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Fixed points for α-admissible contractive mappings via simulation functions

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Abstract

Based on concepts of α-admissible mappings and simulation functions, we establish some fixed point results in the setting of metric-like spaces. We show that many known results in the literature are simple consequences of our obtained results. We also provide some concrete examples to illustrate the obtained results.

Keywords: Metric-like, fixed point, simulation functions, α-admissible mappings.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

As generalizations of standard metric spaces, metric-like spaces were considered first by Hitzler and Seda [10] under the name of dislocated metric spaces and partial metric spaces were introduced by Matthews [13] in 1994 to study the denotational semantics of dataflow networks. Many authors obtained (common) fixed point results in the setting of above spaces, for example see [1, 2, 4, 5, 7–9, 16]. Let us recall some notations and definitions we will need in the sequel.

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Definition 1.1. Let X be a nonempty set. A function $\sigma : X \times X \to [0,\infty)$ is said to be a metric-like (or a dislocated metric) on X, if for any $x, y, z \in X$, the following conditions hold:

(σ_1) $\sigma(x,y) = 0 \Longrightarrow x = y$;

(σ_2) $\sigma(x,y) = \sigma(y,x)$;

(σ_3) $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$.

The pair (X,σ) is then called a metric-like space.

Now, let (X,σ) be a metric-like space. A sequence $\{x_n\}$ in X converges to $x \in X$, if and only if

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x).$$

A sequence $\{x_n\}$ is Cauchy in (X,σ), if and only if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. Moreover, (X,σ) is complete, if and only if for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that $\lim_{n\to+\infty} \sigma(x, x_n) = \sigma(x,x) = \lim_{n,m\to+\infty} \sigma(x_n, x_m)$.

Lemma 1.2 ([4, 5])**.** *Let (X,σ) be a metric-like space and $\{x_n\}$ be a sequence that converges to x with $\sigma(x,x) = 0$. Then, for each $y \in X$ one has*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x,y).$$

Definition 1.3. A partial metric on a nonempty set X is a function $p : X \times X \to [0,\infty)$, such that for all $x, y, z \in X$

(PM1) $p(x,x) = p(x,y) = p(y,y)$, then $x = y$;
(PM2) $p(x,x) \le p(x,y)$;
(PM3) $p(x,y) = p(y,x)$;
(PM4) $p(x,z) + p(y,y) \le p(x,y) + p(y,z)$.

The pair (X,p) is then called a partial metric space.

It is known that each partial metric is a metric-like, but the converse is not true in general.

Example 1.4. Let $X = \{0,1\}$ and $\sigma : X \times X \to [0,\infty)$ defined by

$$\sigma(0,0) = 2, \quad \sigma(x,y) = 1 \quad \text{if}\,(x,y) \ne (0,0).$$

Then, (X,σ) is a metric-like space. Note that σ is not a partial metric on X because $\sigma(0,0) \nleq \sigma(1,0)$.

In 2012, Samet et al. [17] introduced the concept of α-admissible mappings.

Definition 1.5 ([17])**.** For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be given mappings. We say that T is α-admissible, if for all $x, y \in X$, we have

$$\alpha(x,y) \ge 1 \Longrightarrow \alpha(Tx, Ty) \ge 1.$$

The concept of α-admissible mappings has been used in many works, see for example [6, 14]. Later, Karapinar et al. [11] introduced the notion of triangular α-admissible mappings.

Definition 1.6 ([11])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be given mappings. A mapping $T : X \to X$ is called a triangular α-admissible if

(T_1) T is α-admissible;

(T_2) $\alpha(x,y) \ge 1$ and $\alpha(y,z) \ge 1 \Rightarrow \alpha(x,z) \ge 1$, $x, y, z \in X$.

Very recently, Khojasteh et al. [12] introduced a new class of mappings called simulation functions. By using the above concept, they [12] proved several fixed point theorems and showed that many known results in the literature are simple consequences of their obtained results. Later, Argoubi et al. [3] slightly modified the definition of simulation functions by withdrawing a condition.

Let $\mathcal{Z}^*$ be the set of simulation functions in the sense of Argoubi et al. [3].

Definition 1.7 ([3])**.** A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$, satisfying the following conditions:

(ζ_1) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Example 1.8 ([3])**.** Let $\zeta_\lambda : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (t,s) = (0,0), \\ \lambda s - t & \text{otherwise}, \end{cases}$$

where $\lambda \in (0,1)$. Then, $\zeta_\lambda \in \mathcal{Z}^*$.

Example 1.9. Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = \psi(s) - \varphi(t)$ for all $t, s \geq 0$, where $\psi : [0,\infty) \to \mathbb{R}$ is an upper semi-continuous function and $\varphi : [0,\infty) \to \mathbb{R}$ is a lower semi-continuous function such that $\psi(t) < t \leq \varphi(t)$, for all $t > 0$. Then, $\zeta \in \mathcal{Z}^*$.

2. Fixed points via simulation functions

The first main result is as follows.

