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Bifurcations of twisted double homoclinic loops with resonant condition

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Abstract

In this paper, the bifurcation problems of twisted double homoclinic loops with resonant condition are studied for $(m+n)$-dimensional nonlinear dynamic systems. In the small tubular neighborhoods of the homoclinic orbits, the foundational solutions of the linear variational systems are selected as the local coordinate systems. The Poincaré maps are constructed by using the composition of two maps, one is in the small tubular neighborhood of the homoclinic orbit, and another is in the small neighborhood of the equilibrium point of system. By the analysis of bifurcation equations, the existence, uniqueness and existence regions of the large homoclinic loops, large periodic orbits are obtained, respectively. Moreover, the corresponding bifurcation diagrams are given. ©2016 all rights reserved.

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1. Introduction and hypotheses

In the studies of many research areas and its application problems of nonlinear science, there are a large number of nonlinear dynamical systems with complex dynamical behaviors. Homoclinic, heteroclinic orbits and the corresponding bifurcation phenomena are the very important source of complex dynamical behavior, and have been becoming a hot topic in the study of nonlinear dynamical systems. Using classical Cartesian coordinate system and the successor function method, many scholars have studied the bifurcation problems of low dimensional systems and achieved many breakthrough results. Recently, the research of

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bifurcation problems of homoclinic and heteroclinic loops have been increasing widespread, and the research scope have been developing from low dimensional systems to high dimensional systems. Since the 1980s, Wiggins, Kovacic, Luo, Han, et al. studied some low dimensional systems and some systems with special forms (e.g. Hamilton) by using the well-known Melnikov methods [2, 3, 15, 18, 20, 21]. In 1990, Chow et al. studied the high dimensional non-degenerate homoclinic orbits bifurcations in [1]. Since then, the related studies were mostly by use of the traditional construction method of Poincaré map. In 1998, by using the generalized Floquet method to construct the local coordinate system and the Poincaré map, Zhu and Xia discussed the bifurcation problems of non-degenerated homoclinic and heteroclinic loops in [23, 24]. In 2000, Jin and Zhu [8] studied the bifurcations of degenerate homoclinic loop for higher dimensional system by using the foundational solutions of the linear variational system of the unperturbed system along the homoclinic orbit as the local coordinate systems to construct the Poincaré map. This method not only has important theoretical significance, but also has good maneuverability in the application. From then on, Jin, Zhu, Huang, Liu, et al. studied the bifurcations and stability of homoclinic and heteroclinic loops for higher dimensional systems [4–6, 9–14, 16]. In [7, 22], Jin and Zhang studied the double homoclinic loops bifurcations under the non-twisted condition. For twisted double homoclinic loops, Lu studied the bifurcation problems under the non-resonant condition in [17]. In this paper, we study the bifurcations of twisted double homoclinic loops under the resonant condition for the higher dimensional systems. In this case, we obtain the existence, uniqueness and existence regions of the large homoclinic loops, large periodic orbits, respectively. Moreover, the corresponding diagrams are given.

Suppose that the C^r system

$$\dot{z} = f(z), \tag{1.1}$$

where $r \geq 5$, $z \in \mathbf{R}^{m+n}$ satisfies the following hypotheses.

(H1) (Hyperbolicity) $z = 0$ is the hyperbolic equilibrium point of system (1.1), the stable manifold W_0^s and the unstable manifold W_0^u of $z = 0$ are m-dimensional and n-dimensional, respectively. λ_1 and $-\rho_1$ are simple eigenvalues of $Df(0)$, such that any other eigenvalue σ of $Df(0)$ satisfies either $\mathrm{Re}\sigma < -\rho_0 < -\rho_1 < 0$ or $\mathrm{Re}\sigma > \lambda_0 > \lambda_1 > 0$, where λ_0 and ρ_0 are some positive numbers.

(H2) (Non-degeneration) System (1.1) has a double homoclinic loops $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_i = \{z = r_i(t) : t \in \mathbf{R}, r_i(\pm\infty) = 0\}$, $i = 1, 2$. For any $P \in \Gamma$, $\mathrm{codim}(T_P W_0^u + T_P W_0^s) = 1$, where $T_P W_0^s$ and $T_P W_0^u$ are the tangent spaces of W_0^s and W_0^u at P, respectively.

(H3) (Strong inclination) Denote $e_i^{\pm} = \lim\limits_{t \to \mp\infty} \dot{r}_i(t)/|\dot{r}_i(t)|$, e_i^+ and e_i^- are the unit eigenvectors corresponding to λ_1 and $-\rho_1$, respectively. Let $T_0 W_0^u = T_0 W_0^{uu} \oplus e_i^+$, $T_0 W_0^s = T_0 W_0^{ss} \oplus e_i^-$, where $T_0 W_0^s$ and $T_0 W_0^u$ are the tangent spaces of W_0^s and W_0^u at $z = 0$, W_0^{ss} and W_0^{uu} are the strong stable manifold and the strong unstable manifold of $z = 0$, $T_0 W_0^{ss}$ and $T_0 W_0^{uu}$ are the tangent spaces of W_0^{ss} and W_0^{uu} at $z = 0$, respectively. The following strong inclination hold:

$$\lim_{t \to +\infty} (T_{r_i(t)} W_0^s + T_{r_i(t)} W_0^u) = T_0 W_0^s \oplus T_0 W_0^{uu}, \quad \lim_{t \to -\infty} (T_{r_i(t)} W_0^s + T_{r_i(t)} W_0^u) = T_0 W_0^{ss} \oplus T_0 W_0^u,$$

where $i = 1, 2$.

Remark 1.1. Obviously, $e_1^+ = -e_2^+$, $e_1^- = -e_2^-$, $T_0 W_0^{ss}$ is the generalized eigenspace corresponding to those eigenvalues with smaller real part than $-\rho_0$, $T_0 W_0^{uu}$ is the generalized eigenspace corresponding to those eigenvalues with larger real part than λ_0.

(H4) (Resonance condition) $\rho_1 = \lambda_1$.

Now, we consider the bifurcation problems of the following C^r perturbed system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.2}$$

where $\mu \in \mathbf{R}^l$, $l \geq 5$, $0 \leq |\mu| \ll 1$, $g(0, \mu) = g(z, 0) = 0$.

2. Local coordinate systems

Suppose that **(H1)**$\sim$**(H4)** are established, then, in the small enough neighborhood U of $z = 0$, we can introduce successively two transformations (see [22]) such that system (1.2) has the following form:

$$\begin{cases} \dot{x} = [\lambda_1(\mu) + h.o.t.]\, x + O(u)[O(y) + O(v)], \\ \dot{y} = [-\rho_1(\mu) + h.o.t.]\, y + O(v)[O(x) + O(u)], \\ \dot{u} = [B_1(\mu) + h.o.t.]\, u + O(x)[O(x) + O(y) + O(v)], \\ \dot{v} = [-B_2(\mu) + h.o.t.]\, v + O(y)[O(x) + O(y) + O(u)], \end{cases} \tag{2.1}$$

where $z = (x, y, u^*, v^*)^*$, $x \in R^1$, $y \in R^1$, $u \in \mathbf{R}^{n-1}$, $v \in \mathbf{R}^{m-1}$, $*$ means transposition, $\lambda_1(0) = \rho_1(0) = \lambda_1$, $\mathrm{Re}\sigma(B_1(\mu)) > \lambda_0$, $\mathrm{Re}\sigma(-B_2(\mu)) < -\rho_0$, and the "h.o.t." means higher order term. Thus, the unstable manifold, stable manifold, strong unstable manifold, strong stable manifold and local homoclinic orbits have the following forms, respectively

$$\begin{aligned} W^u_{loc} &= \{y = 0, v = 0\}, & W^s_{loc} &= \{x = 0, u = 0, \}, \\ W^{uu}_{loc} &= \{x = 0, y = 0, v = 0\}, & W^{ss}_{loc} &= \{x = 0, u = 0, y = 0\}, \\ \Gamma_i \cap W^u_{loc} &= \{y = 0, v = 0, u = u_i(x)\}, & \Gamma_i \cap W^s_{loc} &= \{x = 0, u = 0, v = v_i(y)\}, \end{aligned}$$

where $i = 1, 2$, $u_i(0) = \dot{u}_i(0) = 0$, $v_i(0) = \dot{v}_i(0) = 0$.

Denote $r_i(t) = (r^x_i(t), r^y_i(t), (r^u_i(t))^*, (r^v_i(t))^*)^*$, $i = 1, 2$. Suppose that $r_1(-T_1) = (\delta, 0, \delta^*_{1,u}, 0^*)^*$, $r_1(T_1) = (0, \delta, 0^*,\ \delta^*_{1,v})^*$, $r_2(-T_2) = (-\delta, 0, \delta^*_{2,u}, 0^*)^*$, $r_2(T_2) = (0, -\delta, 0^*,\ \delta^*_{2,v})^*$, where, $T_i > 0$, $i = 1, 2$, δ is small enough, such that $\{(x, y, u, v) : |x|,|y|,|u|,|v| < 2\delta\} \subset U$. Obviously, $|\delta_{i,u}|$, $|\delta_{i,v}|$ at least are $O(\delta^\omega)$, $\omega = \min\{\frac{Re\sigma(B_2(\mu))}{\rho_1(\mu)}, \frac{Re\sigma(B_1(\mu))}{\lambda_1(\mu)}\} > 1$.

Consider the linear variational system

$$\dot{z} = (Df(r_i(t)))z. \tag{2.2}$$

Similar to [6, 7, 16, 17, 22], (2.2) has a fundamental solution matrix $Z_i(t) = (z^1_i(t), z^2_i(t),\ z^3_i(t),\ z^4_i(t))$ satisfying

$$\begin{aligned} &z^1_i(t) \in (T_{r_i(t)}W^s)^c \cap (T_{r_i(t)}W^u)^c, \\ &z^2_i(t) = (-1)^i \dot{r}_i(t)/|\dot{r}^y_i(T_i)| \in T_{r_i(t)}W^s \cap T_{r_i(t)}W^u, \\ &z^3_i(t) = (z^{3,1}_i(t), \cdots, z^{3,n-1}_i(t)) \in (T_{r_i(t)}W^s)^c \cap (T_{r_i(t)}W^u) = T_{r_i(t)}W^{uu}, \\ &z^4_i(t) = (z^{4,1}_i(t), \cdots, z^{4,m-1}_i(t)) \in (T_{r_i(t)}W^s) \cap (T_{r_i(t)}W^u)^c = T_{r_i(t)}W^{ss}, \end{aligned}$$

and

$$Z_i(T_i) = \begin{pmatrix} 1 & 0 & w^{31}_i & 0 \\ 0 & 1 & w^{32}_i & 0 \\ 0 & 0 & w^{33}_i & 0 \\ w^{14}_i, & w^{24}_i & w^{34}_i & I \end{pmatrix}, \quad Z_i(-T_i) = \begin{pmatrix} w^{11}_i & w^{21}_i & 0 & w^{41}_i \\ w^{12}_i & 0 & 0 & w^{42}_i \\ w^{13}_i & w^{23}_i & I & w^{43}_i \\ 0, & 0 & 0 & w^{44}_i \end{pmatrix},$$

where $i = 1, 2$, $w^{21}_i < 0$, $w^{12}_i \neq 0$, $\det w^{33}_i \neq 0$, $\det w^{44}_i \neq 0$, and $|w^{1j}_i(w^{12}_i)^{-1}| \ll 1$, $j \neq 2$; $|w^{2j}_i(w^{21}_i)^{-1}| \ll 1$, $j = 3, 4$; $|w^{3j}_i(w^{33}_i)^{-1}| \ll 1$, $j \neq 3$; $|w^{4j}_i(w^{44}_i)^{-1}| \ll 1$, $j \neq 4$.

Denote $\Phi_i(t) = (\phi^1_i(t), \phi^2_i(t),\ \phi^3_i(t), \phi^4_i(t)) = (Z^{-1}_i(t))^*$, $i = 1, 2$, so, $\Phi_i(t)$ is a fundamental solution matrix of the adjoint system $\dot{\phi} = -(Df(r_i(t)))^*\phi$ of (2.2), and $\phi^1_i(t) \in (T_{r_i(t)}W^s)^c \cap (T_{r_i(t)}W^u)^c$ is bounded and tends to zero exponentially as $t \to \pm\infty$ [7, 8, 17, 19, 22, 23].

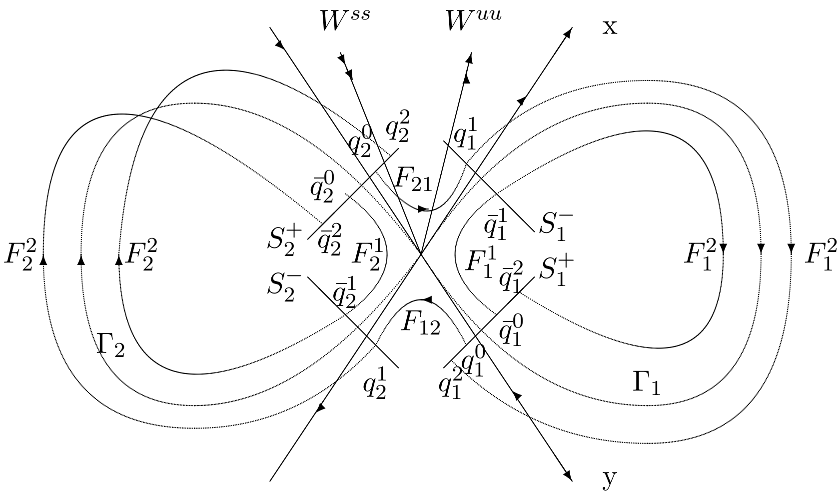
We select $z^1_i(t)$, $z^2_i(t)$, $z^3_i(t)$, $z^4_i(t)$ as the local coordinate systems along Γ_i, $i = 1, 2$.

Let $\Delta_i = w^{12}_i/|w^{12}_i|$, $i = 1, 2$. We say that Γ_i is non-twisted if $\Delta_i = 1$, and twisted if $\Delta_i = -1$. In this paper, we consider the case of twisted.

3. Poincaré maps and the bifurcation equations with single twisted orbit

(H5) (Single twisted condition) $\Delta_1 = 1$, $\Delta_2 = -1$.

Denote $h_i(t) = r_i(t) + Z_i(t)N_i(t)$, $N_i(t) = (n_i^1, 0, (n_i^3)^*, (n_i^4)^*)^*$, $i = 1, 2$, and let $S_i^- = \{z = h_i(-T_i) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$, $S_i^+ = \{z = h_i(T_i) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$ be the cross sections of Γ_i at $t = -T_i$ and $t = T_i$, respectively.



$\Delta_1 = 1, \Delta_2 = -1$

Figure 1

Now, we set up Poincaré maps.

In U, denote $F_{21} : S_2^+ \to S_1^-$, $F_{21}(q_2^{2j}) = q_1^{2j+1}$; $F_{12} : S_1^+ \to S_2^-$, $F_{12}(q_1^{2j}) = q_2^{2j+1}$; $F_1^1 : S_1^+ \to S_1^-$, $F_1^1(\bar{q}_1^{2j}) = \bar{q}_1^{2j+1}$; $F_2^1 : S_2^+ \to S_2^-$, $F_2^1(\bar{q}_2^{2j}) = \bar{q}_2^{2j+1}$ where $i = 1, 2$, $j = 0, 1, \cdots$.

In the tubular neighborhood of Γ_i, denote by F_i^2, the map from S_i^- to S_i^+. Due to $\Delta_1 = 1$, $\Delta_2 = -1$, we denote $F_1^2(q_1^{2j+1}) = q_1^{2j+2}$, $F_1^2(\bar{q}_1^{2j+1}) = \bar{q}_1^{2j+2}$; $F_2^2(q_2^{2j+1}) = \bar{q}_2^{2j+2}$, $F_2^2(\bar{q}_2^{2j+1}) = q_2^{2j+2}$ where $i = 1, 2$, $j = 0, 1, 2, \cdots$ (Figure 1).

At first, we set up the relationship between the Cartesian coordinates and the normal coordinates of the points in the neighborhood of homoclinic loop. Let

$$\begin{aligned}
q_i^{2j}(x_i^{2j}, y_i^{2j}, (u_i^{2j})^*, (v_i^{2j})^*)^* &= r_i(T_i) + Z_i(T_i)N_i^{2j},\\
\bar{q}_i^{2j}(\bar{x}_i^{2j}, \bar{y}_i^{2j}, (\bar{u}_i^{2j})^*, (\bar{v}_i^{2j})^*)^* &= r_i(T_i) + Z_i(T_i)\bar{N}_i^{2j},\\
q_i^{2j+1}(x_i^{2j+1}, y_i^{2j+1}, (u_i^{2j+1})^*, (v_i^{2j+1})^*)^* &= r_i(-T_i) + Z(-T_i)N_i^{2j+1},\\
\bar{q}_i^{2j+1}(\bar{x}_i^{2j+1}, \bar{y}_i^{2j+1}, (\bar{u}_i^{2j+1})^*, (\bar{v}_i^{2j+1})^*)^* &= r_i(-T_i) + Z(-T_i)\bar{N}_i^{2j+1},\\
N_i^{2j} &= (n_i^{2j,1}, 0, (n_i^{2j,3})^*, (n_i^{2j,4})^*)^*,\\
N_i^{2j+1} &= (n_i^{2j+1,1}, 0, (n_i^{2j+1,3})^*, (n_i^{2j+1,4})^*)^*,\\
\bar{N}_i^{2j} &= (\bar{n}_i^{2j,1}, 0, (\bar{n}_i^{2j,3})^*, (\bar{n}_i^{2j,4})^*)^*,\\
\bar{N}_i^{2j+1} &= (\bar{n}_i^{2j+1,1}, 0, (\bar{n}_i^{2j+1,3})^*, (\bar{n}_i^{2j+1,4})^*)^*.
\end{aligned}$$

By $Z_i^{-1}(T_i)$, $Z_i^{-1}(-T_i)$ and some simple calculations, we get

$$y_1^{2j} \approx \delta, \;\; x_1^{2j+1} \approx \delta, \;\; y_2^{2j} \approx -\delta, \;\; x_2^{2j+1} \approx -\delta, \tag{3.1}$$

and

$$\begin{cases} n_i^{2j+1,1} = (w_i^{12})^{-1}[y_i^{2j+1} - w_i^{42}(w_i^{44})^{-1}v_i^{2j+1}], \\ n_i^{2j+1,3} = u_i^{2j+1} - \delta_{iu} + b_i(w_i^{12})^{-1}y_i^{2j+1} + a_i(w_i^{44})^{-1}v_i^{2j+1}, \\ n_i^{2j+1,4} = (w_i^{44})^{-1}v_i^{2j+1}, \end{cases} \tag{3.2}$$

$$\begin{cases} n_i^{2j,1} = x_i^{2j} - w_i^{31}(w_i^{33})^{-1}u_i^{2j}, \\ n_i^{2j,3} = (w_i^{33})^{-1}u_i^{2j}, \\ n_i^{2j,4} = -w_i^{14}x_i^{2j} + c_i(w_i^{33})^{-1}u_i^{2j} + v_i^{2j} - \delta_{iv}, \end{cases} \tag{3.3}$$

where, $b_i = w_i^{11}w_i^{23}(w_i^{21})^{-1} - w_i^{13}$, $a_i^1 = -w_i^{41} + w_i^{11}(w_i^{12})^{-1}w_i^{42}$, $a_i^3 = -w_i^{43} + w_i^{13}(w_i^{12})^{-1}w_i^{42}$, $a_i = a_i^3 - w_i^{23}(w_i^{21})^{-1}a_i^1$, $c_i = (w_i^{14}w_i^{31} + w_i^{24}w_i^{32} - w_i^{34})$.

