx_{n+2}) &\leq KE_{\mu,F}(fx_n, fx_{n+1}) + KE_{\mu,F}(fx_{n+1}, fx_{n+2}) \\ &\leq KE_{\mu,F}(fx_n, fx_{n+1}) + \phi(KE_{\mu,F}(fx_n, fx_{n+1})) \\ &\leq KE_F(fx_n, fx_{n+1}) + \phi(KE_F(fx_n, fx_{n+1})) \\ &\leq K\frac{\epsilon - \phi(\epsilon)}{K} + \phi\left(K\frac{\epsilon - \phi(\epsilon)}{K}\right) \\ &\leq \epsilon. \end{aligned}$$

We can do this argument for each $\lambda \in (0,1)$ so that

$$E_F(fx_n, fx_{n+2}) \leq \epsilon.$$

For any $\lambda \in (0,1)$, there exists $\mu \in (0,1)$ such that

$$\begin{aligned} E_{\lambda,F}(fx_n, x_{n+3}) &\leq KE_{\mu,F}(fx_n, fx_{n+1}) + KE_{\mu,F}(fx_{n+1}, fx_{n+3}) \\ &\leq KE_{\mu,F}(fx_n, fx_{n+1}) + \phi(KE_{\mu,F}(fx_n, fx_{n+2})) \\ &\leq KE_F(fx_n, fx_{n+1}) + \phi(KE_F(fx_n, fx_{n+2})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\ &= \epsilon, \end{aligned}$$

where note that we used the fact that

$$F_{fx_{n+1}, fx_{n+3}}(\phi(t)) \geq F_{gx_{n+1}, gx_{n+3}}(t) = F_{fx_n, fx_{n+2}}(t),$$

and so

$$E_{\lambda,F}(fx_{n+1}, fx_{n+3}) \leq \phi(E_{\mu,F}(fx_n, fx_{n+2})).$$

Thus we have

$$E_F(fx_n, fx_{n+3}) \leq \epsilon.$$

By the induction, it follows that

$$E_F(fx_n, fx_{n+k}) \leq \epsilon,$$

for each $k \in \{1, 2, \cdots\}$. Thus $\{fx_n\}$ is a Cauchy sequence in X and so, by the completeness of X, $\{fx_n\}$ converges to a point in X, say it z. Also, $\{gx_n\}$ converges to $z \in X$.

Suppose that the mapping f is continuous. Then $\lim_{n\to\infty} ffx_n = fz$ and $\lim_n fgx_n = fz$. Furthermore, since f and g are weakly commuting, we have

$$F_{fgx_n, gfx_n}(t) \geq F_{fx_n, gx_n}(t).$$

By letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} gfx_n = fz$ by the continuity of $\mathcal{F}$.

Now, we prove that $z = fz$, that is, z is a fixed point of f. Suppose $z \neq fz$. By (c), it follows that, for any $t > 0$,

$$F_{fx_n,ffx_n}(\phi^{k+1}(t)) \geq F_{gx_n,gfx_n}(\phi^k(t))$$

for each $k \in \mathbb{N}$. Let $n \to \infty$ in the above inequality, then we have

$$F_{z,fz}(\phi^{k+1}(t)) \geq F_{z,fz}\phi^k(t)).$$

Also, we get

$$F_{z,fz}(\phi^k(t)) \geq F_{z,fz}(\phi^{k-1}(t)),$$

and

$$F_{z,fz}(\phi(t)) \geq F_{z,fz}(t).$$

Therefore, we obtain

$$F_{z,fz}(\phi^{k+1}(t)) \geq F_{z,fz}(t).$$

On the other hand, we observe (see Remark 2.2)

$$F_{z,fz}(\phi^{k+1}(t)) \leq F_{z,fz}(t).$$

Then $F_{z,fz}(t) = C$ and by Lemma 2.3, $z = fz$. Since $f(X) \subseteq g(X)$, we can find $z_1 \in X$ such that $z = fz = gz_1$. Now, we see

$$F_{ffx_n,fz_1}(t) \geq F_{gfx_n,gz_1}(\phi^{-1}(t)).$$

By taking the limit as $n \to \infty$, we have

$$F_{fz,fz_1}(t) \geq F_{fz,gz_1}(\phi^{-1}(t)) = \varepsilon_0(t),$$

which implies that $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. Also, for any $t > 0$, since f and g are weakly commuting, we obtain