Theorem 2.1. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose that there exist a simulation function $\zeta \in \mathcal{Z}^*$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\zeta\left(\sigma(Tx,Ty), M(x,y)\right) \geq 0 \tag{2.1}$$

for all $x, y \in X$ satisfying $\alpha(x,y) \geq 1$, where

$$M(x,y) = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty) + \sigma(y,Tx)}{4}\}.$$

Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.*

Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$, for all $n \geq 0$.

We split the proof into several steps.

(Step 1): $\alpha(x_n, x_m) \geq 1$, for all $m > n \geq 0$.
We have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$. Since T is α-admissible, by the induction we have

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } \ n \geq 0.$$

T is triangular α-admissible, then

$$\alpha(x_n, x_{n+1}) \geq 1, \ \text{ and } \ \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

Thus, by the induction

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } \ m > n \geq 0.$$

(Step 2): We shall prove

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{2.2}$$

By Step 1, we have $\alpha(x_n, x_m) \geq 1$, for all $m > n \geq 0$. Then, from (2.1)

$$\zeta\left(\sigma(x_n, x_{n+1}), M(x_{n-1}, x_n)\right) = \zeta\left(\sigma(Tx_{n-1}, Tx_n), M(x_{n-1}, x_n)\right) \geq 0,$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{\sigma(x_{n-1}, x_n), \sigma(x_{n-1}, Tx_{n-1}), \sigma(x_n, Tx_n), \frac{\sigma(x_{n-1}, Tx_n) + \sigma(x_n, Tx_{n-1})}{4}\} \\ &= \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)}{4}\}. \end{aligned}$$

By a triangular inequality, we have

$$\begin{aligned} \frac{\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)}{4} &\leq \frac{3\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})}{4} \\ &\leq \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}. \end{aligned}$$

Thus

$$M(x_{n-1}, x_n) = \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}.$$

It follows that

$$\zeta\left(\sigma(x_n, x_{n+1}), \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}\right) \geq 0. \tag{2.3}$$

If $\sigma(x_n, x_{n+1}) = 0$ for some n, then $x_n = x_{n+1} = Tx_n$, that is, x_n is a fixed point of T and so the proof is finished. Suppose now that

$$\sigma(x_n, x_{n+1}) > 0, \qquad \text{for all } n = 0, 1, \cdots.$$

Therefore, from condition (ζ_1), we have

$$\begin{aligned} 0 &\leq \zeta\left(\sigma(x_n, x_{n+1}), \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}\right) \\ &< \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} - \sigma(x_n, x_{n+1}), \quad \text{for all } n \geq 1. \end{aligned}$$

Then

$$\sigma(x_n, x_{n+1}) < \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}, \quad \text{for all } n \geq 1.$$

Necessarily, we have

$$\max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} = \sigma(x_{n-1}, x_n), \quad \text{for all } n \geq 1. \tag{2.4}$$

Consequently, we obtain

$$\sigma(x_n, x_{n+1}) < \sigma(x_{n-1}, x_n), \quad \text{for all } n \geq 1, \tag{2.5}$$

which implies that $\{\sigma(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers, so there exists $t \geq 0$ such that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = t.$$

Suppose that $t > 0$. By (2.3), (2.4) and the condition (ζ_2),

$$0 \leq \limsup_{n\to\infty} \zeta\left(\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\right) < 0,$$

which is a contradiction. Then, we conclude that $t = 0$.

(Step 3): Now, we shall prove that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.6}$$

Suppose to the contrary that there exists $\varepsilon > 0$, for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k,

$$\sigma(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{2.7}$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (2.7). Then

$$\sigma(x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{2.8}$$

By using (2.7), (2.8) and the triangular inequality, we get

$$\begin{aligned}\varepsilon \leq \sigma(x_{n(k)}, x_{m(k)}) &\leq \sigma(x_{n(k)}, x_{m(k)-1}) + \sigma(x_{m(k)-1}, x_{m(k)})\\ &< \sigma(x_{m(k)-1}, x_{m(k)}) + \varepsilon.\end{aligned}$$

By (2.2)

$$\lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.9}$$

We also have

$$\sigma(x_{n(k)}, x_{m(k)-1}) - \sigma(x_{n(k)}, x_{n(k)-1}) - \sigma(x_{m(k)}, x_{m(k)-1}) \leq \sigma(x_{n(k)-1}, x_{m(k)}),$$

and

$$\sigma(x_{n(k)-1}, x_{m(k)}) \leq \sigma(x_{n(k)-1}, x_{n(k)}) + \sigma(x_{n(k)}, x_{m(k)}).$$

Letting $k \to \infty$ in the above inequalities and by using (2.2) and (2.9), we obtain

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{2.10}$$

Moreover, the triangular inequality gives that

$$|\sigma(x_{n(k)-1}, x_{m(k)}) - \sigma(x_{n(k)-1}, x_{m(k)-1})| \leq \sigma(x_{m(k)-1}, x_{m(k)}).$$