As well, the relationship between the two kinds of coordinates of $\bar{q}_i^{2j}$, $\bar{q}_i^{2j+1}$ also satisfies (3.1), (3.2), and (3.3).

Now, we consider the map F_i^2. Substituting transformation $z = h_i(t)$ into (1.2), and using $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$, we get

$$Z_i(t)(\dot{n_i^1}, 0, \dot{(n_i^3)}^*, \dot{(n_i^4)}^*)^* = g_\mu(r_i(t), 0)\mu + h.o.t..$$

Multiplying the both sides of the above equation by $\Phi_i^*(t)$ and using $\Phi_i^*(t)Z_i(t) = I$, we have

$$(\dot{n_i^1}, 0, \dot{(n_i^3)}^*, \dot{(n_i^4)}^*)^* = \Phi_i^*(t)g_\mu(r_i(t), 0)\mu + h.o.t..$$

Integrating it, we have F_i^2 defined by the following

$$\begin{cases} n_1^{2j+2,k} = n_1^{2j+1,k} + M_1^k\mu + h.o.t., \\ \bar{n}_1^{2j+2,k} = \bar{n}_1^{2j+1,k} + M_1^k\mu + h.o.t., \\ \bar{n}_2^{2j+2,k} = n_2^{2j+1,k} + M_2^k\mu + h.o.t., \\ n_2^{2j+2,k} = \bar{n}_2^{2j+1,k} + M_2^k\mu + h.o.t., \end{cases} \quad k = 1, 3, 4, \tag{3.4}$$

where, $M_i^k = \int_{-\infty}^{+\infty}(\phi_i^k(t))^* g_\mu(r_i(t), 0)dt$, $k = 1, 3, 4$, $i = 1, 2$ [7, 8, 17, 22, 23].

Next, we consider the map in U. Without loss of generality, we may assume that the resonance condition has the following form for the system (1.2).

$\rho_1(\mu) = (1 + \alpha(\mu))\lambda_1(\mu)$, where, $\alpha(\mu) \in R^1$, $|\alpha(\mu)| \ll 1$, $\alpha(0) = 0$.

Assume that τ_{21} is the time from q_2^0 to q_1^1, τ_{12} is the time from q_1^0 to q_2^1, τ_1 is the time from $\bar{q}_1^0$ to $\bar{q}_1^1$, and τ_2 is the time from $\bar{q}_2^0$ to $\bar{q}_2^1$. Set $s_j = e^{-\lambda_1(\mu)\tau_j}$, $j = 21, 12, 1, 2$, which are called the Silnikov times. By (2.1) we have

$$x = e^{\lambda_1(\mu)(t-T_i-\tau_i)}x_1 + h.o.t., \quad y = e^{-(1+\alpha(\mu))\lambda_1(\mu)(t-T_i)}y_0 + h.o.t.,$$
$$u = e^{B_1(\mu)(t-T_i-\tau_i)}u_1 + h.o.t., \quad v = e^{-B_2(\mu)(t-T_i)}v_0 + h.o.t..$$

Neglecting the higher order terms, the above formulas defined the following maps for x_1, y_0, u_1, v_0 and t take the corresponding values.

$$F_1^1: \quad \bar{x}_1^0 \approx \delta s_1, \ \bar{y}_1^1 \approx \delta s_1^{(1+\alpha(\mu))}, \ \bar{u}_1^0 \approx s_1^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_1^1, \ \bar{v}_1^1 \approx s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^0, \tag{3.5}$$

$$F_2^1: \quad \bar{x}_2^0 \approx -\delta s_2, \bar{y}_2^1 \approx -\delta s_2^{(1+\alpha(\mu))}, \bar{u}_2^0 \approx s_2^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_2^1, \bar{v}_2^1 \approx s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^0, \tag{3.6}$$

$$F_{21}: \quad x_2^0 \approx \delta s_{21}, y_1^1 \approx -\delta s_{21}^{(1+\alpha(\mu))}, u_2^0 \approx s_{21}^{B_1(\mu)/\lambda_1(\mu)}u_1^1, v_1^1 \approx s_{21}^{B_2(\mu)/\lambda_1(\mu)}v_2^0, \tag{3.7}$$

$$F_{12}: \quad x_1^0 \approx -\delta s_{12}, y_2^1 \approx \delta s_{12}^{(1+\alpha(\mu))}, u_1^0 \approx s_{12}^{B_1(\mu)/\lambda_1(\mu)}u_2^1, v_2^1 \approx s_{12}^{B_2(\mu)/\lambda_1(\mu)}v_1^0. \tag{3.8}$$

At last, by (3.1) $\sim$ (3.4) and (3.5) $\sim$ (3.8), we can get Poincaré maps as follows.

$\bar{F}_1 = F_1^2 \circ F_1^1\colon\ S_1^+ \mapsto S_1^+,\ \bar{F}_1(\bar{q}_1^0) = \bar{q}_1^2$:

$$\begin{cases} \bar{n}_1^{2,1} = (w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} + M_1^1\mu + h.o.t., \\ \bar{n}_1^{2,3} = \bar{u}_1^1 - \delta_{1u} + b_1(w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} + M_1^3\mu + h.o.t., \\ \bar{n}_1^{2,4} = (w_1^{44})^{-1} s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^0 + M_1^4\mu + h.o.t.. \end{cases}$$

$F_1 = F_1^2 \circ F_{21}\colon\ S_2^+ \mapsto S_1^+,\ F_1(q_2^0) = q_1^2$:

$$\begin{cases} n_1^{2,1} = -(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} + M_1^1\mu + h.o.t., \\ n_1^{2,3} = u_1^1 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} + M_1^3\mu + h.o.t., \\ n_1^{2,4} = (w_1^{44})^{-1} s_{21}^{B_2(\mu)/\lambda_1(\mu)} v_2^0 + M_1^4\mu + h.o.t.. \end{cases}$$

$F_2 = F_2^2 \circ F_{12}\colon\ S_1^+ \mapsto S_2^+,\ F_2(q_1^0) = \bar{q}_2^2$:

$$\begin{cases} \bar{n}_2^{2,1} = (w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t., \\ \bar{n}_2^{2,3} = u_2^1 - \delta_{2u} + b_2(w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} + M_2^3\mu + h.o.t., \\ \bar{n}_2^{2,4} = (w_2^{44})^{-1} s_{12}^{B_2(\mu)/\lambda_1(\mu)} v_1^0 + M_2^4\mu + h.o.t.. \end{cases}$$

$\bar{F}_2 = F_2^2 \circ F_2^1\colon\ S_2^+ \mapsto S_2^+,\ \bar{F}_2(\bar{q}_2^0) = q_2^2$:

$$\begin{cases} n_2^{2,1} = -(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t., \\ n_2^{2,3} = \bar{u}_2^1 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} + M_2^3\mu + h.o.t., \\ n_2^{2,4} = (w_2^{44})^{-1} s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^0 + M_2^4\mu + h.o.t.. \end{cases}$$

Meanwhile, we get the successor functions as follows:
$\bar{G}_1(s_1, \bar{u}_1^1, \bar{v}_1^0) = (\bar{G}_1^1, \bar{G}_1^3, \bar{G}_1^4) = (\bar{F}_1(\bar{q}_1^0) - \bar{q}_1^0)$ is given by

$$\begin{cases} \bar{G}_1^1 = \delta[(w_1^{12})^{-1} s_1^{(1+\alpha(\mu))} - s_1] + M_1^1\mu + h.o.t., \\ \bar{G}_1^3 = \bar{u}_1^1 - \delta_{1u} + b_1(w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} - (w_1^{33})^{-1} s_1^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_1^1 + M_1^3\mu + h.o.t., \\ \bar{G}_1^4 = -\bar{v}_1^0 + \delta_{1v} + w_1^{14}\delta s_1 + (w_1^{44})^{-1} s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^0 + M_1^4\mu + h.o.t.. \end{cases}$$

$\bar{G}_2(s_2, \bar{u}_2^1, \bar{v}_2^0) = (\bar{G}_2^1, \bar{G}_2^3, \bar{G}_2^4) = (\bar{F}_2(\bar{q}_2^0) - \bar{q}_2^0)$ is given by

$$\begin{cases} \bar{G}_2^1 = \delta[-(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} + s_2] + M_2^1\mu + h.o.t., \\ \bar{G}_2^3 = \bar{u}_2^1 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} - (w_2^{33})^{-1} s_2^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_2^1 + M_2^3\mu + h.o.t., \\ \bar{G}_2^4 = -\bar{v}_2^0 + \delta_{2v} - w_2^{14}\delta s_2 + (w_2^{44})^{-1} s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^0 + M_2^4\mu + h.o.t.. \end{cases}$$

$G_1(s_{12}, s_{21}, u_1^1, u_2^1, v_1^0, v_2^0) = (G_1^1, G_1^3, G_1^4) = (F_1(q_2^0) - q_1^0)$ is given by

$$\begin{cases} G_1^1 = \delta[-(w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} + s_{12}] + M_1^1\mu + h.o.t., \\ G_1^3 = u_1^1 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} - (w_1^{33})^{-1} s_{12}^{B_1(\mu)/\lambda_1(\mu)} u_2^1 + M_1^3\mu + h.o.t., \\ G_1^4 = -v_1^0 + \delta_{1v} - w_1^{14}\delta s_{12} + (w_1^{44})^{-1} s_{21}^{B_2(\mu)/\lambda_1(\mu)} v_2^0 + M_1^4\mu + h.o.t.. \end{cases} \tag{3.9}$$

$G_2(s_{12}, s_2, u_2^1, \bar{u}_2^1, v_1^0, \bar{v}_2^0) = (G_2^1, G_2^3, G_2^4) = (F_2(q_1^0) - \bar{q}_2^0)$ is given by

$$\begin{cases} G_2^1 = \delta[(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} + s_2] + M_2^1\mu + h.o.t., \\ G_2^3 = u_2^1 - \delta_{2u} + b_2(w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} - (w_2^{33})^{-1} s_2^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_2^1 + M_2^3\mu + h.o.t., \\ G_2^4 = -\bar{v}_2^0 + \delta_{2v} - w_2^{14}\delta s_2 + (w_2^{44})^{-1} s_{12}^{B_2(\mu)/\lambda_1(\mu)} v_1^0 + M_2^4\mu + h.o.t.. \end{cases} \tag{3.10}$$

$\tilde{G}_2(s_{21}, s_2, u_1^1, \bar{u}_2^1, v_2^0, \bar{v}_2^0) = (\tilde{G}_2^1, \tilde{G}_2^3, \tilde{G}_2^4) = (\bar{F}_2(\bar{q}_2^0) - q_2^0)$ is given by

$$\begin{cases} \tilde{G}_2^1 = \delta[-(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} - s_{21}] + M_2^1\mu + h.o.t., \\ \tilde{G}_2^3 = \bar{u}_2^1 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} - (w_2^{33})^{-1} s_{21}^{B_1(\mu)/\lambda_1(\mu)} u_1^1 + M_2^3\mu + h.o.t., \\ \tilde{G}_2^4 = -v_2^0 + \delta_{2v} + w_2^{14}\delta s_{21} + (w_2^{44})^{-1} s_2^{B_2(\mu)/\lambda_1(\mu)} \bar{v}_2^0 + M_2^4\mu + h.o.t.. \end{cases} \tag{3.11}$$

Thus, we get the three bifurcation equations as follows:

$$\bar{G}_1(s_1, \bar{u}_1^1, \bar{v}_1^0) = (\bar{G}_1^1, \bar{G}_1^3, \bar{G}_1^4) = 0, \tag{3.12}$$

$$\bar{G}_2(s_2, \bar{u}_2^1, \bar{v}_2^0) = (\bar{G}_2^1, \bar{G}_2^3, \bar{G}_2^4) = 0, \tag{3.13}$$

$$G(s_{12}, s_{21}, s_2, u_1^1, u_2^1, v_1^0, v_2^0, \bar{u}_2^1, \bar{v}_2^0) = (G_1, G_2, \tilde{G}_2) = 0. \tag{3.14}$$

Obviously, for system (1.2), there is a one to one correspondence between the solutions of the bifurcation equations satisfying $s_j \geq 0$, $j = 1, 2, 21, 12$, and the 1-homoclinic loops and 1-periodic orbits bifurcated from Γ_1, Γ_2, $\Gamma = \Gamma_1 \cup \Gamma_2$, respectively.

We call the 1-homoclinic loop and 1-periodic orbit bifurcated from the single homoclinic loop Γ_i as small homoclic loop and small period orbit, respectively; call the 1-homoclinic loop and 1-periodic orbit bifurcated from $\Gamma = \Gamma_1 \cup \Gamma_2$ as large homoclic loop and large period orbit, respectively.

4. Bifurcations with the single twisted orbit

At first, by the analysis of the existence of solutions of the equations (3.12) and (3.13) which satisfy $s_j \geq 0$, $j = 1, 2$, we can get the bifurcations of the single homoclinic loop Γ_i, $i = 1, 2$, for the case of non-twisted and the case of twisted, respectively. About the details, one can see [6, 11] and their references.

Now, in this paper, we discuss the large 1-homoclinic loops and large 1-periodic orbits bifurcated by $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, discuss the solutions $Q(s_{12}, s_{21}, s_2, u_1^1, u_2^1, v_1^0, v_2^0, \bar{u}_2^1, \bar{v}_2^0)$ of the bifurcation equation (3.14) which satisfy $s_{12} \geq 0$, $s_{21} \geq 0$, $s_2 \geq 0$.

By (3.9), (3.10), (3.11), for $0 \leq s_{12}, s_{21}, s_2, |\mu| \ll 1$, the equation $(G_1^3, G_1^4, G_2^3, G_2^4, \tilde{G}_2^3, \tilde{G}_2^4) = 0$ has always a unique solution $u_1^1 = u_1^1(s_{21}, s_{12}, s_2, \mu)$, $u_2^1 = u_2^1(s_{21}, s_{12}, s_2, \mu)$, $v_1^0 = v_1^0(s_{21}, s_{12}, s_2, \mu)$, $v_2^0 = v_2^0(s_{21}, s_{12}, s_2, \mu)$, $\bar{u}_2^1 = \bar{u}_2^1(s_{21}, s_{12}, s_2, \mu)$, $\bar{v}_2^0 = \bar{v}_2^0(s_{21}, s_{12}, s_2, \mu)$. Substituting it into $(G_1^1, G_2^1, \tilde{G}_2^1) = 0$, we have

$$\begin{cases} \delta[-(w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} + s_{12}] + M_1^1\mu + h.o.t. = 0, \\ \delta[(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} + s_2] + M_2^1\mu + h.o.t. = 0, \\ \delta[-(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} - s_{21}] + M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.1}$$

Thus, for system (1.2), there is a one to one correspondence between the large 1-homoclinic loops and large 1-periodic orbits bifurcated from $\Gamma = \Gamma_1 \cup \Gamma_2$ and the solutions of the bifurcation equation (4.1) satisfying $s_{12} \geq 0$, $s_{12} \geq 0$, $s_2 \geq 0$, respectively.