$$F_{fz,gz}(t) = F_{fgz_1,gfz_1}(t) \geq F_{fz_1,gz_1}(t) = \varepsilon_0(t),$$

which again implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now, to prove the uniqueness of the common fixed point z, suppose that $z' \neq z$ is another common fixed point of f and g. Then, for any $t > 0$ and $n \in \mathbb{N}$, we have

$$F_{z,z'}(\phi^{n+1}(t)) = F_{fz,fz'}(\phi^{n+1}(t)) \geq F_{gz,gz'}(\phi^n(t)) = F_{z,z'}(\phi^n(t)).$$

Also, we infer

$$F_{z,z'}(\phi^n(t)) \geq F_{z,z'}(\phi^{n-1}(t)),$$

and

$$F_{z,z'}(\phi(t)) \geq F_{z,z'}(t).$$

Therefore, we obtain

$$F_{z,z'}(\phi^{n+1}(t)) \geq F_{z,z'}(t).$$

On the other hand, we have

$$F_{z,z'}(t) \leq F_{z,z'}(\phi^{n+1}(t)).$$

Then we have $F_{z,z'}(t) = C$ and so, by Lemma 2.3, $z = z'$, which is a contradiction. Therefore, z is the unique common fixed point of f and g.

(2) The argument is as in the case (1) except in this case we use Remark 1.11 in [8]. This completes the proof. □

In Theorem 2.4, if we take $g = I_X$ (the identity on X), then we have the following:

Corollary 2.5. *Let $(X, \mathcal{F}, T)$ be a complete Menger PMT-space and f be a self-mapping of X satisfying the following conditions:*

(a) *f is continuous;*

(b) *$F_{fx,fy}(\phi(t)) \geq F_{x,y,}(t)$, where $\phi \in \Phi$.*

Then we have the following:

(1) *If (1.1) holds and there exists $x_0 \in X$ such that*

$$E_F(x_0, fx_0) = \sup\{E_{\gamma,F}(x_0, fx_0) : \gamma \in (0,1)\} < \infty,$$

then f has a unique common fixed point.

(2) *If (1.2) holds, then f has a unique common fixed point.*

3. Applications on solutions of integral equations

Let $X = C([1,3], (-\infty, 2.1443888))$ and define

$$F_{x,y}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \inf_{\ell \in [1,3]} \frac{t}{t+(x(\ell)-y(\ell))^2}, & \text{if } t > 0, \end{cases}$$

for all $x, y \in X$. It is easily seen that $(X, \mathcal{F}, \min)$ is a complete PTM-space with $K = 2$.

Define a mapping $T : X \to X$ by

$$T(x(\ell)) = 4 + \int_1^\ell (x(u) - u^2)\ e^{1-u} du.$$

Put $g(x) = T(x)$ and $f(x) = T^2(x)$. Since $fg = gf$, f and g are (weakly) commuting. Now, it follows that, for $x, y \in X$ and $t > 0$,

$$\begin{aligned} F_{fx,fy}(t) &= F_{T(Tx(\ell)),T(Ty(\ell))}(t) \\ &= \inf_{\ell\in[1,3]} \frac{t}{t + |\int_1^\ell (Tx(u) - Ty(u))\ e^{1-u} du|^2} \\ &\geq \frac{t}{t + \frac{1}{e^4}|\int_1^3 (Tx(u) - Ty(u)) du|^2} \\ &= F_{gx,gy}(t), \end{aligned}$$

and hence

$$F_{fx,fy}\left(\frac{t}{e^4}\right) \geq F_{gx,gy}(t).$$

Thus all the conditions of Theorem 2.4 are satisfied for $\phi(t) = \frac{t}{e^4}$ and so f and g have a unique common fixed point, which is a unique solution of the integral equations:

$$x(\ell) = 4 + \int_1^\ell (x(u) - u^2)\ e^{1-u} du,$$

and

$$x(\ell) = (1-\ell)^2 e^{1-\ell} + \int_1^\ell \int_1^u (x(v) - v^2)\ e^{2-(u+v)} dv du.$$

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