Let again $k \to \infty$ in the above inequality and by using (2.2) and (2.10), we have

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{2.11}$$

By (2.1) and as $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$ for all $k \geq 1$, we get

$$0 \leq \zeta\left(\sigma(x_{n(k)}, x_{m(k)}), M(x_{n(k)-1}, x_{m(k)-1})\right),$$

where

$$M(x_{n(k)-1}, x_{m(k)-1}) = \max\{\sigma(x_{n(k)-1}, x_{m(k)-1}), \sigma(x_{n(k)-1}, x_{n(k)}), \sigma(x_{m(k)-1}, x_{m(k)}), \frac{\sigma(x_{n(k)-1}, x_{m(k)}) + \sigma(x_{m(k)-1}, x_{n(k)})}{4}\}.$$

From (2.9), (2.10), (2.11) and (2.2)

$$\lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon.$$

On the other hand, if $x_n = x_m$ for some $n < m$, then $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. Equation (2.5) leads to

$$0 < \sigma(x_n, x_{n+1}) = \sigma(x_m, x_{m+1}) < \sigma(x_{m-1}, x_m) < \cdots < \sigma(x_n, x_{n+1}),$$

which is a contradiction. Then $x_n \neq x_m$ for all $n < m$. The condition (ζ_2) implies that

$$0 \leq \limsup_{k\to\infty} \zeta\left(\sigma(x_{n(k)}, x_{m(k)}), M(x_{n(k)-1}, x_{m(k)-1})\right) < 0,$$

which is a contradiction. This completes the proof of (2.6).

It follows that $\{x_n\}$ is a Cauchy sequence. Since (X, σ) is complete, there exists some $z \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.12}$$

(Step 4): Now, we shall prove that z is a fixed point of T.

If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$ for all k, then $\sigma(z, Tz) = \sigma(z, x_{n_k+1})$ for all k. Let $k \to \infty$ and use (2.12) to get $\sigma(z, Tz) = 0$, that is, $z = Tz$ and the proof is finished. So, without loss of generality, we may suppose that $x_n \neq z$ and $Tx_n \neq Tz$ for all nonnegative integers n. Suppose that $\sigma(z, Tz) > 0$. By assumption (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$ for all k. By (2.1) and as $\alpha(x_{n(k)}, z) \geq 1$ for all $k \geq 1$, we get

$$0 \leq \zeta\left(\sigma(x_{n(k)+1}, Tz), M(x_{n(k)}, z)\right) = \zeta\left(\sigma(Tx_{n(k)}, Tz), M(x_{n(k)}, z)\right),$$

where

$$M(x_{n(k)}, z) = \max\{\sigma(x_{n(k)}, z), \sigma(x_{n(k)}, x_{n(k)+1}), \sigma(z, Tz), \frac{\sigma(x_{n(k)}, Tz) + \sigma(z, x_{n(k)+1})}{4}\}.$$

By Lemma 1.2 and (2.12)

$$\lim_{k\to\infty} \sigma(x_{n(k)+1}, Tz) = \lim_{k\to\infty} M(x_{n(k)}, z) = \sigma(z, Tz) > 0.$$

From the condition (ζ_2)

$$0 \leq \limsup_{k\to\infty} \zeta\left(\sigma(x_{n(k)+1}, Tz), M(x_{n(k)}, z)\right) < 0,$$

which is a contradiction and hence $\sigma(z, Tz) = 0$, that is, $Tz = z$ and so z is a fixed point of T. This ends the proof of Theorem 2.1. □

By using the same techniques, we obtain the following result.

Theorem 2.2. *Let (X, p) be a complete partial metric space. Let $T : X \to X$ be a given mapping. Suppose there exist a simulation function $\zeta \in \mathcal{Z}^*$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\zeta\left(p(Tx,Ty),M_p(x,y)\right)\geq 0 \tag{2.13}$$

for all $x,y\in X$ satisfying $\alpha(x,y)\geq 1$, where

$$M_p(x,y)=\max\{p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Ty)+p(y,Tx)}{2}\}.$$

Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\geq 1$ for all k.*

Then, T has a fixed point $z\in X$ such that $p(z,z)=0$.

Now, we prove the uniqueness fixed point result. For this, we need the following additional condition.

(U): For all $x,y\in Fix(T)$, we have $\alpha(x,y)\geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.3. *By adding condition* (U) *to the hypotheses of Theorem* 2.2*, we obtain that z is the unique fixed point of T.*

Proof. We argue by contradiction, that is, there exist $z,w\in X$ such that $z=Tz$ and $w=Tw$ with $z\neq w$. By assumption (U), we have $\alpha(z,w)\geq 1$. So, by (2.13) and by using the condition (ζ_2), we get that

$$\begin{aligned}0\leq\zeta\left(p(Tz,Tw),M_p(z,w)\right)&=\zeta\left(p(z,w),\max\{p(z,w),p(z,z),p(w,w)\}\right)\\&=\zeta\left(p(z,w),p(z,w)\right)<p(z,w)-p(z,w)=0,\end{aligned}$$

which is a contradiction. Hence, $z=w$. □

We also state the following result.