4.1. $\alpha(\mu) = 0$

Theorem 4.1. *Suppose that* **(H1)∼(H5)** *hold. If* $\alpha(\mu) = 0$, $(w_1^{12})^{-1}(w_2^{12})^{-2} \neq 1$, *rank*$\{M_1^1, M_2^1\} = 2$, *then, for* $|\mu| \ll 1$, *system* (1.2) *has at most one large 2-1 homoclinic loop, or one large 1-1 homoclinic loop, or one large 2-1 double homoclinic loop, or one large 1-1 double homoclinic loop, or a large 2-1 periodic loop in the small neighbourhood of* $\Gamma = \Gamma_1 \cap \Gamma_2$. *Moreover, these orbits do not coexist.*

Proof. In this case, (4.1) becomes

$$\begin{cases} -(w_1^{12})^{-1} s_{21} + s_{12} + \delta^{-1} M_1^1\mu + h.o.t. = 0, \\ (w_2^{12})^{-1} s_{12} + s_2 + \delta^{-1} M_2^1\mu + h.o.t. = 0, \\ -(w_2^{12})^{-1} s_2 - s_{21} + \delta^{-1} M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.2}$$

That is

$$\begin{pmatrix} (w_1^{12})^{-1} & -1 & 0 \\ 0 & -(w_2^{12})^{-1} & -1 \\ 1 & 0 & (w_2^{12})^{-1} \end{pmatrix} \begin{pmatrix} s_{21} \\ s_{12} \\ s_2 \end{pmatrix} = \delta^{-1} \begin{pmatrix} M_1^1\mu \\ M_2^1\mu \\ M_2^1\mu \end{pmatrix} + h.o.t..$$

Denote $B = \begin{pmatrix} (w_1^{12})^{-1} & -1 & 0 \\ 0 & -(w_2^{12})^{-1} & -1 \\ 1 & 0 & (w_2^{12})^{-1} \end{pmatrix}$. If $\|B\| = 1 - (\omega_1^{12})^{-1}(\omega_2^{12})^{-2} \neq 0$, then, (4.2) has a unique solution $0 \leq s_{12}(\mu), s_{21}(\mu), s_2(\mu) \ll 1$ satisfying $s_{12}(0) = s_{21}(0) = s_2(0) = 0$, that is,

$$\begin{aligned} \begin{pmatrix} s_{21} \\ s_{12} \\ s_2 \end{pmatrix} &= \delta^{-1}B^{-1} \begin{pmatrix} M_1^1\mu \\ M_2^1\mu \\ M_2^1\mu \end{pmatrix} + h.o.t. = \delta^{-1}\|B\|^{-1}B^* \begin{pmatrix} M_1^1\mu \\ M_2^1\mu \\ M_2^1\mu \end{pmatrix} + h.o.t. \\ &= \delta^{-1}\|B\|^{-1} \begin{pmatrix} \left[-(w_2^{12})^{-2}M_1^1 + \left((w_2^{12})^{-1}+1\right)M_2^1\right]\mu \\ \left[-M_1^1 + (w_1^{12})^{-1}\left((w_2^{12})^{-1}+1\right)M_2^1\right]\mu \\ \left[(w_2^{12})^{-1}M_1^1 - \left(1+(w_1^{12}w_2^{12})^{-1}\right)M_2^1\right]\mu \end{pmatrix} + h.o.t.. \end{aligned} \tag{4.3}$$

If $s_{21} = 0$, $s_{12} > 0$, $s_2 > 0$, or $s_{12} = 0$, $s_{21} > 0$, $s_2 > 0$, or $s_2 = 0$, $s_{12} > 0$, $s_{21} > 0$, then, system (1.2) has a large 2-1 homoclinic loop (Figures 2, 3, 4).

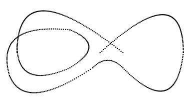
If $s_{21} = 0$, $s_2 = 0$, $s_{12} > 0$, or $s_{12} = 0$, $s_2 = 0$, $s_{21} > 0$, then, system (1.2) has a large 1-1 homoclinic loop (Figures 5, 6).

If $s_{21} = 0$, $s_{12} = 0$, $s_2 > 0$, then, system (1.2) has a large 2-1 double homoclinic loop (Figure 7).

If $s_{12} = 0$, $s_{21} = 0$, $s_2 = 0$, then, system (1.2) has a double 1-1 homoclinic loop (Figure 8).

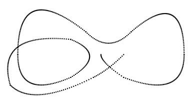
If $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$, then, system (1.2) has a large 2-1 periodic loop (Figure 9).

Thus, the theorem is established. □



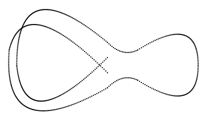
$s_{21} = 0,$
$s_{12} > 0, s_2 > 0$

Figure 2



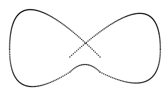
$s_{12} = 0,$
$s_{21} > 0, s_2 > 0$

Figure 3



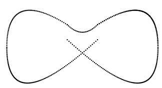
$s_2 = 0,$
$s_{21} > 0, s_{12} > 0$

Figure 4



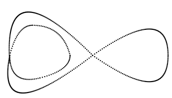
$s_2 = 0, s_{21} = 0,$
$s_{12} > 0$

Figure 5



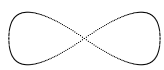
$s_2 = 0, s_{12} = 0,$
$s_{21} > 0$

Figure 6



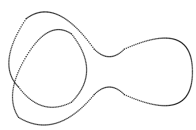
$s_{21} = 0, s_{12} = 0,$
$s_2 > 0$

Figure 7



$s_{21} = 0, s_{12} = 0,$
$s_2 = 0$

Figure 8



$s_{21} > 0, s_{12} > 0,$
$s_2 > 0$

Figure 9

Theorem 4.2. *Suppose that* **(H1)**∼**(H5)** *hold. If* $\alpha(\mu) = 0$, $(w_1^{12})^{-1}(w_2^{12})^{-2} \neq 1$, *rank*$\{M_1^1, M_2^1\} = 2$, *then, for* $|\mu| \ll 1$, *there exist surfaces* L_{21}^0, L_{12}^0, L_2^0, $L_{21,2}^0$, $L_{12,2}^0$, $L_{21,12}^0$, $L_{21,12,2}^0$, *and a region* H, *such that*

(1) *For* $\mu \in L^0_{21}$, *(4.2) has a solution* $s_{21} = 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 2*).*

(2) *For* $\mu \in L^0_{12}$, *(4.2) has a solution* $s_{12} = 0$, $s_{21} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 3*).*

(3) *For* $\mu \in L^0_{2}$, *(4.2) has a solution* $s_{2} = 0$, $s_{12} > 0$, $s_{21} > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 4*).*

(4) *For* $\mu \in L^0_{21,2}$, *(4.2) has a solution* $s_{21} = 0$, $s_2 = 0$, $s_{12} > 0$, *that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 5*).*

(5) *For* $\mu \in L^0_{12,2}$, *(4.2) has a solution* $s_{12} = 0$, $s_2 = 0$, $s_{21} > 0$, *that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 6*).*

(6) *For* $\mu \in L^0_{21,12}$, *(4.2) has a solution* $s_{21} = 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 double homoclinic loop (Figure* 7*).*

(7) *For* $\mu \in L^0_{21,12,2}$, *(4.2) has solution* $s_{12} = 0$, $s_{21} = 0$, $s_2 = 0$, *that is, system* (1.2) *has a double 1-1 homoclinic loop* $\Gamma^0 = \Gamma^0_1(\mu) \cup \Gamma^0_2(\mu)$ *(Figure* 8*).*

(8) *For* $\mu \in H$, *(4.2) has a solution* $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 periodic loop (Figure* 9*).*

Proof. By (4.3), we get

(1). In the region $R^0_{21} = \{\mu : \|B\|^{-1} \cdot \left[-M^1_1\mu + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1} + 1\right) M^1_2\mu\right] + h.o.t. > 0\} \cap \{\mu : \|B\|^{-1} \cdot \left[(w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1)^{-1}(w^{12}_2)^{-1}\right) M^1_2\mu\right] + h.o.t. > 0\}$, there is an $(l-1)$-dimensional surface

$$L^0_{21} = \{\mu : -(w^{12}_2)^{-2}M^1_1\mu + \left[(w^{12}_2)^{-1} + 1\right] M^1_2\mu + h.o.t. = 0\},$$

which has normal vector $-(w^{12}_2)^{-2}M^1_1 + \left[(w^{12}_2)^{-1} + 1\right] M^1_2$ at $\mu = 0$, such that for $\mu \in L^0_{21}$, (4.2) has a solution $s_{21} = 0$, $s_{12} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 homoclinic loop.

(2). In the region $R^0_{12} = \{\mu : \|B\|^{-1} \cdot \left[(w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right) M^1_2\mu\right] + h.o.t. > 0\} \cap \{\mu : \|B\|^{-1} \cdot \left[-(w^{12}_2)^{-2}M^1_1\mu + \left((w^{12}_2)^{-1} + 1\right) M^1_2\mu\right] + h.o.t. > 0\}$, there is a $(l-1)$-dimensional surface

$$L^0_{12} = \{\mu : -M^1_1\mu + (w^{12}_1)^{-1}\left[(w^{12}_2)^{-1} + 1\right] M^1_2\mu + h.o.t. = 0\},$$

which has normal vector $-M^1_1 + (w^{12}_1)^{-1}\left[(w^{12}_2)^{-1} + 1\right] M^1_2$ at $\mu = 0$, such that for $\mu \in L^0_{12}$, (4.2) has a solution $s_{12} = 0$, $s_{21} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 homoclinic loop.

(3). In the region $R^0_2 = \{\mu : \|B\|^{-1}\left[-(w^{12}_2)^{-2}M^1_1\mu + \left((w^{12}_2)^{-1} + 1\right) M^1_2\mu\right] + h.o.t. > 0\} \cap \{\mu : \|B\|^{-1}\left[-M^1_1\mu + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1} + 1\right) M^1_2\mu\right] + h.o.t. > 0\}$, there is an $(l-1)$-dimensional surface

$$L^0_2 = \{\mu : (w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right) M^1_2\mu + h.o.t. = 0\},$$

which has normal vector $(w^{12}_2)^{-1}M^1_1 - \left(1 + (w^{12}_1)^{-1}(w^{12}_2)^{-1}\right) M^1_2$ at $\mu = 0$, such that for $\mu \in L^0_2$, (4.2) has a solution $s_2 = 0$, $s_{12} > 0$, $s_{21} > 0$, that is, system (1.2) has a large 2-1 homoclinic loop.

(4). In the region $R^0_{21,2} = \{\mu : \|B\|^{-1}\left[-M^1_1\mu + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1} + 1\right) M^1_2\mu\right] + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\begin{aligned} L^0_{21,2} = L^0_{21} \cap L^0_2 = &\{\mu : -(w^{12}_2)^{-2}M^1_1\mu + \left((w^{12}_2)^{-1} + 1\right) M^1_2\mu + h.o.t. = 0\} \\ &\cap \{\mu : (w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right) M^1_2\mu + h.o.t. = 0\}, \end{aligned}$$

which has normal plane span$\{-(w^{12}_2)^{-2}M^1_1 + \left((w^{12}_2)^{-1} + 1\right) M^1_2, (w^{12}_2)^{-1}M^1_1 - (1 + (w^{12}_1 w^{12}_2)^{-1})M^1_2\}$ at $\mu = 0$, such that for $\mu \in L^0_{21,2}$, (4.2) has a solution $s_{21} = 0$, $s_2 = 0$, $s_{12} > 0$, that is, system (1.2) has a large 1-1 homoclinic loop.

(5). In the region $R^0_{12,2} = \{\mu : \|B\|^{-1}\left[-(w^{12}_2)^{-2}M^1_1\mu + \left((w^{12}_2)^{-1}+1\right)M^1_2\mu\right] + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\begin{aligned} L^0_{12,2} = L^0_{12} \cap L^0_2 =& \{\mu : -M^1_1\mu + (w^{12}_1)^{-1}((w^{12}_2)^{-1}+1)M^1_2\mu + h.o.t. = 0\} \\ &\cap \{\mu : (w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right)M^1_2\mu + h.o.t. = 0\}, \end{aligned}$$

which has normal vector span$\{-M^1_1 + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1}+1\right)M^1_2,\ (w^{12}_2)^{-1}M^1_1 - (1 + (w^{12}_1 w^{12}_2)^{-1})M^1_2\}$ at $\mu = 0$, such that for $\mu \in L^0_{12,2}$, (4.2) has a solution $s_{12} = 0$, $s_2 = 0$, $s_{21} > 0$, that is, system (1.2) has a large 1-1 homoclinic loop.

(6). In the region $R^0_{21,12} = \{\mu : \|B\|^{-1}\left[(w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right)M^1_2\mu\right] + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\begin{aligned} L^0_{21,12} = L^0_{21} \cap L^0_{12} = & \{\mu : -(w^{12}_2)^{-2}M^1_1\mu + \left((w^{12}_2)^{-1}+1\right)M^1_2\mu + h.o.t. = 0\} \\ &\cap \{\mu : -M^1_1\mu + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1}+1\right)M^1_2\mu + h.o.t. = 0\}, \end{aligned}$$

which has normal vector span$\{-(w^{12}_2)^{-2}M^1_1 + ((w^{12}_2)^{-1}+1)M^1_2,\ -M^1_1 + (w^{12}_1)^{-1}((w^{12}_2)^{-1}+1)M^1_2\}$ at $\mu = 0$, such that for $\mu \in L^0_{21,12}$, (4.2) has a solution $s_{21} = 0$, $s_{12} = 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 double homoclinic loop.

(7). There is a surface

$$\begin{aligned} L^0_{21,12,2} = L^0_{21} \cap L^0_{12} \cap L^0_2 = & \{\mu : -(w^{12}_2)^{-2}M^1_1\mu + ((w^{12}_2)^{-1}+1)M^1_2\mu + h.o.t. = 0\} \\ &\cap \left\{\mu : -M^1_1\mu + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1}+1\right)M^1_2\mu + h.o.t. = 0\right\} \\ &\cap \{\mu : (w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right)M^1_2\mu + h.o.t. = 0\}, \end{aligned}$$

which has normal vector

$$\begin{aligned} \mathcal{M} = \text{span}\{&-(w^{12}_2)^{-2}M^1_1 + \left((w^{12}_2)^{-1}+1\right)M^1_2,\ (w^{12}_2)^{-1}M^1_1 - \left(1 + (w^{12}_1)^{-1}(w^{12}_2)^{-1}\right)M^1_2, \\ &- M^1_1 + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1}+1\right)M^1_2\} \end{aligned}$$

at $\mu = 0$, such that, for $\mu \in L^0_{21,12,2}$, (4.2) has solution $s_{12} = 0$, $s_{21} = 0$, $s_2 = 0$, that is, in the small neighborhood of Γ, system (1.2) has a double 1-1 homoclinic loop $\Gamma^0 = \Gamma^0_1(\mu) \cup \Gamma^0_2(\mu)$.

Notice that dim$\mathcal{M}$=2, $\|B^*\| \neq 0$, so, indeed, the surface $L^0_{21,12,2}$ is an $(l-2)$-dimensional surface which has normal plane $\mathcal{M}$ at $\mu = 0$. In fact, $L^0_{21,12,2}$=$\{\mu : M^1_1\mu + h.o.t. = 0\} \cap \{\mu : M^1_2\mu + h.o.t. = 0\}$.

(8). Denote

$$\begin{aligned} H =& \left\{\mu : \|B\|^{-1}\left[-(w^{12}_2)^{-2}M^1_1\mu + \left((w^{12}_2)^{-1}+1\right)M^1_2\mu\right] + h.o.t. > 0\right\} \\ &\cap \left\{\mu : \|B\|^{-1}\left[-M^1_1\mu + (w^{12}_1)^{-1}\left((w^{12}_2)^{-1}+1\right)M^1_2\mu\right] + h.o.t. > 0\right\} \\ &\cap \left\{\mu : \|B\|^{-1}\left[(w^{12}_2)^{-1}M^1_1\mu - \left(1 + (w^{12}_1 w^{12}_2)^{-1}\right)M^1_2\mu\right] + h.o.t. > 0\right\}. \end{aligned}$$

For $\mu \in H$, (4.2) has a solution $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 periodic loop.

About the bifurcation diagrams for the cases $-1 < w^{12}_2 < 0$, $0 < w^{12}_1 < 1$ and $w^{12}_1 > 1$, $w^{12}_2 > 1$, see Figures 10 and 11. □

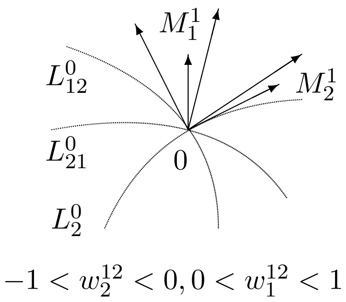


Figure 10

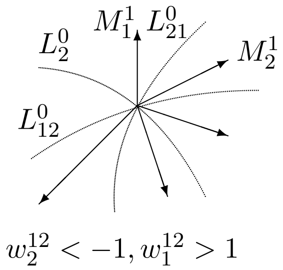


Figure 11

4.2. $0 < \alpha(\mu) \ll 1$

Theorem 4.3. *Suppose that* **(H1)**∼**(H5)** *hold. If* $0 < \alpha(\mu) \ll 1$*, rank*$\{M_1^1, M_2^1\} = 2$*, then, for* $|\mu| \ll 1$*, system* (1.2) *has at most one large 2-1 homoclinic loop, or one large 1-1 homoclinic loop, or one large 2-1 double homoclinic loop, or one large 1-1 double homoclinic loop, or a large 2-1 periodic loop in the small neighborhood of* $\Gamma = \Gamma_1 \cap \Gamma_2$*, and, these orbits do not coexist.*

Moreover, there exist surfaces L_1*,* L_2*,* $L_{1,2}$*,* $L_{21,2}^{12}$*,* $L_{12,2}^{21}$*,* $L_{21,12}^{2}$*,* $L_{21}^{12,2}$*,* $L_{12}^{21,2}$*,* $L_2^{21,12}$*, and a region* $R^{21,12,2}$*, such that:*

For $\mu \in L_1$*, system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_1*.*

For $\mu \in L_2$*, system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_2*.*

For $\mu \in L_{1,2}$*, system* (1.2) *has a unique double homoclinic loops in the small neighborhood of* $\Gamma = \Gamma_1 \cup \Gamma_2$*, that is, double homoclinic loop is preserved (Figure* 8*).*

For $\mu \in L_{21,2}^{12} \cup L_{12,2}^{21}$*, system* (1.2) *has a large 1-1 homoclinic loop (Figures* 5 and 6*).*

For $\mu \in L_{21,12}^{2}$*, system* (1.2) *has a large 2-1 double homoclinic loop (Figure* 7*).*

For $\mu \in L_{21}^{12,2} \cup L_{12}^{21,2} \cup L_2^{21,12}$*, system* (1.2) *has a large 2-1 homoclinic loop (Figures* 2, 3, and 4*).*

For $\mu \in R^{21,12,2}$*, system* (1.2) *has a large 2-1 periodic loop (Figure* 9*).*

Proof. In this case, by (4.1), we have that $\frac{\partial(G_1^1, G_2^1, \tilde{G}_2^1)}{\partial(s_{12}, s_2, s_{21})}|_{s_{12}=s_2=s_{21}=0} = \mathrm{diag}(1, 1, -1)$ is a full rank matrix, so, according to the implicit function theorem, we have that (4.1) has a unique solution

$$\begin{cases} s_{12} = (w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu + h.o.t., \\ s_2 = -(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu + h.o.t., \\ s_{21} = -(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu + h.o.t., \end{cases} \tag{4.4}$$

in the small neighborhood of $s_{12} = s_2 = s_{21} = 0$. Thus, the uniqueness and non-coexistence are proved.

(1). If (4.4) has a solution $s_{12} = s_2 = s_{21} = 0$, then (4.1) is turned to

$$\begin{cases} M_1^1 \mu + h.o.t. = 0, \\ M_2^1 \mu + h.o.t. = 0. \end{cases}$$

If $M_1^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $L_1 = \{\mu : \ M_1^1 \mu + h.o.t. = 0\}$ which has normal vector M_1^1 at $\mu = 0$, such that, for $\mu \in L_1$, system (1.2) has a unique small 1-homoclinic loop $\Gamma_1(\mu)$ in the small neighborhood of Γ_1.

If $M_2^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $L_2 = \{\mu : \ M_2^1 \mu + h.o.t. = 0\}$ which has normal vector M_2^1 at $\mu = 0$, such that, for $\mu \in L_2$, system (1.2) has a unique small 1-homoclinic loop $\Gamma_2(\mu)$ in the small neighborhood of Γ_2.

Thus, if $\mathrm{rank}\{M_1^1, M_2^1\} = 2$, then, there exists an $(l-2)$-dimensional surface $L_{1,2} = L_1 \cap L_2 = \{\mu : M_1^1\mu + h.o.t. = 0,\ M_2^1\mu + h.o.t. = 0\}$ which has normal plane $\mathrm{span}\{M_1^1, M_2^1\}$ at $\mu = 0$, such that, for $\mu \in L_{1,2}$, system (1.2) has a unique double homoclinic loop $\Gamma(\mu) = \Gamma_1(\mu) \cup \Gamma_2(\mu)$ in the small neighborhood of $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, double homoclinic loop is preserved (Figure 8).

(2). If (4.4) has a solution $s_{12} > 0$, $s_2 = s_{21} = 0$, then (4.1) is turned to

$$\begin{cases} s_{12} = -\delta^{-1}M_1^1\mu + h.o.t., \\ s_{12}^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $R_{21,2}^{12} = \{\mu : -\delta^{-1}M_1^1\mu + h.o.t. > 0, \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $L_{21,2}^{12}$ of large 1-1 homoclinic loop as follows (Figure 5).

$$\begin{cases} (-\delta^{-1}M_1^1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.5}$$

(3). If (4.4) has a solution $s_{21} > 0$, $s_2 = s_{12} = 0$, then (4.1) is turned to

$$\begin{cases} -s_{21}^{(1+\alpha(\mu))} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ s_{21} = \delta^{-1}M_2^1\mu + h.o.t.. \end{cases}$$

Thus, in the region $R_{12,2}^{21} = \{\mu : \delta^{-1}M_2^1\mu + h.o.t. > 0, \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $L_{12,2}^{21}$ of large 1-1 homoclinic loop as follows (Figure 6).

$$\begin{cases} (\delta^{-1}M_2^1\mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. = 0, \\ -\delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

(4). If (4.4) has a solution $s_2 > 0$, $s_{21} = s_{12} = 0$, then (4.1) is turned to

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_2 = -\delta^{-1}M_2^1\mu + h.o.t., \\ -s_2^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.6}$$

Thus, in the region $R_{21,12}^{2} = \{\mu : -\delta^{-1}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $L_{21,12}^{2}$ of large 2-1 double homoclinic loop as follows (Figure 7).

$$\begin{cases} -(-\delta^{-1}M_2^1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0, \\ \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{cases} \tag{4.7}$$

(5). If (4.4) has a solution $s_2 > 0$, $s_{12} > 0$, $s_{21} = 0$, then (4.1) is turned to

$$\begin{cases} s_{12} = -\delta^{-1}M_1^1\mu + h.o.t., \\ s_2 = -(w_2^{12})^{-1}s_{12}^{(1+\alpha(\mu))} - \delta^{-1}M_2^1\mu + h.o.t., \\ -s_2^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.8}$$

Thus, in the region $R_{21}^{12,2} = \{\mu : -\delta^{-1}M_1^1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $L_{21}^{12,2}$ of large 2-1 homoclinic loop as follows (Figure 2).

$$-\left[-(w_2^{12})^{-1}(-\delta^{-1}M_1^1\mu)^{(1+\alpha(\mu))} - \delta^{-1}M_2^1\mu\right]^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0. \tag{4.9}$$

(6). If (4.4) has a solution $s_2 > 0$, $s_{21} > 0$, $s_{12} = 0$, then (4.1) is turned to

$$\begin{cases} -s_{21}^{(1+\alpha(\mu))} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0, \\ s_2 = -\delta^{-1} M_2^1 \mu + h.o.t., \\ s_{21} = -(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu + h.o.t.. \end{cases} \tag{4.10}$$

Thus, in the region $R_{12}^{21,2} = \{\mu : -\delta^{-1} M_2^1 \mu + h.o.t. > 0\} \cap \{\mu : -(w_2^{12})^{-1}(-\delta^{-1} M_2^1 \mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $L_{12}^{21,2}$ of large 2-1 homoclinic loop as follows (Figure 3).

$$-\Big[-(w_2^{12})^{-1}(-\delta^{-1} M_2^1 \mu)^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu\Big]^{(1+\alpha(\mu))} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0.$$

(7). If (4.4) has a solution $s_{21} > 0$, $s_{12} > 0$, $s_2 = 0$, then (4.1) is turned to

$$\begin{cases} s_{12} = (w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu + h.o.t., \\ s_{12}^{(1+\alpha(\mu))} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. = 0, \\ s_{21} = \delta^{-1} M_2^1 \mu + h.o.t.. \end{cases} \tag{4.11}$$

Thus, in the region $R_2^{21,12} = \{\mu : \delta^{-1} M_2^1 \mu + h.o.t. > 0\} \cap \{\mu : (w_1^{12})^{-1}(\delta^{-1} M_2^1 \mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $L_2^{21,12}$ of large 2-1 homoclinic loop as follows (Figure 4).

$$\Big[(w_1^{12})^{-1}(\delta^{-1} M_2^1 \mu)^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu\Big]^{(1+\alpha(\mu))} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. = 0. \tag{4.12}$$

(8). If (4.4) has a solution $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$, then differentiating (4.4), and denoting by $(s_i)_\mu$ the gradient of $s_i(\mu)$ with respect to μ, we get

$$\begin{cases} (s_{12})_\mu = (w_1^{12})^{-1}(1+\alpha(\mu)) s_{21}^{\alpha(\mu)} (s_{21})_\mu - \delta^{-1} M_1^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu)) s_{12}^{\alpha(\mu)} (s_{12})_\mu - \delta^{-1} M_2^1 + h.o.t., \\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu)) s_2^{\alpha(\mu)} (s_2)_\mu + \delta^{-1} M_2^1 + h.o.t.. \end{cases} \tag{4.13}$$

(i). If μ is situated in the neighborhood of $L_2^{21,12}$, then, substituting (4.11) into (4.13), we get

$$\begin{cases} (s_{12})_\mu = (w_1^{12})^{-1}(1+\alpha(\mu))(\delta^{-1} M_2^1 \mu + h.o.t.)^{\alpha(\mu)} (s_{21})_\mu - \delta^{-1} M_1^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))(-\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}} (s_{12})_\mu - \delta^{-1} M_2^1 + h.o.t., \\ (s_{21})_\mu = \delta^{-1} M_2^1 + h.o.t.. \end{cases}$$

So, $(s_2)_\mu = -\delta^{-1} M_2^1 + O\left((-\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}\right) + h.o.t.$, this means that $s_2 = s_2(\mu)$ increases along the direction $-M_2^1$ in the small neighborhood of $L_2^{21,12}$.

(ii). If μ is situated in the neighborhood of $L_{21}^{12,2}$, then, substituting (4.8) into (4.13), we get

$$\begin{cases} (s_{12})_\mu = -\delta^{-1} M_1^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))(-\delta^{-1} M_1^1 \mu + h.o.t.)^{\alpha(\mu)} (s_{12})_\mu - \delta^{-1} M_2^1 + h.o.t., \\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))(\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}} (s_2)_\mu + \delta^{-1} M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{21})_\mu = \delta^{-1} M_2^1 + O\left((\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}\right) + h.o.t.$, this means that $s_{21} = s_{21}(\mu)$ increases along the direction M_2^1 in the small neighborhood of $L_{21}^{12,2}$.

(iii). If μ is situated in the neighborhood of $L_{12}^{2,21}$, then, substituting (4.10) into (4.13), we get

$$\begin{cases}(s_{12})_\mu = (w_1^{12})^{-1}(1+\alpha(\mu))(\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}(s_{21})_\mu - \delta^{-1}M_1^1 + h.o.t.,\\ (s_2)_\mu = -\delta^{-1}M_2^1 + h.o.t.,\\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))(-\delta^{-1}M_2^1\mu + h.o.t.)^{\alpha(\mu)}(s_2)_\mu + \delta^{-1}M_2^1 + h.o.t..\end{cases}$$

So, $(s_{12})_\mu = -\delta^{-1}M_1^1 + O\left((\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}\right) + h.o.t.$, this means that $s_{12} = s_{12}(\mu)$ increases along the direction $-M_1^1$ in the small neighborhood of $L_{12}^{2,21}$.

Denote by $R^{21,12,2}$ the region which is bounded by $L_{21}^{12,2}$, $L_{12}^{2,21}$, $L_2^{21,12}$, the vector M_1^1 point out of it from $L_{12}^{2,21}$, the vector M_2^1 point out of it from $L_2^{21,12}$, and the vector M_2^1 point into it from $L_{21}^{12,2}$. By the discussion of above, we get (4.4) has solution $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$ for $\mu \in R^{21,12,2}$, that is, system (1.2) has a large 2-1 periodic loop (Figure 9).

At last, by (4.5), (4.7), (4.9), (4.12) and **(H5)**, we get

$$-\delta^{-1}w_2^{12}M_2^1\mu\,|_{L_2^{21,12}} > -\delta^{-1}w_2^{12}M_2^1\mu\,|_{L_{21,2}^{12}} > -\delta^{-1}w_2^{12}M_2^1\mu\,|_{L_2} > -\delta^{-1}w_2^{12}M_2^1\mu\,|_{L_{21,12}^{2}} > -\delta^{-1}w_2^{12}M_2^1\mu\,|_{L_{21}^{12,2}},$$

$$\delta^{-1}w_1^{12}M_1^1\mu\,|_{L_{12}^{21,2}} > \delta^{-1}w_1^{12}M_1^1\mu\,|_{L_{12,2}^{21}} > \delta^{-1}w_1^{12}M_1^1\mu\,|_{L_1}.$$

Thus, we obtain the bifurcation diagram as Figure 12. □

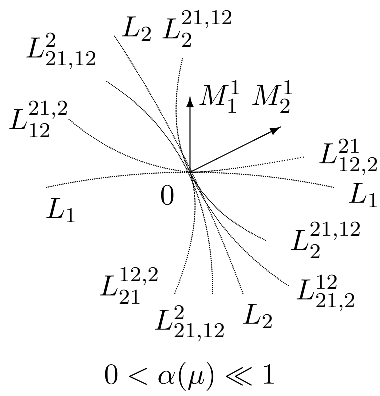


Figure 12

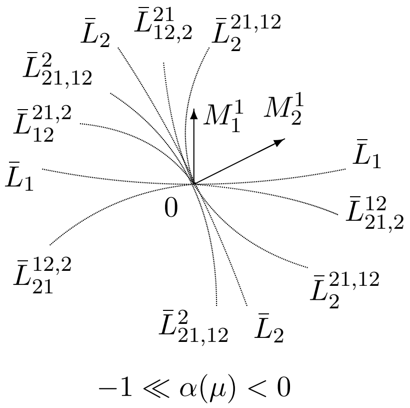


Figure 13

4.3. $-1 \ll \alpha(\mu) < 0$

Theorem 4.4. *Suppose that* **(H1)**$\sim$**(H5)** *hold. If* $-1 \ll \alpha(\mu) < 0$, *rank*$\{M_1^1, M_2^1\} = 2$, *then, for* $|\mu| \ll 1$, *system* (1.2) *has at most one large 2-1 homoclinic loop, or one large 1-1 homoclinic loop, or one large 2-1 double homoclinic loop, or one large 1-1 double homoclinic loop, or a large 2-1 periodic loop in the small neighborhood of* $\Gamma = \Gamma_1 \cap \Gamma_2$, *and, these orbits do not coexist.*

Moreover, there exist surfaces $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_{1,2}$, $\bar{L}_{21,2}^{12}$, $\bar{L}_{12,2}^{21}$, $\bar{L}_{21,12}^{2}$, $\bar{L}_{21}^{12,2}$, $\bar{L}_{12}^{21,2}$, $\bar{L}_2^{21,12}$, *and a region* $\bar{R}^{21,12,2}$, *such that:*

For $\mu \in \bar{L}_1$, *system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_1.

For $\mu \in \bar{L}_2$, *system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_2.

For $\mu \in \bar{L}_{1,2}$, *system* (1.2) *has a unique double homoclinic loops in the small neighborhood of* $\Gamma = \Gamma_1 \cup \Gamma_2$, *that is, the double homoclinic loops are preserved (Figure* 8*).*

For $\mu \in \bar{L}_{21,2}^{12} \cup \bar{L}_{12,2}^{21}$, *system* (1.2) *has a large 1-1 homoclinic loop (Figures* 5 and 6*).*

For $\mu \in \bar{L}_{21,12}^{2}$, *system* (1.2) *has a large 2-1 double homoclinic loop (Figure* 7*).*

For $\mu \in \bar{L}_{21}^{12,2} \cup \bar{L}_{12}^{21,2} \cup \bar{L}_{2}^{21,12}$, *system* (1.2) *has a large 2-1 homoclinic loop (Figures* 2, 3, and 4*).*
For $\mu \in \bar{R}^{21,12,2}$, *system* (1.2) *has a large 2-1 periodic loop (Figure* 9*).*
The bifurcation diagram for this case (see the Figure 13*).*

Proof. In this case, $1+\alpha(\mu) < 1$, by times scale transformations $s_{12} \to (s_{12})^{\frac{1}{1+\alpha(\mu)}}$, $s_{21} \to (s_{21})^{\frac{1}{1+\alpha(\mu)}}$, $s_2 \to (s_2)^{\frac{1}{1+\alpha(\mu)}}$, (4.1) becomes

$$\begin{cases} \delta[-(w_1^{12})^{-1}s_{21} + (s_{12})^{\frac{1}{1+\alpha(\mu)}}] + M_1^1\mu + h.o.t. = 0, \\ \delta[(w_2^{12})^{-1}s_{12} + (s_2)^{\frac{1}{1+\alpha(\mu)}}] + M_2^1\mu + h.o.t. = 0, \\ \delta[-(w_2^{12})^{-1}s_2 - (s_{21})^{\frac{1}{1+\alpha(\mu)}}] + M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.14}$$

Similar to that of Theorem 4.3, (4.14) has a unique solution

$$\begin{cases} s_{21} = w_1^{12}(s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ s_{12} = -w_2^{12}(s_2)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \\ s_2 = -w_2^{12}(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \end{cases} \tag{4.15}$$

in the small neighborhood of $s_{12} = s_2 = s_{21} = 0$. Thus, we get the uniqueness and non-coexistence.

(1). If (4.15) has a solution $s_{12} = s_2 = s_{21} = 0$, then (4.14) is turned to

$$\begin{cases} M_1^1\mu + h.o.t. = 0, \\ M_2^1\mu + h.o.t. = 0. \end{cases}$$

If $M_1^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $\bar{L}_1 = \{\mu : \ M_1^1\mu + h.o.t. = 0\}$ which has normal vector M_1^1 at $\mu = 0$, such that, for $\mu \in \bar{L}_1$, $0 < |\mu| \ll 1$, system (1.2) has a unique 1-homoclinic loop $\Gamma_1(\mu)$ in the small neighborhood of Γ_1.

If $M_2^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $\bar{L}_2 = \{\mu : \ M_2^1\mu + h.o.t. = 0\}$ which has normal vector M_2^1 at $\mu = 0$, such that, for $\mu \in \bar{L}_2$, $0 < |\mu| \ll 1$, system (1.2) has a unique 1-homoclinic loop $\Gamma_2(\mu)$ in the small neighborhood of Γ_2.

So, if $\operatorname{rank}\{M_1^1, M_2^1\} = 2$, then, there exists an $(l-2)$-dimensional surface $\bar{L}_{1,2} = \bar{L}_1 \cap \bar{L}_2 = \{\mu : M_1^1\mu + h.o.t. = 0, \ M_2^1\mu + h.o.t. = 0\}$ which has normal plane $\operatorname{span}\{M_1^1, M_2^1\}$ at $\mu = 0$, such that, for $\mu \in \bar{L}_{1,2}$, $0 < |\mu| \ll 1$, system (1.2) has a unique double homoclinic loop $\Gamma(\mu) = \Gamma_1(\mu) \cup \Gamma_2(\mu)$ in the small neighborhood of $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, the double homoclinic loops are preserved (Figure 8).