Theorem 2.4. *Let (X,σ) be a complete metric-like space. Let $T:X\to X$ be a given mapping. Suppose that there exist a simulation function $\zeta\in\mathcal{Z}^*$ and $\alpha:X\times X\to[0,\infty)$ such that*

$$\zeta\left(\sigma(Tx,Ty),\sigma(x,y)\right)\geq 0 \tag{2.14}$$

for all $x,y\in X$ satisfying $\alpha(x,y)\geq 1$. Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\geq 1$ for all k.*

Then, T has a fixed point $z\in X$ such that $\sigma(z,z)=0$.

Proof. By following the proof of Theorem 2.1, we can construct a sequence $\{x_n\}$ such that $\alpha(x_n,x_m)\geq 1$ for all $m>n\geq 0$. $\{x_n\}$ is also Cauchy in (X,σ) and converges to some $z\in X$ such that (2.12) holds. We claim that z is a fixed point of T. Similarly, if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}=z$ or $Tx_{n_k}=Tz$ for all k, so z is a fixed point of T and the proof is finished. Without loss of generality, we may suppose that $x_n\neq z$ and $Tx_n\neq Tz$ for all nonnegative integer n. By assumption (iii) and by using (2.14) together with the condition (ζ_1), again we deduce that

$$0\leq\zeta\left(\sigma(Tx_{n(k)},Tz),\sigma(x_{n(k)},z)\right)<\sigma(x_{n(k)},z)-\sigma(x_{n(k)+1},Tz).$$

This implies

$$\sigma(x_{n(k)+1},Tz)<\sigma(x_{n(k)},z),\quad\forall k\geq 0.$$

Letting $k\to\infty$ in the above inequality and by Lemma 1.2 and (2.12), we get

$$\sigma(z,Tz)\leq\sigma(z,z)=0,$$

that is, $\sigma(z,Tz)=0$ and so $z=Tz$. □

Theorem 2.5. *By adding condition* (U) *to the hypotheses of Theorem* 2.4*, we obtain that z is the unique fixed point of T.*

Proof. We argue by contradiction, that is, there exist $z, w \in X$ such that $z = Tz$ and $w = Tw$ with $z \neq w$. By assumption (U), we have $\alpha(z, w) \geq 1$. So, by (2.14) and by using the condition (ζ_2), we get that

$$0 \leq \zeta\left(\sigma(Tz, Tw), \sigma(z, w)\right) < \sigma(z, w) - \sigma(Tz, Tw) = 0,$$

which is a contradiction. Hence, $z = w$. □

Example 2.6. Take $X = [0, \infty)$ endowed with the metric-like $\sigma(x, y) = x + y$. Consider the mapping $T : X \to X$ given by

$$Tx = \begin{cases} \frac{x^2}{2} & \text{if } x \in [0, 1] \\ x + 1 & \text{if } x > 1. \end{cases}$$

Note that (X, σ) is a complete metric-like space. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta(t, s) = s - \frac{2+t}{1+t}t$ for all $s, t \geq 0$. Note that T is α-admissible. In fact, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $x, y \in [0, 1]$. Thus,

$$\alpha(Tx, Ty) = \alpha(\frac{x^2}{2}, \frac{y^2}{2}) = 1.$$

T is also triangular α-admissible. In fact, let $x, y, z \in X$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$, this implies that $x, y, z \in [0, 1]$. It follows that $\alpha(x, z) \geq 1$.

Now, we show that the contraction condition (2.14) is verified. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case, we have

$$\begin{aligned} \zeta(\sigma(Tx, Ty), \sigma(x, y)) &= \sigma(x, y) - \frac{2 + \sigma(Tx, Ty)}{1 + \sigma(Tx, Ty)}\sigma(Tx, Ty) \\ &= x + y - \frac{(4 + x^2 + y^2)(x^2 + y^2)}{4 + 2(x^2 + y^2)} \\ &= \frac{4(1 - x)x + 4(1 - y)y + (2 - x)x^3 + 2(1 - x)xy^2 + (2 - y)y^3 + 2x^2y}{4 + 2(x^2 + y^2)} \geq 0. \end{aligned}$$

Now, we show that condition (iii) of Theorem 2.4 is verified. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$. Then, $\{x_n\} \subset [0, 1]$ and $x_n + x \to 2x$ as $n \to \infty$. Thus, $x_n \to x$ as $n \to \infty$ in $(X, |.|)$. This implies that $x \in [0, 1]$ and so $\alpha(x_n, x) = 1$ for all n. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T1) = \alpha(1, \frac{1}{2}) = 1$. Thus, all hypotheses of Theorem 2.4 are verified. Here $x = 0$ is the unique fixed point of T.