(2). If (4.15) has a solution $s_{12} > 0$, $s_2 = s_{21} = 0$, then (4.14) is turned to

$$\begin{cases} (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \\ \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{R}_{21,2}^{12} = \{\mu : -\delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0, \delta^{-1}M_1^1\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\bar{L}_{21,2}^{12}$ of large 1-1 homoclinic loop as follows (Figure 5).

$$\begin{cases} (-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

(3). If (4.15) has a solution $s_{21} > 0$, $s_2 = s_{12} = 0$, then (4.14) is turned to

$$\begin{cases} s_{21} = \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ -(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{R}^{21}_{12,2} = \{\mu : \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. > 0, \delta^{-1}M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{L}^{21}_{12,2}$ of large 1-1 homoclinic loop as follows (Figure 6).

$$\begin{cases} \delta^{-1}M^1_2\mu + h.o.t. = 0, \\ -(\delta^{-1}w^{12}_1 M^1_1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(4). If (4.15) has a solution $s_2 > 0$, $s_{21} = s_{12} = 0$, then (4.14) is turned to

$$\begin{cases} \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ (s_2)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0, \\ s_2 = \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.. \end{cases}$$

Thus, in the region $\bar{R}^2_{21,12} = \{\mu : \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{L}^2_{21,12}$ of large 2-1 double homoclinic loop as follows (Figure 7).

$$\begin{cases} \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ (\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(5). If (4.15) has a solution $s_2 > 0$, $s_{12} > 0$, $s_{21} = 0$, then (4.14) is turned to

$$\begin{cases} (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_{12} = -w^{12}_2(s_2)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t., \\ s_2 = \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.. \end{cases} \tag{4.16}$$

Thus, in the region $\bar{R}^{12,2}_{21} = \{\mu : \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}M^1_1\mu + h.o.t. < 0\} \cap \{\mu : -w^{12}_2(\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{L}^{12,2}_{21}$ of large 2-1 homoclinic loop as follows (Figure 2).

$$\left[-w^{12}_2(\delta^{-1}w^{12}_2 M^1_2\mu)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w^{12}_2 M^1_2\mu\right]^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0.$$

(6). If (4.15) has a solution $s_2 > 0$, $s_{21} > 0$, $s_{12} = 0$, then (4.14) is turned to

$$\begin{cases} s_{21} = \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t., \\ (s_2)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0, \\ s_2 = -w^{12}_2(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.. \end{cases} \tag{4.17}$$

Thus, in the region $\bar{R}^{21,2}_{12} = \{\mu : \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}M^1_2\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\bar{L}^{21,2}_{12}$ of large 2-1 homoclinic loop as follows (Figure 3).

$$\left[-w^{12}_2(\delta^{-1}w^{12}_1 M^1_1\mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w^{12}_2 M^1_2\mu\right]^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0.$$

(7). If (4.15) has a solution $s_{21} > 0$, $s_{12} > 0$, $s_2 = 0$, then (4.14) is turned to

$$\begin{cases} s_{21} = w^{12}_1(s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t., \\ s_{12} = -\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t., \\ -(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases} \tag{4.18}$$

Thus, in the region $\bar{R}_2^{21,12} = \{\mu : w_1^{12}(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{L}_2^{21,12}$ of large 2-1 homoclinic loop as follows (Figure 4).

$$-\left[w_1^{12}(-\delta^{-1}w_2^{12}M_2^1\mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu\right]^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0.$$

(8). If (4.15) has a solution $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$, then, differentiating (4.15), and denoting by $(s_i)_\mu$ the gradient of $s_i(\mu)$ with respect to μ, we get

$$\begin{cases} (s_{21})_\mu = (w_1^{12})\dfrac{1}{1+\alpha(\mu)}(s_{12})^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{12})_\mu + \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_2^{12})\dfrac{1}{1+\alpha(\mu)}(s_2)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_2)_\mu - \delta^{-1}w_2^{12}M_2^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})\dfrac{1}{1+\alpha(\mu)}(s_{21})^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{21})_\mu + \delta^{-1}w_2^{12}M_2^1 + h.o.t.. \end{cases} \tag{4.19}$$

(i). If μ is situated in the neighborhood of $\bar{L}_2^{21,12}$, then, substituting (4.18) into (4.19), we get

$$\begin{cases} (s_{21})_\mu = \dfrac{w_1^{12}}{1+\alpha(\mu)}(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{12})_\mu + \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_{12})_\mu = -\delta^{-1}w_2^{12}M_2^1 + h.o.t., \\ (s_2)_\mu = \dfrac{-w_2^{12}}{1+\alpha(\mu)}(\delta^{-1}M_2^1\mu + h.o.t.)^{-\alpha(\mu)}(s_{21})_\mu + \delta^{-1}w_2^{12}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_2)_\mu = \delta^{-1}w_2^{12}M_2^1 + O\left((\delta^{-1}M_2^1\mu + h.o.t.)^{-\alpha(\mu)}\right) + h.o.t.$, this means that $s_2 = s_2(\mu)$ increases along the direction $-M_2^1$ in the small neighborhood of $\bar{L}_2^{21,12}$.

(ii). If μ is situated in the neighborhood of $\bar{L}_{21}^{12,2}$, then, substituting (4.16) into (4.19), we get

$$\begin{cases} (s_{21})_\mu = \dfrac{w_1^{12}}{1+\alpha(\mu)}(-\delta^{-1}M_1^1\mu + h.o.t.)^{-\alpha(\mu)}(s_{12})_\mu + \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_{12})_\mu = \dfrac{-w_2^{12}}{1+\alpha(\mu)}(\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_2)_\mu - \delta^{-1}w_2^{12}M_2^1 + h.o.t., \\ (s_2)_\mu = \delta^{-1}w_2^{12}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{21})_\mu = \delta^{-1}w_1^{12}M_1^1 + O\left((-\delta^{-1}M_1^1\mu + h.o.t.)^{-\alpha(\mu)}\right) + h.o.t.$, this means that $s_{21} = s_{21}(\mu)$ increases along the direction M_1^1 in the small neighborhood of $\bar{L}_{21}^{12,2}$.

(iii). If μ is situated in the neighborhood of $\bar{L}_{12}^{2,21}$, then, substituting (4.17) into (4.19), we get

$$\begin{cases} (s_{21})_\mu = \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_{12})_\mu = \dfrac{-w_2^{12}}{1+\alpha(\mu)}(-\delta^{-1}M_2^1\mu + h.o.t.)^{-\alpha(\mu)}(s_2)_\mu - \delta^{-1}w_2^{12}M_2^1 + h.o.t., \\ (s_2)_\mu = \dfrac{-w_2^{12}}{1+\alpha(\mu)}(\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{21})_\mu + \delta^{-1}w_2^{12}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{12})_\mu = -\delta^{-1}w_2^{12}M_2^1 + O\left((-\delta^{-1}M_2^1\mu + h.o.t.)^{-\alpha(\mu)}\right) + h.o.t.$, this means that $s_{12} = s_{12}(\mu)$ increases along the direction M_2^1 in the small neighborhood of $\bar{L}_{12}^{2,21}$.

Denote by $\bar{R}^{21,12,2}$ the region which is bounded by $\bar{L}_2^{21,12}$, $\bar{L}_{21}^{12,2}$, $\bar{L}_{12}^{2,21}$, the vector M_2^1 point out of it from $\bar{L}_2^{21,12}$, the vector M_1^1 point into it from $\bar{L}_{21}^{12,2}$, and the vector M_2^1 point into it from $\bar{L}_{12}^{2,21}$. By the discussion of above, we get (4.15) has solution $s_{21} > 0$, $s_{12} > 0$, $s_2 > 0$ for $\mu \in \bar{R}^{21,12,2}$, that is, system (1.2) has a large 2-1 periodic loop (Figure 9). □

5. Poincaré maps and the bifurcation equations with double twisted orbits

(H6) (Double twisted conditions) $\Delta_1 = -1$, $\Delta_2 = -1$.

In this case, in the tubular neighborhood of Γ_i, due to $\Delta_1 = -1$, $\Delta_2 = -1$, we have $F_i^2(q_i^{2j+1}) = \bar{q}_i^{2j+2}$, $F_i^2(\bar{q}_i^{2j+1}) = q_i^{2j+2}$ defined by

$$\begin{cases} \bar{n}_i^{2j+2,k} = n_i^{2j+1,k} + M_i^k\mu + h.o.t., \\ n_i^{2j+2,k} = \bar{n}_i^{2j+1,k} + M_i^k\mu + h.o.t., \end{cases}$$

where, $M_i^k = \int_{-\infty}^{+\infty} (\phi_i^k(t))^* g_\mu(r_i(t), 0)dt$, $k = 1, 3, 4$, $i = 1, 2$, $j = 0, 1, 2, \cdots$ (Figure 14).

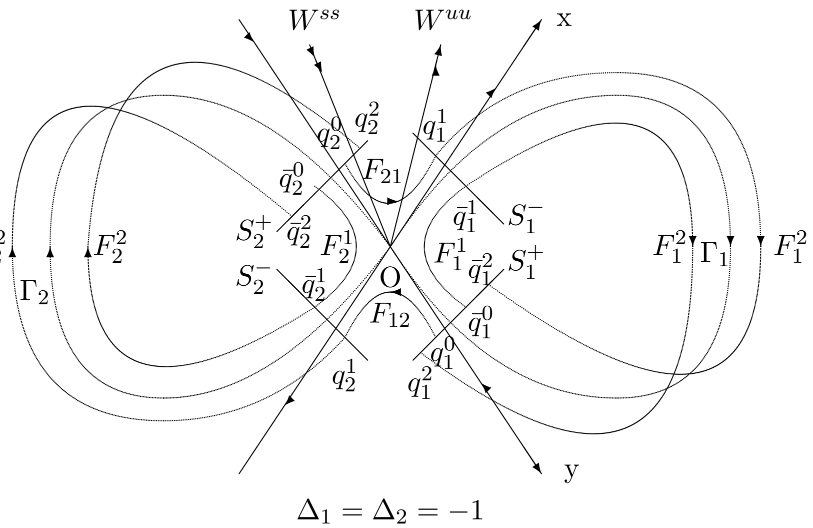


Figure 14

Moreover, the Poincaré maps F_1, $\bar{F}_1$, F_2 and $\bar{F}_2$ have the following forms.

$F_1 = F_1^2 \circ F_{21}$: $S_2^+ \mapsto S_1^+$, $F_1(q_2^0) = \bar{q}_1^2$:

$$\begin{cases} \bar{n}_1^{2,1} = n_1^{1,1} + M_1^1\mu + h.o.t. = -(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} + M_1^1\mu + h.o.t., \\ \bar{n}_1^{2,3} = n_1^{1,3} + M_1^3\mu + h.o.t. = u_1^1 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} + M_1^3\mu + h.o.t., \\ \bar{n}_1^{2,4} = n_1^{1,4} + M_1^4\mu + h.o.t. = (w_1^{44})^{-1} s_{21}^{B_2(\mu)/\lambda_1(\mu)} v_2^0 + M_1^4\mu + h.o.t.. \end{cases}$$

$\bar{F}_1 = F_1^2 \circ F_1^1$: $S_1^+ \mapsto S_1^+$, $\bar{F}_1(\bar{q}_1^0) = q_1^2$:

$$\begin{cases} n_1^{2,1} = \bar{n}_1^{1,1} + M_1^1\mu + h.o.t. = (w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} + M_1^1\mu + h.o.t., \\ n_1^{2,3} = \bar{n}_1^{1,3} + M_1^3\mu + h.o.t. = \bar{u}_1^1 - \delta_{1u} + b_1(w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} + M_1^3\mu + h.o.t., \\ n_1^{2,4} = \bar{n}_1^{1,4} + M_1^4\mu + h.o.t. = (w_1^{44})^{-1} s_1^{B_2(\mu)/\lambda_1(\mu)} \bar{v}_1^0 + M_1^4\mu + h.o.t.. \end{cases}$$

$F_2 = F_2^2 \circ F_{12}$: $S_1^+ \mapsto S_2^+$, $F_2(q_1^0) = \bar{q}_2^2$:

$$\begin{cases} \bar{n}_2^{2,1} = n_2^{1,1} + M_2^1\mu + h.o.t. = (w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t., \\ \bar{n}_2^{2,3} = n_2^{1,3} + M_2^3\mu + h.o.t. = u_2^1 - \delta_{2u} + b_2(w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} + M_2^3\mu + h.o.t., \\ \bar{n}_2^{2,4} = n_2^{1,4} + M_2^4\mu + h.o.t. = (w_2^{44})^{-1} s_{12}^{B_2(\mu)/\lambda_1(\mu)} v_1^0 + M_2^4\mu + h.o.t.. \end{cases}$$

$\bar{F}_2 = F_2^2 \circ F_2^1$: $S_2^+ \mapsto S_2^+$, $\bar{F}_2(\bar{q}_2^0) = q_2^2$:

$$\begin{cases} n_2^{2,1} = \bar{n}_2^{1,1} + M_2^1\mu + h.o.t. = -(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t., \\ n_2^{2,3} = \bar{n}_2^{1,3} + M_2^3\mu + h.o.t. = \bar{u}_2^1 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} + M_2^3\mu + h.o.t., \\ n_2^{2,4} = \bar{n}_2^{1,4} + M_2^4\mu + h.o.t. = (w_2^{44})^{-1} s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^0 + M_2^4\mu + h.o.t.. \end{cases}$$

Let $\bar{q}_1^2 = \bar{q}_1^0$, $q_1^2 = q_1^0$, $\bar{q}_2^2 = \bar{q}_2^0$, $q_2^2 = q_2^0$, we get the successor functions as follows.
$G_1(s_1, s_{21}, u_1^1, \bar{u}_1^1, \bar{v}_1^0, v_2^0) = (G_1^1, G_1^3, G_1^4) = (F_1(q_2^0) - \bar{q}_1^0)$ is given by

$$\begin{cases} G_1^1 = \delta[-(w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} - s_1] + M_1^1\mu + h.o.t., \\ G_1^3 = u_1^1 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} - (w_1^{33})^{-1} s_1^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_1^1 + M_1^3\mu + h.o.t., \\ G_1^4 = -\bar{v}_1^0 + \delta_{1v} + w_1^{14}\delta s_1 + (w_1^{44})^{-1} s_{21}^{B_2(\mu)/\lambda_1(\mu)} v_2^0 + M_1^4\mu + h.o.t.. \end{cases} \tag{5.1}$$

$\tilde{G}_1(s_1, s_{12}, \bar{u}_1^1, u_2^1, v_1^0, \bar{v}_1^0) = (\tilde{G}_1^1, \tilde{G}_1^3, \tilde{G}_1^4) = (\bar{F}_1(\bar{q}_1^0) - q_1^0)$ is given by

$$\begin{cases} \tilde{G}_1^1 = \delta[(w_1^{12})^{-1} s_1^{(1+\alpha(\mu))} + s_{12}] + M_1^1\mu + h.o.t., \\ \tilde{G}_1^3 = \bar{u}_1^1 - \delta_{1u} + b_1(w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} - (w_1^{33})^{-1} s_{12}^{B_1(\mu)/\lambda_1(\mu)} u_2^1 + M_1^3\mu + h.o.t., \\ \tilde{G}_1^4 = -v_1^0 + \delta_{1v} - w_1^{14}\delta s_{12} + (w_1^{44})^{-1} s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^0 + M_1^4\mu + h.o.t.. \end{cases} \tag{5.2}$$

$G_2(s_{12}, s_2, u_2^1, \bar{u}_2^1, v_1^0, \bar{v}_2^0) = (G_2^1, G_2^3, G_2^4) = (F_2(q_1^0) - \bar{q}_2^0)$ is given by

$$\begin{cases} G_2^1 = \delta[(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} + s_2] + M_2^1\mu + h.o.t., \\ G_2^3 = u_2^1 - \delta_{2u} + b_2(w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} - (w_2^{33})^{-1} s_2^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_2^1 + M_2^3\mu + h.o.t., \\ G_2^4 = -\bar{v}_2^0 + \delta_{2v} - w_2^{14}\delta s_2 + (w_2^{44})^{-1} s_{12}^{B_2(\mu)/\lambda_1(\mu)} v_1^0 + M_2^4\mu + h.o.t.. \end{cases} \tag{5.3}$$

$\tilde{G}_2(s_{21}, s_2, u_1^1, \bar{u}_2^1, v_2^0, \bar{v}_2^0) = (\tilde{G}_2^1, \tilde{G}_2^3, \tilde{G}_2^4) = (\bar{F}_2(\bar{q}_2^0) - q_2^0)$ is given by

$$\begin{cases} \tilde{G}_2^1 = \delta[-(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} - s_{21}] + M_2^1\mu + h.o.t., \\ \tilde{G}_2^3 = \bar{u}_2^1 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} - (w_2^{33})^{-1} s_{21}^{B_1(\mu)/\lambda_1(\mu)} u_1^1 + M_2^3\mu + h.o.t., \\ \tilde{G}_2^4 = -v_2^0 + \delta_{2v} + w_2^{14}\delta s_{21} + (w_2^{44})^{-1} s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^0 + M_2^4\mu + h.o.t.. \end{cases} \tag{5.4}$$

Thus, we get the bifurcation equations as follows.

$$G(s_{21}, s_1, s_{12}, s_2, u_1^1, \bar{u}_1^1, u_2^1, \bar{u}_2^1, v_1^0, \bar{v}_1^0, v_2^0, \bar{v}_2^0) = (G_1, \tilde{G}_1, G_2, \tilde{G}_2) = 0. \tag{5.5}$$

6. Bifurcations with the double twisted orbits

Now, we discuss the solutions $Q(s_{21}, s_1, s_{12}, s_2, u_1^1, \bar{u}_1^1, u_2^1, \bar{u}_2^1, v_1^0, \bar{v}_1^0, v_2^0, \bar{v}_2^0)$ of the bifurcation equation (5.5) which satisfy $s_{12} \geq 0$, $s_{21} \geq 0$, $s_2 \geq 0$, $s_1 \geq 0$.

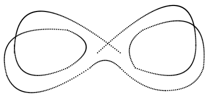
By (5.1)$\sim$(5.4), for $0 \leq s_{12}, s_{21}, s_2, s_1, |\mu| \ll 1$, the equation $(G_1^3, G_1^4, \tilde{G}_1^3, \tilde{G}_1^4, G_2^3, G_2^4, \tilde{G}_2^3, \tilde{G}_2^4) = 0$ has always a unique solution $u_1^1 = u_1^1(s_{21}, s_{12}, s_2, s_1, \mu)$, $\bar{u}_1^1 = \bar{u}_1^1(s_{21}, s_{12}, s_2, s_1, \mu)$, $u_2^1 = u_2^1(s_{21}, s_{12}, s_2, s_1, \mu)$, $\bar{u}_2^1 = \bar{u}_2^1(s_{21}, s_{12}, s_2, s_1, \mu)$, $v_1^0 = v_1^0(s_{21}, s_{12}, s_2, s_1, \mu)$, $\bar{v}_1^0 = \bar{v}_1^0(s_{21}, s_{12}, s_2, s_1, \mu)$, $v_2^0 = v_2^0(s_{21}, s_{12}, s_2, s_1, \mu)$, $\bar{v}_2^0 = \bar{v}_2^0(s_{21}, s_{12}, s_2, s_1, \mu)$. Substituting it into $(G_1^1, \tilde{G}_1^1, G_2^1, \tilde{G}_2^1) = 0$, we have

$$\begin{cases} \delta[-(w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} - s_1] + M_1^1\mu + h.o.t. = 0, \\ \delta[(w_1^{12})^{-1} s_1^{(1+\alpha(\mu))} + s_{12}] + M_1^1\mu + h.o.t. = 0, \\ \delta[(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} + s_2] + M_2^1\mu + h.o.t. = 0, \\ \delta[-(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} - s_{21}] + M_2^1\mu + h.o.t. = 0. \end{cases} \tag{6.1}$$

Thus, for system (1.2), there is a one to one correspondence between the large homoclinic loops and large periodic orbits bifurcated from $\Gamma = \Gamma_1 \cup \Gamma_2$ and the solutions of the bifurcation equation (6.1) satisfying $s_{12} \geq 0$, $s_{12} \geq 0$, $s_2 \geq 0$, $s_1 \geq 0$, respectively.

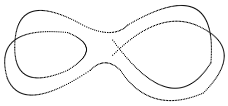
6.1. $\alpha(\mu) = 0$

Theorem 6.1. *Suppose that* **(H1)**$\sim$**(H4)** *and* **(H6)** *hold. If* $\alpha(\mu) = 0$, $(w_1^{12}w_2^{12})^{-2} \neq 1$, *rank*$\{M_1^1, M_2^1\} = 2$, *then, for* $|\mu| \ll 1$, *system* (1.2) *has at most one large 2-2 homoclinic loop, or one large 2-1 homoclinic loop, or one large 1-2 homoclinic loop, or one large 2-2 double homoclinic loops, or one large 1-1 homoclinic loop, or one large 1-2 double homoclinic loops, or one large 2-1 double homoclinic loops, or a large 2-2 periodic loop, or one large 1-1 double homoclinic loops in the small neighborhood of* $\Gamma = \Gamma_1 \cap \Gamma_2$. *Moreover, these orbits do not coexist.*



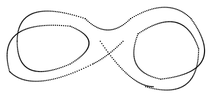
$s_{21} = 0, s_1 > 0$
$s_{12} > 0, s_2 > 0$

Figure 15



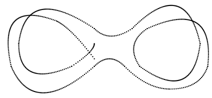
$s_{21} > 0, s_1 = 0$
$s_{12} > 0, s_2 > 0$

Figure 16



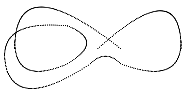
$s_{21} > 0, s_1 > 0$
$s_{12} = 0, s_2 > 0$

Figure 17



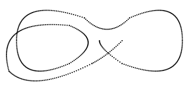
$s_{21} > 0, s_1 > 0$
$s_{12} > 0, s_2 = 0$

Figure 18



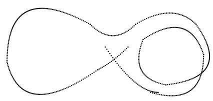
$s_{21} = 0, s_1 = 0$
$s_{12} > 0, s_2 > 0$

Figure 19



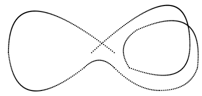
$s_{21} > 0, s_1 = 0$
$s_{12} = 0, s_2 > 0$

Figure 20



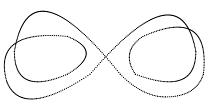
$s_{21} > 0, s_1 > 0$
$s_{12} = 0, s_2 = 0$

Figure 21



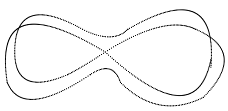
$s_{21} = 0, s_1 > 0$
$s_{12} > 0, s_2 = 0$

Figure 22



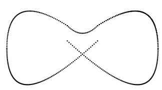
$s_{21} = 0, s_1 > 0$
$s_{12} = 0, s_2 > 0$

Figure 23



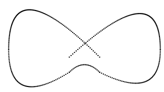
$s_{21} > 0, s_1 = 0$
$s_{21} > 0, s_2 = 0$
Figure 24

Figure 24



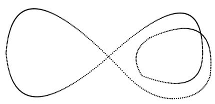
$s_{21} > 0, s_1 = 0$
$s_{12} = 0, s_2 = 0$

Figure 25



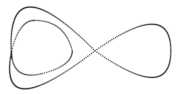
$s_{21} = 0, s_1 = 0$
$s_{12} > 0, s_2 = 0$

Figure 26



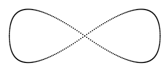
$s_{21}=0, s_1>0$
$s_{12}=0, s_2=0$

Figure 27



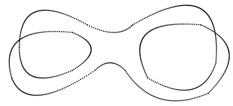
$s_{21}=0, s_1=0$
$s_{12}=0, s_2>0$

Figure 28



$s_{21}=0, s_1=0$
$s_{12}=0, s_2=0$

Figure 29



$s_{21}>0, s_1>0$
$s_{12}>0, s_2>0$

Figure 30

Proof. In this case, (6.1) becomes

$$\begin{cases} \delta[-(w_1^{12})^{-1}s_{21}-s_1]+M_1^1\mu+h.o.t.=0,\\ \delta[(w_1^{12})^{-1}s_1+s_{12}]+M_1^1\mu+h.o.t.=0,\\ \delta[(w_2^{12})^{-1}s_{12}+s_2]+M_2^1\mu+h.o.t.=0,\\ \delta[-(w_2^{12})^{-1}s_2-s_{21}]+M_2^1\mu+h.o.t.=0. \end{cases} \tag{6.2}$$

That is

$$\begin{pmatrix} (w_1^{12})^{-1} & 1 & 0 & 0\\ 0 & -(w_1^{12})^{-1} & -1 & 0\\ 0 & 0 & -(w_2^{12})^{-1} & -1\\ 1 & 0 & 0 & (w_2^{12})^{-1} \end{pmatrix}\begin{pmatrix} s_{21}\\ s_1\\ s_{12}\\ s_2\end{pmatrix}=\delta^{-1}\begin{pmatrix} M_1^1\mu\\ M_1^1\mu\\ M_2^1\mu\\ M_2^1\mu\end{pmatrix}+h.o.t..$$

Denote $B=\begin{pmatrix} (w_1^{12})^{-1} & 1 & 0 & 0\\ 0 & -(w_1^{12})^{-1} & -1 & 0\\ 0 & 0 & -(w_2^{12})^{-1} & -1\\ 1 & 0 & 0 & (w_2^{12})^{-1} \end{pmatrix}$, if $\|B\|=(w_1^{12}w_2^{12})^{-2}-1\neq 0$, then, (6.2) has a unique solution $0\le s_{21}(\mu),\ s_1(\mu),\ s_{12}(\mu),\ s_2(\mu)\ll 1$,

$$\begin{pmatrix} s_{21}\\ s_1\\ s_{12}\\ s_2\end{pmatrix}=\delta^{-1}B^{-1}\begin{pmatrix} M_1^1\mu\\ M_1^1\mu\\ M_2^1\mu\\ M_2^1\mu\end{pmatrix}+h.o.t.=\delta^{-1}\|B\|^{-1}B^*\begin{pmatrix} M_1^1\mu\\ M_1^1\mu\\ M_2^1\mu\\ M_2^1\mu\end{pmatrix}+h.o.t.,$$

satisfying $s_{21}(0)=s_1(0)=s_{12}(0)=s_2(0)=0$, where,

$$B^*=\begin{pmatrix} (w_1^{12})^{-1}(w_2^{12})^{-2} & (w_2^{12})^{-2} & -(w_2^{12})^{-1} & -1\\ -1 & -(w_1^{12})^{-1}(w_2^{12})^{-2} & (w_1^{12})^{-1}(w_2^{12})^{-1} & (w_1^{12})^{-1}\\ (w_1^{12})^{-1} & 1 & -(w_1^{12})^{-2}(w_2^{12})^{-1} & -(w_1^{12})^{-2}\\ -(w_1^{12})^{-1}(w_2^{12})^{-1} & -(w_2^{12})^{-1} & 1 & (w_1^{12})^{-2}(w_2^{12})^{-1} \end{pmatrix}.$$

Thus, we get the uniqueness and non-coexistence.

If $s_{21}=0$, $s_1>0$, $s_{12}>0$, $s_2>0$, or $s_{21}>0$, $s_1=0$, $s_{12}>0$, $s_2>0$, or $s_{21}>0$, $s_1>0$, $s_{12}=0$, $s_2>0$, or $s_{21}>0$, $s_1>0$, $s_{12}>0$, $s_2=0$, then, system (1.2) has a large 2-2 homoclinic loop (Figures 15, 16, 17, and 18).

If $s_{21}=0$, $s_1=0$, $s_{12}>0$, $s_2>0$, or $s_{21}>0$, $s_1=0$, $s_{12}=0$, $s_2>0$, then, system (1.2) has a large 2-1 homoclinic loop (Figures 19 and 20).

If $s_{21}>0$, $s_1>0$, $s_{12}=0$, $s_2=0$, or $s_{21}=0$, $s_1>0$, $s_{12}>0$, $s_2=0$, then, system (1.2) has a large 1-2 homoclinic loop (Figures 21 and 22).

If $s_{21}=0$, $s_1>0$, $s_{12}=0$, $s_2>0$, or $s_{21}>0$, $s_1=0$, $s_{12}>0$, $s_2=0$, then, system (1.2) has a large 2-2 double homoclinic loop (Figures 23 and 24).

If $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$, or $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, then, system (1.2) has a large 1-1 homoclinic loop (Figures 25 and 26).

If $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, then, system (1.2) has a large 1-2 double homoclinic loop (Figure 27).

If $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, then, system (1.2) has a large 2-1 double homoclinic loop (Figure 28).

If $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$, then, system (1.2) has a 1-1 homoclinic loop, that is, the double homoclinic loop is preserved (Figure 29).

If $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, then, system (1.2) has a large 2-2 periodic loop (Figure 30).

Thus, the theorem is established. □

Denote

$$\begin{cases} s_{21} = \delta^{-1}\mathcal{M}_{21}\mu + h.o.t., \\ s_1 = \delta^{-1}\mathcal{M}_1\mu + h.o.t., \\ s_{12} = \delta^{-1}\mathcal{M}_{12}\mu + h.o.t., \\ s_2 = \delta^{-1}\mathcal{M}_2\mu + h.o.t., \end{cases} \tag{6.3}$$

where

$$\begin{cases} \mathcal{M}_{21} = \|B\|^{-1}\left[((w_1^{12})^{-1}(w_2^{12})^{-2} + (w_2^{12})^{-2})M_1^1 - ((w_2^{12})^{-1}+1)M_2^1\right], \\ \mathcal{M}_1 = \|B\|^{-1}\left[\left(-1-(w_1^{12})^{-1}(w_2^{12})^{-2}\right)M_1^1 + \left((w_1^{12})^{-1}(w_2^{12})^{-1} + (w_1^{12})^{-1}\right)M_2^1\right], \\ \mathcal{M}_{12} = \|B\|^{-1}\left[\left((w_1^{12})^{-1}+1\right)M_1^1 - \left((w_1^{12})^{-2}(w_2^{12})^{-1} + (w_1^{12})^{-2}\right)M_2^1\right], \\ \mathcal{M}_2 = \|B\|^{-1}\left[\left(-(w_1^{12})^{-1}(w_2^{12})^{-1} - (w_2^{12})^{-1}\right)M_1^1 + \left(1+(w_1^{12})^{-2}(w_2^{12})^{-1}\right)M_2^1\right]. \end{cases}$$

Due to $\|B^*\| \neq 0$, if $\mathrm{rank}\{M_1^1, M_2^1\} = 2$, then, $\mathrm{rank}\{\mathcal{M}_i, \mathcal{M}_j\} = 2$ for $i \neq j$, $i = 21, 1, 12, 2$, $j = 21, 1, 12, 2$. Notice that $\mathcal{M}_i \in \mathrm{span}\{M_1^1, M_2^1\}$, $i = 21, 1, 12, 2$, so, $\mathrm{rank}\{\mathcal{M}_{21}, \mathcal{M}_1, \mathcal{M}_{12}, \mathcal{M}_2\} = 2$.

Thus, we get the following theorem.

Theorem 6.2. *Suppose that* **(H1)**∼**(H4)** *and* **(H6)** *hold. If* $\alpha(\mu) = 0$, $(w_1^{12}w_2^{12})^{-2} \neq 1$, *rank*$\{M_1^1, M_2^1\} = 2$, *then , for* $|\mu| \ll 1$, *there exist surfaces* $\mathcal{L}_{21}^0$, $\mathcal{L}_1^0$, $\mathcal{L}_{12}^0$, $\mathcal{L}_2^0$, $\mathcal{L}_{21,1}^0$, $\mathcal{L}_{1,12}^0$, $\mathcal{L}_{12,2}^0$, $\mathcal{L}_{2,21}^0$, $\mathcal{L}_{21,12}^0$, $\mathcal{L}_{1,2}^0$, $\mathcal{L}_{1,12,2}^0$, $\mathcal{L}_{21,1,2}^0$, $\mathcal{L}_{21,12,2}^0$, $\mathcal{L}_{21,1,12}^0$, $\mathcal{L}_{21,1,12,2}^0$, *and a region* $\mathcal{H}$, *such that:*

(1) *For* $\mu \in \mathcal{L}_{21}^0$, (6.2) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 15*).*

(2) *For* $\mu \in \mathcal{L}_1^0$, (6.2) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 16*).*

(3) *For* $\mu \in \mathcal{L}_{12}^0$, (6.2) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 17*).*

(4) *For* $\mu \in \mathcal{L}_2^0$, (6.2) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 18*).*

(5) *For* $\mu \in \mathcal{L}_{21,1}^0$, (6.2) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 19*).*

(6) *For* $\mu \in \mathcal{L}_{1,12}^0$, (6.2) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 20*).*

(7) *For* $\mu \in \mathcal{L}_{12,2}^0$, (6.2) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-2 homoclinic loop (Figure* 21*).*

(8) *For* $\mu \in \mathcal{L}_{2,21}^0$, (6.2) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-2 homoclinic loop (Figure* 22*).*

(9) *For* $\mu \in \mathcal{L}^0_{21,12}$*,* (6.2) *has a solution* $s_{21} = 0$*,* $s_1 > 0$*,* $s_{12} = 0$*,* $s_2 > 0$*, that is, system* (1.2) *has a large 2-2 double homoclinic loop (Figure* 23*).*

(10) *For* $\mu \in \mathcal{L}^0_{1,2}$*,* (6.2) *has a solution* $s_{21} > 0$*,* $s_1 = 0$*,* $s_{12} > 0$*,* $s_2 = 0$*, that is, system* (1.2) *has a large 2-2 double homoclinic loop (Figure* 24*).*

(11) *For* $\mu \in \mathcal{L}^0_{1,12,2}$*,* (6.2) *has a solution* $s_{21} > 0$*,* $s_1 = 0$*,* $s_{12} = 0$*,* $s_2 = 0$*, that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 25*).*

(12) *For* $\mu \in \mathcal{L}^0_{21,1,2}$*,* (6.2) *has a solution* $s_{21} = 0$*,* $s_1 = 0$*,* $s_{12} > 0$*,* $s_2 = 0$*, that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 26*).*

(13) *For* $\mu \in \mathcal{L}^0_{21,12,2}$*,* (6.2) *has a solution* $s_{21} = 0$*,* $s_1 > 0$*,* $s_{12} = 0$*,* $s_2 = 0$*, that is, system* (1.2) *has a large 1-2 double homoclinic loop (Figure* 27*).*

(14) *For* $\mu \in \mathcal{L}^0_{21,1,12}$*,* (6.2) *has a solution* $s_{21} = 0$*,* $s_1 = 0$*,* $s_{12} = 0$*,* $s_2 > 0$*, that is, system* (1.2) *has a large 2-1 double homoclinic loop (Figure* 28*).*

(15) *For* $\mu \in \mathcal{L}^0_{21,1,12,2}$*,* $0 < |\mu| \ll 1$*,* (6.2) *has solution* $s_{21} = 0$*,* $s_1 = 0$*,* $s_{12} = 0$*,* $s_2 = 0$*, that is, in the small neighborhood of* Γ*, system* (1.2) *has a 1-1 double homoclinic loop* $\Gamma^0 = \Gamma^0_1(\mu) \cup \Gamma^0_2(\mu)$ *(Figure* 29*).*

(16) *For* $\mu \in \mathcal{H}$*,* (6.2) *has a solution* $s_{21} > 0$*,* $s_1 > 0$*,* $s_{12} > 0$*,* $s_2 > 0$*, that is, system* (1.2) *has a large 2-2 periodic loop (Figure* 30*).*

Proof. By (6.3), we get

(1). In the region $\{\mu : \delta^{-1}\mathcal{M}_1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. > 0\}$, there is an $(l-1)$-dimensional surface

$$\mathcal{L}^0_{21} = \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\},$$

which has normal vector $\mathcal{M}_{21}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{21}$, (6.2) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-2 homoclinic loop.