On the other, Theorem 5.1 in [15] is not applicable for the partial metric $p(x, y) = \max\{x, y\}$. Indeed, for $x = 2$ and $y = 3$, we have

$$\zeta(p(T2, T3), p(2, 3)) = \zeta(4, 3) = -\frac{9}{5} < 0.$$

Also, the Banach contraction principle is not applicable because, for $x = 2$ and $y = 3$, we have

$$\sigma(T2, T3) = 7 > 5 = \sigma(2, 3).$$

Now, we present the following result in the setting of metric-like spaces which generalizes the result obtained by [15].

Theorem 2.7. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose that there exist a simulation function $\zeta \in \mathcal{Z}^*$ and a lower semi-continuous function $\varphi : X \to [0,\infty)$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\zeta\left(\sigma(Tx,Ty)+\varphi(Tx)+\varphi(Ty),\sigma(x,y)+\varphi(x)+\varphi(y)\right)\geq 0 \tag{2.15}$$

for all $x,y\in X$ satisfying $\alpha(x,y)\geq 1$. Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n \to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\geq 1$ for all k.*

Then, T has a fixed point $z\in X$ such that $\sigma(z,z)=0$ and $\varphi(z)=0$.

Proof. By following the proof of Theorem 2.1, we construct a sequence $\{x_n\}$ such that $\alpha(x_n,x_m)\geq 1$ for all $m>n\geq 0$. We shall prove

$$\lim_{n\to\infty}\sigma(x_n,x_{n+1})=0.$$

Since $\alpha(x_n,x_m)\geq 1$ for all $m>n\geq 0$, it follows from (2.15) that

$$\zeta\left(\sigma(Tx_{n-1},Tx_n)+\varphi(Tx_{n-1})+\varphi(Tx_n),\sigma(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)\right)\geq 0.$$

It means that

$$\zeta\left(\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\sigma(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)\right)\geq 0.$$

If $\sigma(x_n,x_{n+1})=0$ for some n, then $x_n=x_{n+1}=Tx_n$, that is, x_n is a fixed point of T and so the proof is finished. Suppose now that

$$\sigma(x_n,x_{n+1})>0, \quad \text{for all } \ n=0,1,\cdots.$$

Therefore, from condition (ζ_1), we have

$$\begin{aligned}0&\leq \zeta\left(\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\sigma(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)\right)\\ &<\sigma(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)-[\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})], \quad \text{for all } n\geq 1.\end{aligned}$$

This leads to

$$\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})<\sigma(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), \quad \text{for all } n\geq 1, \tag{2.16}$$

which implies that $\{\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$ is a decreasing sequence of positive real numbers, so there exists $t\geq 0$ such that

$$\lim_{n\to\infty}[\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})]=t.$$

Suppose that $t>0$. From the condition (ζ_2),

$$0\leq \limsup_{n\to\infty}\zeta\left(\sigma(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\sigma(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)\right)<0,$$

which is a contradiction. Then, we conclude that $t=0$. Since $\varphi\geq 0$, we get that

$$\lim_{n\to\infty}\sigma(x_n,x_{n+1})=0.$$

Also,

$$\lim_{n\to\infty}\varphi(x_n)=0. \tag{2.17}$$

From (2.16), mention that $x_n \neq x_m$ for all $n < m$. Now, we shall prove that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.18}$$

Suppose to the contrary that there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

$$\sigma(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{2.19}$$

Moreover, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (2.19). By following again the proof of Theorem 2.1 we see that (2.9), (2.10) and (2.11) hold. Put $a_k = \sigma(x_{n(k)}, x_{m(k)})$ and $b_k = \sigma(x_{n(k)-1}, x_{m(k)-1})$. By (2.15) and as $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$ for all $k \geq 1$, we get

$$0 \leq \zeta\left(a_k + \varphi(x_{n(k)}) + \varphi(x_{m(k)}), b_k + \varphi(x_{n(k)-1}) + \varphi(x_{m(k)-1})\right).$$

By (2.9), (2.10), (2.11) and (2.17), we have

$$\lim_{k\to\infty} [a_k + \varphi(x_{n(k)}) + \varphi(x_{m(k)})] = \lim_{k\to\infty} [b_k + \varphi(x_{n(k)-1}) + \varphi(x_{m(k)-1})] = \varepsilon.$$

From the condition (ζ_2), it follows that

$$0 \leq \limsup_{k\to\infty} \zeta\left(a_k + \varphi(x_{n(k)}) + \varphi(m(k)), b_k + \varphi(x_{n(k)-1}) + \varphi(x_{m(k)-1})\right) < 0,$$

which is a contradiction. This completes the proof of (2.18).

Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, σ) is complete, there exists some $z \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0.$$

By referring to (2.17) and taking into account that φ is lower semi-continuous, we have

$$0 \leq \varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0,$$

and so $\varphi(z) = 0$. Now, we claim that z is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$ for all k, then z is a fixed point of T and the proof is finished. Without loss of generality, we may suppose that $x_n \neq z$ and $Tx_n \neq Tz$ for all nonnegative integer n. By assumption (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$ for all k. By using (2.15) and the condition (ζ_1), we deduce that

$$\begin{aligned}0 &\leq \zeta\left(\sigma(x_{n(k)+1}, Tz) + \varphi(x_{n(k)+1}) + \varphi(Tz), \sigma(x_{n(k)}, z) + \varphi(x_{n(k)}) + \varphi(z)\right)\\ &< \sigma(x_{n(k)}, z) + \varphi(x_{n(k)}) + \varphi(z) - [\sigma(x_{n(k)+1}, Tz) + \varphi(x_{n(k)+1}) + \varphi(Tz)].\end{aligned}$$

This implies

$$\sigma(x_{n(k)+1}, Tz) + \varphi(x_{n(k)+1}) + \varphi(Tz) < \sigma(x_{n(k)}, z) + \varphi(x_{n(k)}) + \varphi(z), \quad \forall k \geq 0.$$

By letting $k \to \infty$ in the above inequality and by taking into account that $\varphi \geq 0$ and $\varphi(z) = 0$,

$$\sigma(z, Tz) + \varphi(Tz) \leq \sigma(z, z) + \varphi(z) = 0,$$

that is, $\sigma(z, Tz) + \varphi(Tz) = 0$ and so $\sigma(z, Tz) = 0$. This ends the proof of Theorem 2.7.

□

Theorem 2.8. *By adding condition* (U) *to the hypotheses of Theorem* 2.7*, we obtain that z is the unique fixed point of T.*

Proof. We argue by contradiction, that is, there exist $z, w \in X$ such that $z = Tz$ and $w = Tw$ with $z \neq w$. By assumption (U), we have $\alpha(z, w) \geq 1$. So, by (2.15) and by using the condition (ζ_2), we get that

$$\begin{aligned} 0 &\leq \zeta\left(\sigma(Tz, Tw) + \varphi(Tz) + \varphi(Tw), \sigma(z, w) + \varphi(z) + \varphi(w)\right) \\ &= \zeta\left(\sigma(z, w) + \varphi(z) + \varphi(w), \sigma(z, w) + \varphi(z) + \varphi(w)\right) \\ &< \sigma(z, w) + \varphi(z) + \varphi(w) - [\sigma(z, w) + \varphi(z) + \varphi(w)] = 0, \end{aligned}$$

which is a contradiction. Hence, $z = w$. □

Example 2.9. Take $X = [0, \infty)$ endowed with the metric-like $\sigma(x, y) = x^2 + y^2$. Consider the mapping $T : X \to X$ given by

$$Tx = \begin{cases} \frac{x^2}{x+1} & \text{if } x \in [0, 1], \\ x^2 & \text{if } x > 1. \end{cases}$$

Note that (X, σ) is a complete metric-like space. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta(t, s) = \frac{1}{2}s - t$ for all $s, t \geq 0$ and $\varphi(x) = x$ for all $x \in X$. Note that T is α-admissible. In fact, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $x, y \in [0, 1]$. Thus,

$$\alpha(Tx, Ty) = \alpha(\frac{x^2}{x+1}, \frac{y^2}{y+1}) = 1.$$

T is also triangular α-admissible.

Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case, we have

$$\begin{aligned} \sigma(Tx, Ty) + \varphi(Tx) + \varphi(Ty) &= (\frac{x^2}{x+1})^2 + (\frac{y^2}{y+1})^2 + \frac{x^2}{x+1} + \frac{y^2}{y+1} \\ &\leq \frac{1}{4}(x^2 + y^2) + \frac{1}{2}(x + y) \\ &\leq \frac{1}{2}(x^2 + y^2 + x + y) \\ &= \frac{1}{2}(\sigma(x, y) + \varphi(x) + \varphi(y)). \end{aligned}$$

It follows that

$$\begin{aligned} &\zeta(\sigma(Tx, Ty) + \varphi(Tx) + \varphi(Ty), \sigma(x, y) + \varphi(x) + \varphi(y)) \\ &\quad = \frac{1}{2}(\sigma(x, y) + \varphi(x) + \varphi(y)) - [\sigma(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] \geq 0. \end{aligned}$$

Now, we show that condition (iii) of Theorem 2.7 is verified. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$. Then, $\{x_n\} \subset [0, 1]$ and $x_n^2 + x^2 \to 2x^2$ as $n \to \infty$. Thus, $x_n \to x$ as $n \to \infty$ in $(X, |.|)$. This implies that $x \in [0, 1]$ and so $\alpha(x_n, x) = 1$ for all n. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T1) = \alpha(1, \frac{1}{2}) = 1$. Thus, all hypotheses of Theorem 2.7 are verified. Here, $x = 0$ is the unique fixed point of T and $\varphi(0) = 0$.

On the other hand, Theorem 3.2 in [15] is not applicable for the standard metric d. Indeed, for $x = 2$ and $y = 3$, we have

$$\zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) = -15 < 0.$$

Moreover, $\sigma(T\sqrt{2}, T\sqrt{3}) = 13 > 5 = \sigma(\sqrt{2}, \sqrt{3})$, then T is not a Banach contraction on X.