(2). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. > 0\}$, there is an $(l-1)$-dimensional surface

$$\mathcal{L}^0_{1} = \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. = 0\},$$

which has normal vector $\mathcal{M}_{1}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{1}$, (6.2) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-2 homoclinic loop.

(3). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. > 0\}$, there is an $(l-1)$-dimensional surface

$$\mathcal{L}^0_{12} = \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\},$$

which has normal vector $\mathcal{M}_{12}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{12}$, (6.2) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, that is, system (1.2) has a large 2-2 homoclinic loop.

(4). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\}$, there is an $(l-1)$-dimensional surface

$$\mathcal{L}^0_{2} = \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. = 0\},$$

which has normal vector $\mathcal{M}_{2}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{2}$, (6.2) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, that is, system (1.2) has a large 2-2 homoclinic loop.

(5). In the region $\{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{21,1} = \mathcal{L}^0_{21} \cap \mathcal{L}^0_{1} = \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{21}, \mathcal{M}_1\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{21,1}$, (6.2) has a solution $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 homoclinic loop.

(6). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{1,12} = \mathcal{L}^0_{1} \cap \mathcal{L}^0_{12} = \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{1}, \mathcal{M}_{12}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{1,12}$, (6.2) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 homoclinic loop.

(7). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{12,2} = \mathcal{L}^0_{12} \cap \mathcal{L}^0_{2} = \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{12}, \mathcal{M}_{2}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{12,2}$, (6.2) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, that is, system (1.2) has a large 1-2 homoclinic loop.

(8). In the region $\{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{2,21} = \mathcal{L}^0_{2} \cap \mathcal{L}^0_{21} = \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{2}, \mathcal{M}_{21}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{2,21}$, (6.2) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, that is, system (1.2) has a large 1-2 homoclinic loop.

(9). In the region $\{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{21,12} = \mathcal{L}^0_{21} \cap \mathcal{L}^0_{12} = \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{21}, \mathcal{M}_{12}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{21,12}$, (6.2) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, that is, system (1.2) has a large 2-2 double homoclinic loop.

(10). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{1,2} = \mathcal{L}^0_{1} \cap \mathcal{L}^0_{2} = \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{1}, \mathcal{M}_{2}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{1,2}$, (6.2) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, that is, system (1.2) has a large 2-2 double homoclinic loop.

(11). In the region $\{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\begin{aligned}\mathcal{L}^0_{1,12,2} = \mathcal{L}^0_{1} \cap \mathcal{L}^0_{12} \cap \mathcal{L}^0_{2} =& \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\} \\ & \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. = 0\},\end{aligned}$$

which has normal plane span$\{\mathcal{M}_{1}, \mathcal{M}_{12}, \mathcal{M}_{2}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{1,12,2}$, (6.2) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$, that is, system (1.2) has a large 1-1 homoclinic loop.

(12). In the region $\{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\begin{aligned}\mathcal{L}^0_{21,1,2} = \mathcal{L}^0_{21} \cap \mathcal{L}^0_{1} \cap \mathcal{L}^0_{2} =& \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{1}\mu + h.o.t. = 0\} \\ & \cap \{\mu : \delta^{-1}\mathcal{M}_{2}\mu + h.o.t. = 0\},\end{aligned}$$

which has normal plane span$\{\mathcal{M}_{21}, \mathcal{M}_{1}, \mathcal{M}_{2}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{21,1,2}$, (6.2) has a solution $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, that is, system (1.2) has a large 1-1 homoclinic loop.

(13). In the region $\{\mu : \delta^{-1}\mathcal{M}_1\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{21,12,2} = \mathcal{L}^0_{21} \cap \mathcal{L}^0_{12} \cap \mathcal{L}^0_2 = \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{21}, \mathcal{M}_{12}, \mathcal{M}_2\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{21,12,2}$, (6.2) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, that is, system (1.2) has a large 1-2 double homoclinic loop.

(14). In the region $\{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. > 0\}$, there is an $(l-2)$-dimensional surface

$$\mathcal{L}^0_{21,1,12} = \mathcal{L}^0_{21} \cap \mathcal{L}^0_1 \cap \mathcal{L}^0_{12} = \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_1\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\},$$

which has normal plane span$\{\mathcal{M}_{21}, \mathcal{M}_1, \mathcal{M}_{12}\}$ at $\mu = 0$, such that for $\mu \in \mathcal{L}^0_{21,1,12}$, (6.2) has a solution $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, that is, system (1.2) has a large 2-1 double homoclinic loop.

(15). There is a surface

$$\begin{aligned}\mathcal{L}^0_{21,1,12,2} &= \mathcal{L}^0_{21} \cap \mathcal{L}^0_1 \cap \mathcal{L}^0_{12} \cap \mathcal{L}^0_2 \\ &= \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_1\mu + h.o.t. = 0\} \\ &\quad \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. = 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. = 0\},\end{aligned}$$

which has normal vector $\mathcal{M} = \text{span}\{\mathcal{M}_{21}, \mathcal{M}_1, \mathcal{M}_{12}, \mathcal{M}_2\}$ at $\mu = 0$, such that, for $\mu \in \mathcal{L}^0_{21,1,12,2}$, $0 < |\mu| \ll 1$, (6.2) has solution $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$, that is, in the small neighborhood of Γ, system (1.2) has a 1-1 double homoclinic loop $\Gamma^0 = \Gamma^0_1(\mu) \cup \Gamma^0_2(\mu)$.

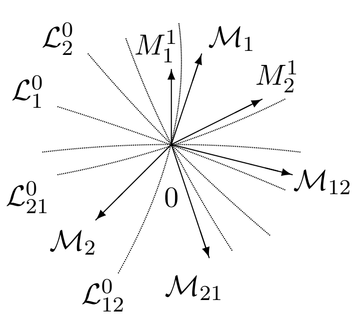
Notice that dim$\mathcal{M}$=2, $\|B^*\| \neq 0$, so, indeed, the surface $\mathcal{L}^0_{21,1,12,2}$ is a $(l-2)$-dimensional surface which has normal plane $\mathcal{M}$ at $\mu = 0$. In fact, $\mathcal{L}^0_{21,1,12,2}$=$\{\mu : M^1_1\mu + h.o.t. = 0\} \cap \{\mu : M^1_2\mu + h.o.t. = 0\}$.

(16). Denote

$$\mathcal{H} = \{\mu : \delta^{-1}\mathcal{M}_{21}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_{12}\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}\mathcal{M}_2\mu + h.o.t. > 0\}.$$

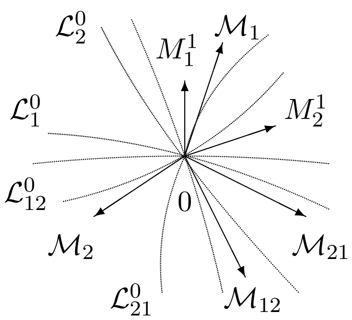
For $\mu \in \mathcal{H}$, (6.2) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, that is, system (1.2) has a large 2-2 periodic loop.

About the bifurcation diagrams for the cases $-1 < w^{12}_2 < 0$, $0 < w^{12}_1 < 1$ and $w^{12}_1 > 1$, $w^{12}_2 > 1$, see Figures 31 and 32. □



$-1 < w^{12}_1 < 0$, $-1 < w^{12}_2 < 0$

Figure 31



$w^{12}_1 < -1$, $w^{12}_2 < -1$

Figure 32

6.2. $0 < \alpha(\mu) \ll 1$

Theorem 6.3. *Suppose that* **(H1)**$\sim$**(H4)** *and* **(H6)** *hold. If* $0 < \alpha(\mu) \ll 1$*, rank*$\{M_1^1, M_2^1\} = 2$*, then, for* $|\mu| \ll 1$*, system* (1.2) *has at most one large 2-2 homoclinic loop, or one large 2-1 homoclinic loop, or one large 1-2 homoclinic loop, or one large 2-2 double homoclinic loops, or one large 1-1 homoclinic loop, or one large 1-2 double homoclinic loops, or one large 2-1 double homoclinic loops, or a large 2-2 periodic loop, or one large 1-1 double homoclinic loops in the small neighborhood of* $\Gamma = \Gamma_1 \cap \Gamma_2$*, and, these orbits do not coexist.*

Moreover, there exist surfaces $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_{21,1,12,2}$, $\mathcal{L}^{21}_{1,12,2}$, $\mathcal{L}^{12}_{21,1,2}$, $\mathcal{L}^{1}_{21,12,2}$, $\mathcal{L}^{2}_{21,1,12}$, $\mathcal{L}^{12,2}_{21,1}$, $\mathcal{L}^{21,2}_{1,12}$, $\mathcal{L}^{21,1}_{12,2}$, $\mathcal{L}^{1,12}_{2,21}$, $\mathcal{L}^{1,2}_{21,12}$, $\mathcal{L}^{21,12}_{1,2}$, $\mathcal{L}^{1,12,2}_{21}$, $\mathcal{L}^{21,12,2}_{1}$, $\mathcal{L}^{21,1,2}_{12}$, $\mathcal{L}^{21,1,12}_{2}$*, and a region* $\mathcal{R}$*, such that*

(1) *For* $\mu \in \mathcal{L}_1$*, system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_1*.*

For $\mu \in \mathcal{L}_2$*, system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_2*.*

For $\mu \in \mathcal{L}_{21,1,12,2}$, (6.1) *has solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$*, that is, in the small neighborhood of* $\Gamma = \Gamma_1 \cup \Gamma_2$*, system* (1.2) *has a unique 1-1 double homoclinic loops* $\Gamma^0 = \Gamma_1^0(\mu) \cup \Gamma_2^0(\mu)$*, that is, double homoclinic loops are preserved (Figure* 29*).*

(2) *For* $\mu \in \mathcal{L}^{21}_{1,12,2}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$*, that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 25*).*

(3) *For* $\mu \in \mathcal{L}^{12}_{21,1,2}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$*, that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 26*).*

(4) *For* $\mu \in \mathcal{L}^{1}_{21,12,2}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$*, that is, system* (1.2) *has a large 1-2 double homoclinic loop (Figure* 27*).*

(5) *For* $\mu \in \mathcal{L}^{2}_{21,1,12}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-1 double homoclinic loop (Figure* 28*).*

(6) *For* $\mu \in \mathcal{L}^{12,2}_{21,1}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 19*).*

(7) *For* $\mu \in \mathcal{L}^{21,2}_{1,12}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 20*).*

(8) *For* $\mu \in \mathcal{L}^{21,1}_{12,2}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$*, that is, system* (1.2) *has a large 1-2 homoclinic loop (Figure* 21*).*

(9) *For* $\mu \in \mathcal{L}^{1,12}_{2,21}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$*, that is, system* (1.2) *has a large 1-2 homoclinic loop (Figure* 22*).*

(10) *For* $\mu \in \mathcal{L}^{1,2}_{21,12}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-2 double homoclinic loop (Figure* 23*).*

(11) *For* $\mu \in \mathcal{L}^{21,12}_{1,2}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$*, that is, system* (1.2) *has a large 2-2 double homoclinic loop (Figure* 24*).*

(12) *For* $\mu \in \mathcal{L}^{1,12,2}_{21}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 15*).*

(13) *For* $\mu \in \mathcal{L}^{21,12,2}_{1}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 16*).*

(14) *For* $\mu \in \mathcal{L}^{21,1,2}_{12}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$*, that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 17*).*

(15) *For* $\mu \in \mathcal{L}_2^{21,1,12}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 18*).*

(16) *For* $\mu \in \mathcal{R}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 periodic loop (Figure* 30*).*

Proof. In this case, by (6.1), we have that $\frac{\partial(G_1^1,\tilde{G}_1^1,G_2^1,\tilde{G}_2^1)}{\partial(s_1,s_{12},s_2,s_{21})}|_{s_1=s_{12}=s_2=s_{21}=0} = \mathrm{diag}(-1,1,1,-1)$ is a full rank matrix, so, according to the implicit function theorem, we have that (6.1) has a unique solution

$$\begin{cases} s_1 = -(w_1^{12})^{-1}s_{21}^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu + h.o.t., \\ s_{12} = -(w_1^{12})^{-1}s_1^{(1+\alpha(\mu))} - \delta^{-1}M_1^1\mu + h.o.t., \\ s_2 = -(w_2^{12})^{-1}s_{12}^{(1+\alpha(\mu))} - \delta^{-1}M_2^1\mu + h.o.t., \\ s_{21} = -(w_2^{12})^{-1}s_2^{(1+\alpha(\mu))} + \delta^{-1}M_2^1\mu + h.o.t., \end{cases} \tag{6.4}$$

in the small neighborhood of $s_1 = s_{12} = s_2 = s_{21} = 0$. Thus, the uniqueness and non-coexistence are proved.

(1). If (6.4) has a solution $s_{21} = s_1 = s_{12} = s_2 = 0$, then (6.1) is turned to

$$\begin{cases} M_1^1\mu + h.o.t. = 0, \\ M_2^1\mu + h.o.t. = 0. \end{cases} \tag{6.5}$$

Thus, if $\mathrm{rank}\{M_1^1, M_2^1\} = 2$, then, there exists an $(l-2)$-dimensional surface $\mathcal{L}_{21,1,12,2}$ defined by (6.5) which has normal plane $\mathrm{span}\{M_1^1, M_2^1\}$ at $\mu = 0$, such that, for $\mu \in \mathcal{L}_{21,1,12,2}$, system (1.2) has a unique double homoclinic loop $\Gamma(\mu) = \Gamma_1(\mu) \cup \Gamma_2(\mu)$ in the small neighborhood of $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, the double homoclinic loops are preserved.

Furthermore, if $M_1^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $\mathcal{L}_1 = \{\mu : M_1^1\mu + h.o.t. = 0\}$ which has normal vector M_1^1 at $\mu = 0$, such that, for $\mu \in \mathcal{L}_1$, system (1.2) has a unique small 1-homoclinic loop $\Gamma_1(\mu)$ in the small neighborhood of Γ_1.

Similarly, if $M_2^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $\mathcal{L}_2 = \{\mu : M_2^1\mu + h.o.t. = 0\}$ which has normal vector M_2^1 at $\mu = 0$, such that, for $\mu \in \mathcal{L}_2$, system (1.2) has a unique small 1-homoclinic loop $\Gamma_2(\mu)$ in the small neighborhood of Γ_2.

(2). If (6.4) has a solution $s_{21} > 0$, $s_1 = s_{12} = s_2 = 0$, then (6.1) is turned to

$$\begin{cases} -s_{21}^{(1+\alpha(\mu))} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. = 0, \\ \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ s_{21} = \delta^{-1}M_2^1\mu + h.o.t.. \end{cases}$$

Thus, in the region $\mathcal{R}_{1,12,2}^{21} = \{\mu : \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0, \delta^{-1}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}_{1,12,2}^{21}$ of large 1-1 homoclinic loop as follows.

$$\begin{cases} -(\delta^{-1}M_2^1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. = 0, \\ \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

(3). If (6.4) has a solution $s_{12} > 0$, $s_{21} = s_1 = s_2 = 0$, then (6.1) is turned to

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1}M_1^1\mu + h.o.t., \\ s_{12}^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0, \\ \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{12}_{21,1,2} = \{\mu : -\delta^{-1}M^1_1\mu + h.o.t. > 0, \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{12}_{21,1,2}$ of large 1-1 homoclinic loop as follows.

$$\begin{cases} (-\delta^{-1}M^1_1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. = 0, \\ \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(4). If (6.4) has a solution $s_1 > 0$, $s_{21} = s_{12} = s_2 = 0$, then (6.1) is turned to

$$\begin{cases} s_1 = \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_1^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{1}_{21,12,2} = \{\mu : \delta^{-1}M^1_1\mu + h.o.t. > 0,\ \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{1}_{21,12,2}$ of large 1-2 double homoclinic loop as follows.

$$\begin{cases} (\delta^{-1}M^1_1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(5). If (6.4) has a solution $s_2 > 0$, $s_{21} = s_1 = s_{12} = 0$, then (6.1) is turned to

$$\begin{cases} \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_2 = -\delta^{-1}M^1_2\mu + h.o.t., \\ -s_2^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{2}_{21,1,12} = \{\mu : -\delta^{-1}M^1_2\mu + h.o.t. > 0,\ \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{2}_{21,1,12}$ of large 2-1 double homoclinic loop as follows.

$$\begin{cases} \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ -(-\delta^{-1}M^1_2\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. = 0. \end{cases}$$

(6). If (6.4) has a solution $s_{12} > 0$, $s_2 > 0$, $s_{21} = s_1 = 0$, then (6.1) is turned to

$$\begin{cases} \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1}M^1_1\mu + h.o.t., \\ s_2 = -(w^{12}_2)^{-1}s_{12}^{(1+\alpha(\mu))} - \delta^{-1}M^1_2\mu + h.o.t., \\ -s_2^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{12,2}_{21,1} = \{\mu : -\delta^{-1}M^1_1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{12,2}_{21,1}$ of large 2-1 homoclinic loop as follows.

$$\begin{cases} \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ -\left(-(w^{12}_2)^{-1}\left(-\delta^{-1}M^1_1\mu\right)^{(1+\alpha(\mu))} - \delta^{-1}M^1_2\mu\right)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. = 0. \end{cases}$$

(7). If (6.4) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, then (6.1) is turned to

$$\begin{cases} -s_{21}^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_2 = -\delta^{-1}M^1_2\mu + h.o.t., \\ s_{21} = -(w^{12}_2)^{-1}s_2^{(1+\alpha(\mu))} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{21,2}_{1,12} = \{\mu : -\delta^{-1}M^1_2\mu + h.o.t. > 0\} \cap \{\mu : -(w^{12}_2)^{-1}(-\delta^{-1}M^1_2\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}M^1_2\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{21,2}_{1,12}$ of large 2-1 homoclinic loop as follows.