3. Consequences

In this section, as consequences of our obtained results, we provide various fixed point results in the literature including fixed point theorems in partially ordered metric-like spaces.

Corollary 3.1. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose that there exist $k \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \le k \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = ks - t$ for all $s,t \ge 0$ in Theorem 2.1. □

Corollary 3.2. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose that there exist $k \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \le k\sigma(x,y)$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Corollary 3.3. *Let (X,p) be a complete partial metric space. Let $T : X \to X$ be a given mapping. Suppose that there exist $k \in (0,1)$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$p(Tx,Ty) \le k \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\}$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $p(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = ks - t$ for all $s,t \ge 0$ in Theorem 2.4. □

Corollary 3.4. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose that there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\begin{aligned}\sigma(Tx,Ty) \le{}& \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}\\ &- \varphi(\max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\})\end{aligned}$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \varphi(s) - t$ for all $s,t \ge 0$ in Theorem 2.1. □

Corollary 3.5. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \le \sigma(x,y) - \varphi(\sigma(x,y))$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \varphi(s) - t$ for all $s,t \ge 0$ in Theorem 2.4. □

Corollary 3.6. *Let (X,p) be a complete partial metric space. Let $T : X \to X$ be a given mapping. Suppose there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$p(Tx,Ty) \le \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\} \\ - \varphi(\max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\})$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $p(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \varphi(s) - t$ for all $s,t \ge 0$ in Theorem 2.2. □

Corollary 3.7. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \le \varphi(\max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}) \\ \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\})$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s\varphi(s) - t$ for all $s,t \ge 0$ in Theorem 2.1. □

Corollary 3.8. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \le \varphi(\sigma(x,y))\sigma(x,y)$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s\varphi(s) - t$ for all $s,t \ge 0$ in Theorem 2.4. □

Corollary 3.9. *Let (X,σ) be a complete partial metric space. Let $T : X \to X$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim\limits_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$p(Tx,Ty) \le \varphi(\max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\}) \\ \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\})$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s\varphi(s) - t$ for all $s,t \ge 0$ in Theorem 2.2. □

Corollary 3.10. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \le \varphi(\max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\})$$

for all $x,y \in X$, satisfying $\alpha(x,y) \ge 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = \varphi(s) - t$ for all $s,t \geq 0$ in Theorem 2.1. $\square$

Corollary 3.11. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) \leq \varphi(\sigma(x,y))$$

for all $x,y \in X$, satisfying $\alpha(x,y) \geq 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = \varphi(s) - t$ for all $s,t \geq 0$ in Theorem 2.4. $\square$

Corollary 3.12. *Let (X,p) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$p(Tx,Ty) \leq \varphi(\max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\})$$

for all $x,y \in X$, satisfying $\alpha(x,y) \geq 1$. Then, T has a fixed point $z \in X$ such that $p(z,z) = 0$.

Proof. It suffices to take simulation function $\zeta(t,s) = \varphi(s) - t$, for all $s,t \geq 0$ in Theorem 2.2. $\square$

Corollary 3.13. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist $k \in (0,1)$ and a lower semi-continuous function $\varphi : X \to [0,\infty)$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \leq k[\sigma(x,y) + \varphi(x) + \varphi(y)]$$

for all $x,y \in X$, satisfying $\alpha(x,y) \geq 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = ks - t$ for all $s,t \geq 0$ in Theorem 2.7. $\square$

Corollary 3.14. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist two lower semi-continuous function $\varphi,\psi : X \to [0,\infty)$ with $\psi(t) > 0$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\sigma(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \leq \sigma(x,y) + \varphi(x) + \varphi(y) - \psi(\sigma(x,y) + \varphi(x) + \varphi(y))$$

for all $x,y \in X$, satisfying $\alpha(x,y) \geq 1$. Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \psi(s) - t$ for all $s,t \geq 0$ in Theorem 2.7. $\square$

Remark 3.15. We can obtain other fixed point results in the class of metric-like spaces via α-admissible mappings by choosing an appropriate simulation function. Moreover, if we take $\alpha(x,y) = 1$ we can obtain known fixed point results in the literature.

Corollary 3.16. *Let (X,σ) be a complete metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\sigma(Tx,Ty), M(x,y)\right) \geq 0$$

for all $x,y \in X$, where

$$M(x,y) = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}.$$

Then, T has a fixed point $z \in X$ such that $\sigma(z,z) = 0$.

Proof. It suffices to take $\alpha(x,y)=1$ in Theorem 2.1. □

Corollary 3.17. *Let (X,σ) be a complete metric-like space. Let $T: X \to X$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\sigma(Tx,Ty),\sigma(x,y)\right) \geq 0$$

for all $x,y \in X$. Then, T has a unique fixed point $z \in X$ such that $\sigma(z,z)=0$.

Corollary 3.18 ([15], Theorem 5.1)**.** *Let (X,σ) be a complete partial metric space. Let $T: X \to X$ be a given mapping. Suppose there exists a simulation function ζ such that*

$$\zeta\left(p(Tx,Ty),p(x,y)\right) \geq 0, \quad \text{for all } x,y \in X.$$

Then, T has a unique fixed point $z \in X$ such that $p(z,z)=0$.

Corollary 3.19. *Let (X,σ) be a complete metric-like space. Let $T: X \to X$ be a given mapping. Suppose there exist a simulation function $\zeta \in \mathcal{Z}^*$ and a lower semi-continuous function $\varphi : X \to [0,\infty)$ such that*

$$\zeta\left(\sigma(Tx,Ty)+\varphi(Tx)+\varphi(Ty),\sigma(x,y)+\varphi(x)+\varphi(y)\right) \geq 0, \quad \text{for all } x,y \in X.$$

Then, T has a unique fixed point $z \in X$ such that $\sigma(z,z)=0$ and $\varphi(z)=0$.

Proof. It suffices to take $\alpha(x,y)=1$ in Theorem 2.7. □

Corollary 3.20 ([15], Theorem 3.2)**.** *Let (X,d) be a complete metric space. Let $T: X \to X$ be a mapping. Suppose there exist a simulation function ζ and a lower semi-continuous function $\varphi : X \to [0,\infty)$ such that*

$$\zeta\left(\sigma(Tx,Ty)+\varphi(Tx)+\varphi(Ty),\sigma(x,y)+\varphi(x)+\varphi(y)\right) \geq 0, \quad \text{for all } x,y \in X.$$

Then, T has a unique fixed point $z \in X$ such that $\varphi(z)=0$.

Now, we give some fixed point results in partially ordered metric-like spaces as consequences of our results.

Definition 3.21. Let X be a nonempty set. We say that $(X,\sigma,\preceq)$ is a partially ordered metric-like space if (X,σ) is a metric-like space and $(X,\preceq)$ is a partially ordered set.

Definition 3.22. Let $T: X \to X$ be a given mapping. We say that T is non-decreasing if

$$(x,y) \in X \times X,\ x \preceq y \Rightarrow Tx \preceq Ty.$$

Corollary 3.23. *Let $(X,\sigma,\preceq)$ be a complete partially ordered metric-like space. Let $T: X \to X$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(\sigma(Tx,Ty),M(x,y)\right) \geq 0$$

for all $x,y \in X$ satisfying $x \preceq y$, where

$$M(x,y) = \max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}.$$

Assume that

(i) *T is non-decreasing;*
(ii) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Then, T has a fixed point $z \in X$ such that $\sigma(z, z) = 0$.

Proof. Let $\alpha : X \times X \to X$ be such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y; \\ 0 & \text{otherwise.} \end{cases}$$

Then, all hypotheses of Theorem 2.1 are satisfied and hence T has a fixed point. □

Corollary 3.24. *Let $(X, p, \preceq)$ be a complete partially ordered partial metric space. Let $T : X \to X$ be a given mapping. Suppose there exists a simulation function $\zeta \in \mathcal{Z}^*$ such that*

$$\zeta\left(p(Tx, Ty), M_p(x, y)\right) \geq 0$$

for all $x, y \in X$ satisfying $x \preceq y$, where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

Assume that

(i) *T is non-decreasing;*
(ii) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Then, T has a fixed point $z \in X$ such that $p(z, z) = 0$.

Corollary 3.25 ([3], Theorem 3.7)**.** *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $f : X \to X$ be a given mapping. Suppose the following conditions hold:*

(i) *f is non-decreasing;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*
(iii) *if $\{x_n\}$ is a non-decreasing sequence with $x_n \to z$, then $x_n \preceq z$ for all $n \in \mathbb{N}$;*
(iv) *there exists a simulation function ζ such that for every $(x, y) \in X \times X$ with $x \preceq y$, we have*

$$\zeta\left(d(fx, fy), M(f, x, y)\right) \geq 0,$$

where

$$M(f, x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\}.$$

Then, $\{f^n x_0\}$ converges to a fixed point of f.

Corollary 3.26. *Let $(X, \sigma, \preceq)$ be a complete partially ordered metric-like space. Let $T : X \to X$ be a given mapping. Suppose there exist a simulation function $\zeta \in \mathcal{Z}^*$ and a lower semi-continuous function $\varphi : X \to [0, \infty)$ such that*

$$\zeta\left(\sigma(Tx, Ty) + \varphi(Tx) + \varphi(Ty), \sigma(x, y) + \varphi(x) + \varphi(y)\right) \geq 0$$

for all $x, y \in X$ satisfying $x \preceq y$. Assume that

(i) *T is non-decreasing;*
(ii) *there exists an elements $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Then, T has a fixed point $z \in X$ such that $\sigma(z, z) = 0$ and $\varphi(z) = 0$.

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