$$\begin{cases} -\left(-(w^{12}_2)^{-1}(-\delta^{-1}M^1_2\mu)^{(1+\alpha(\mu))} + \delta^{-1}M^1_2\mu\right)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_1\mu + h.o.t. = 0. \end{cases}$$

(8). If (6.4) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, then (6.1) is turned to

$$\begin{cases} s_1 = -(w^{12}_1)^{-1}s_{21}^{(1+\alpha(\mu))} + \delta^{-1}M^1_1\mu + h.o.t., \\ s_1^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0, \\ s_{21} = \delta^{-1}M^1_2\mu + h.o.t.. \end{cases}$$

Thus, in the region $\mathcal{R}^{21,1}_{12,2} = \{\mu : \delta^{-1}M^1_2\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{21,1}_{12,2}$ of large 1-2 homoclinic loop as follows.

$$\begin{cases} \left(-(w^{12}_1)^{-1}(\delta^{-1}M^1_2\mu)^{(1+\alpha(\mu))} + \delta^{-1}M^1_1\mu\right)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(9). If (6.4) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, then (6.1) is turned to

$$\begin{cases} s_1 = \delta^{-1}M^1_1\mu + h.o.t., \\ s_{12} = -(w^{12}_1)^{-1}s_1^{(1+\alpha(\mu))} - \delta^{-1}M^1_1\mu + h.o.t., \\ s_{12}^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2M^1_2\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{1,12}_{21,2} = \{\mu : \delta^{-1}M^1_1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w^{12}_2M^1_2\mu + h.o.t. < 0\} \cap \{\mu : -(w^{12}_1)^{-1}(\delta^{-1}M^1_1\mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1}M^1_1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{1,12}_{21,2}$ of large 1-2 homoclinic loop as follows.

$$\begin{cases} \left(-(w^{12}_1)^{-1}(\delta^{-1}M^1_1\mu)^{(1+\alpha(\mu))} - \delta^{-1}M^1_1\mu\right)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2M^1_2\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(10). If (6.4) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, then (6.1) is turned to

$$\begin{cases} s_1 = \delta^{-1}M^1_1\mu + h.o.t., \\ s_1^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. = 0, \\ s_2 = -\delta^{-1}M^1_2\mu + h.o.t., \\ -s_2^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\mathcal{R}^{1,2}_{21,12} = \{\mu : \delta^{-1}M^1_1\mu + h.o.t. > 0\} \cap \{\mu : -\delta^{-1}M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}^{1,2}_{21,12}$ of large 2-2 double homoclinic loop as follows.

$$\begin{cases} (\delta^{-1}M^1_1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_1M^1_1\mu + h.o.t. = 0, \\ -(-\delta^{-1}M^1_2\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}w^{12}_2M^1_2\mu + h.o.t. = 0. \end{cases}$$

(11). If (6.4) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, then (6.1) is turned to

$$\begin{cases} -s_{21}^{(1+\alpha(\mu))} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1} M_1^1 \mu + h.o.t., \\ s_{12}^{(1+\alpha(\mu))} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. = 0, \\ s_{21} = \delta^{-1} M_2^1 \mu + h.o.t.. \end{cases}$$

Thus, in the region $\mathcal{R}_{1,2}^{21,12} = \{\mu : -\delta^{-1} M_1^1 \mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1} M_2^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}_{1,2}^{21,12}$ of large 2-2 double homoclinic loop as follows.

$$\begin{cases} -(\delta^{-1} M_2^1 \mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0, \\ (-\delta^{-1} M_1^1 \mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

(12). If (6.4) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, then (6.1) is turned to

$$\begin{cases} s_1 = \delta^{-1} M_1^1 \mu + h.o.t., \\ s_{12} = -(w_1^{12})^{-1} s_1^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu + h.o.t., \\ s_2 = -(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu + h.o.t., \\ -s_2^{(1+\alpha(\mu))} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. = 0. \end{cases} \tag{6.6}$$

Thus, in the region $\mathcal{R}_{21}^{1,12,2} = \{\mu : \delta^{-1} M_1^1 \mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. > 0\} \cap \{\mu : -(w_1^{12})^{-1}(\delta^{-1} M_1^1 \mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}_{21}^{1,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned} &-\left(-(w_2^{12})^{-1}\left(-(w_1^{12})^{-1}(\delta^{-1} M_1^1 \mu)^{(1+\alpha(\mu))} - \delta^{-1} M_1^1 \mu\right)^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu\right)^{(1+\alpha(\mu))} \\ &\quad + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. = 0. \end{aligned}$$

(13). If (6.4) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, then (6.1) is turned to

$$\begin{cases} -s_{21}^{(1+\alpha(\mu))} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1} M_1^1 \mu + h.o.t., \\ s_2 = -(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu + h.o.t., \\ s_{21} = -(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu + h.o.t.. \end{cases} \tag{6.7}$$

Thus, in the region $\mathcal{R}_1^{21,12,2} = \{\mu : -\delta^{-1} M_1^1 \mu + h.o.t. > 0\} \cap \{\mu : -(w_2^{12})^{-1}(-\delta^{-1} M_1^1 \mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu + h.o.t. > 0\} \cap \{\mu : -(w_2^{12})^{-1}[-(w_2^{12})^{-1}(-\delta^{-1} M_1^1 \mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu + h.o.t.]^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}_1^{21,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned} &-\left(-(w_2^{12})^{-1}\left(-(w_2^{12})^{-1}(-\delta^{-1} M_1^1 \mu)^{(1+\alpha(\mu))} - \delta^{-1} M_2^1 \mu\right)^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu\right)^{(1+\alpha(\mu))} \\ &\quad + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0. \end{aligned}$$

(14). If (6.4) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, then (6.1) is turned to

$$\begin{cases} s_1 = -(w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} + \delta^{-1} M_1^1 \mu + h.o.t., \\ s_1^{(1+\alpha(\mu))} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. = 0, \\ s_2 = -\delta^{-1} M_2^1 \mu + h.o.t., \\ s_{21} = -(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} + \delta^{-1} M_2^1 \mu + h.o.t.. \end{cases} \tag{6.8}$$

Thus, in the region $\mathcal{R}_{12}^{21,1,2} = \{\mu : -\delta^{-1}M_2^1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. < 0\} \cap \{\mu : -(w_2^{12})^{-1}(-\delta^{-1}M_2^1\mu + h.o.t.)^{(1+\alpha(\mu))} + \delta^{-1}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}_{12}^{21,1,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned}&\left(-(w_1^{12})^{-1}\left(-(w_2^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{(1+\alpha(\mu))} + \delta^{-1}M_2^1\mu\right)^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu\right)^{(1+\alpha(\mu))}\\&\qquad + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. = 0.\end{aligned}$$

(15). If (6.4) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, then (6.1) is turned to

$$\begin{cases} s_1 = -(w_1^{12})^{-1}s_{21}^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu + h.o.t.,\\ s_{12} = -(w_1^{12})^{-1}s_1^{(1+\alpha(\mu))} - \delta^{-1}M_1^1\mu + h.o.t.,\\ s_{12}^{(1+\alpha(\mu))} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0,\\ s_{21} = \delta^{-1}M_2^1\mu + h.o.t.. \end{cases} \tag{6.9}$$

Thus, in the region $\mathcal{R}_2^{21,1,12} = \{\mu : \delta^{-1}M_2^1\mu + h.o.t. > 0\} \cap \{\mu : -(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu + h.o.t. > 0\} \cap \{\mu : -(w_1^{12})^{-1}[-(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu]^{(1+\alpha(\mu))} - \delta^{-1}M_1^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\mathcal{L}_1^{21,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned}&\left(-(w_1^{12})^{-1}\left(-(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu\right)^{(1+\alpha(\mu))} - \delta^{-1}M_1^1\mu\right)^{(1+\alpha(\mu))}\\&\qquad + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. = 0.\end{aligned}$$

(16). If (6.4) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, then, differentiating (6.4), and denoting by $(s_i)_\mu$ as the gradient of $s_i(\mu)$ with respect to μ, we get

$$\begin{cases} (s_1)_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))s_{21}^{\alpha(\mu)}(s_{21})_\mu + \delta^{-1}M_1^1 + h.o.t.,\\ (s_{12})_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))s_1^{\alpha(\mu)}(s_1)_\mu - \delta^{-1}M_1^1 + h.o.t.,\\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))s_{12}^{\alpha(\mu)}(s_{12})_\mu - \delta^{-1}M_2^1 + h.o.t.,\\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))s_2^{\alpha(\mu)}(s_2)_\mu + \delta^{-1}M_2^1 + h.o.t.. \end{cases} \tag{6.10}$$

(i). If μ is situated in the neighborhood of $\mathcal{L}_{21}^{1,21,12}$, then, substituting (6.6) into (6.10), we get

$$\begin{cases} (s_1)_\mu = \delta^{-1}M_1^1 + h.o.t.,\\ (s_{12})_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))(\delta^{-1}M_1^1\mu + h.o.t.)^{\alpha(\mu)}(s_1)_\mu - \delta^{-1}M_1^1 + h.o.t.,\\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))\left(-(w_1^{12})^{-1}(\delta^{-1}M_1^1\mu)^{(1+\alpha(\mu))} - \delta^{-1}M_1^1\mu + h.o.t.\right)^{\alpha(\mu)}(s_{12})_\mu\\ \qquad\qquad - \delta^{-1}M_2^1 + h.o.t.,\\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))\left(\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.\right)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}(s_2)_\mu + \delta^{-1}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{21})_\mu = \delta^{-1}M_2^1 + O((\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}) + h.o.t.$, this means that $s_{21} = s_{21}(\mu)$ increases along the direction M_2^1 in the small neighborhood of $\mathcal{L}_{21}^{1,21,12}$.

(ii). If μ is situated in the neighborhood of $\mathcal{L}_1^{21,12,2}$, then, substituting (6.7) into (6.10), we get

$$\begin{cases} (s_1)_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))\left(\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.\right)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}(s_{21})_\mu + \delta^{-1}M_1^1 + h.o.t., \\ (s_{12})_\mu = -\delta^{-1}M_1^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))(-\delta^{-1}M_1^1\mu + h.o.t.)^{\alpha(\mu)}(s_{12})_\mu - \delta^{-1}M_2^1 + h.o.t., \\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))\left(-(w_2^{12})^{-1}(-\delta^{-1}M_1^1\mu)^{(1+\alpha(\mu))} - \delta^{-1}M_2^1\mu + h.o.t.\right)^{\alpha(\mu)}(s_2)_\mu \\ \qquad + \delta^{-1}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_1)_\mu = \delta^{-1}M_1^1 + O((\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}) + h.o.t.$, this means that $s_1 = s_1(\mu)$ increases along the direction M_1^1 in the small neighborhood of $\mathcal{L}_1^{21,12,2}$.

(iii). If μ is situated in the neighborhood of $\mathcal{L}_{12}^{21,1,2}$, then, substituting (6.8) into (6.10), we get

$$\begin{cases} (s_1)_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))\left(-(w_2^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{(1+\alpha(\mu))} + \delta^{-1}M_2^1\mu + h.o.t.\right)^{\alpha(\mu)}(s_{21})_\mu \\ \qquad + \delta^{-1}M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))\left(-\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.\right)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}(s_1)_\mu - \delta^{-1}M_1^1 + h.o.t., \\ (s_2)_\mu = -\delta^{-1}M_2^1 + h.o.t., \\ (s_{21})_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))(-\delta^{-1}M_2^1\mu + h.o.t.)^{\alpha(\mu)}(s_2)_\mu + \delta^{-1}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{12})_\mu = -\delta^{-1}M_1^1 + O\left(\left(-\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.\right)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}\right) + h.o.t.$, this means that $s_{12} = s_{12}(\mu)$ increases along the direction $-M_1^1$ in the small neighborhood of $\mathcal{L}_{12}^{21,1,2}$.

(iv). If μ is situated in the neighborhood of $\mathcal{L}_2^{21,1,12}$, then, substituting (6.9) into (6.10), we get

$$\begin{cases} (s_1)_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))(\delta^{-1}M_2^1\mu + h.o.t.)^{\alpha(\mu)}(s_{21})_\mu + \delta^{-1}M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_1^{12})^{-1}(1+\alpha(\mu))\left(-(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu)^{(1+\alpha(\mu))} + \delta^{-1}M_1^1\mu + h.o.t.\right)^{\alpha(\mu)}(s_1)_\mu \\ \qquad - \delta^{-1}M_1^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})^{-1}(1+\alpha(\mu))\left(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.\right)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}(s_{12})_\mu - \delta^{-1}M_2^1 + h.o.t., \\ (s_{21})_\mu = \delta^{-1}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_2)_\mu = -\delta^{-1}M_2^1 + O\left(\left(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.\right)^{\frac{\alpha(\mu)}{1+\alpha(\mu)}}\right)$, this means that $s_2 = s_2(\mu)$ increases along the direction $-M_2^1$ in the small neighborhood of $\mathcal{L}_2^{21,1,2}$.

Denote by $\mathcal{R}$ the region which is bounded by $\mathcal{L}_{21}^{1,12,2}$, $\mathcal{L}_1^{21,12,2}$, $\mathcal{L}_{12}^{21,1,2}$, $\mathcal{L}_2^{21,1,2}$, the vector M_2^1 point into it from $\mathcal{L}_{21}^{1,12,2}$, the vector M_1^1 point into it from $\mathcal{L}_1^{21,12,2}$, the vector M_1^1 point out of it from $\mathcal{L}_{12}^{21,1,2}$, and the vector M_2^1 point out of it from $\mathcal{L}_2^{21,1,2}$.

By the discussion of above, we get (6.4) has solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$ for $\mu \in \mathcal{R}$, that is, system (1.2) has a large 2-2 periodic loop.

About the bifurcation diagram, see Figure 33, where $\mathcal{L}_{21,12}^{1,2} = \mathcal{L}_{21,1,12}^{2} \cap \mathcal{L}_{21,12,2}^{1}$, $\mathcal{L}_{1,2}^{21,12} = \mathcal{L}_{1,12,2}^{21} \cap \mathcal{L}_{21,1,2}^{12}$. □

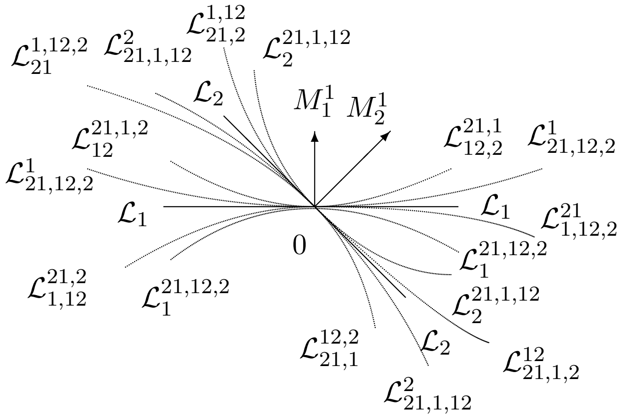


Figure 33

6.3. $-1 \ll \alpha(\mu) < 0$

Theorem 6.4. *Suppose that* **(H1)**$\sim$**(H4)** *and* **(H6)** *hold. If* $-1 \ll \alpha(\mu) < 0$, *rank*$\{M_1^1, M_2^1\} = 2$, *then, for* $|\mu| \ll 1$, *system* (1.2) *has at most one large 2-2 homoclinic loop, or one large 2-1 homoclinic loop, or one large 1-2 homoclinic loop, or one large 2-2 double homoclinic loops, or one large 1-1 homoclinic loop, or one large 1-2 double homoclinic loops, or one large 2-1 double homoclinic loops, or a large 2-2 periodic loop, or one large 1-1 double homoclinic loops in the small neighborhood of* $\Gamma = \Gamma_1 \cap \Gamma_2$, *and, these orbits do not coexist.*

Moreover, there exist surfaces $\bar{\mathcal{L}}_1$, $\bar{\mathcal{L}}_2$, $\bar{\mathcal{L}}_{21,1,12,2}$, $\bar{\mathcal{L}}_{1,12,2}^{21}$, $\bar{\mathcal{L}}_{21,1,2}^{12}$, $\bar{\mathcal{L}}_{21,12,2}^{1}$, $\bar{\mathcal{L}}_{21,1,12}^{2}$, $\bar{\mathcal{L}}_{21,1}^{12,2}$, $\bar{\mathcal{L}}_{1,12}^{21,2}$, $\bar{\mathcal{L}}_{12,2}^{21,1}$, $\bar{\mathcal{L}}_{2,21}^{1,12}$, $\bar{\mathcal{L}}_{21,12}^{1,2}$, $\bar{\mathcal{L}}_{1,2}^{21,12}$, $\bar{\mathcal{L}}_{21}^{1,12,2}$, $\bar{\mathcal{L}}_{1}^{21,12,2}$, $\bar{\mathcal{L}}_{12}^{21,1,2}$, $\bar{\mathcal{L}}_{2}^{21,1,12}$, *and a region* $\bar{\mathcal{R}}$, *such that:*

(1). *For* $\mu \in \bar{\mathcal{L}}_1$, *system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_1.

For $\mu \in \bar{\mathcal{L}}_2$, *system* (1.2) *has a unique small 1-homoclinic loop in the small neighborhood of* Γ_2.

For $\mu \in \bar{\mathcal{L}}_{21,1,12,2}$, (6.1) *has solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$, *that is, in the small neighborhood of* $\Gamma = \Gamma_1 \cup \Gamma_2$, *system* (1.2) *has a unique 1-1 double homoclinic loop* $\Gamma^0 = \Gamma_1^0(\mu) \cup \Gamma_2^0(\mu)$, *that is, the double homoclinic loops are preserved (Figure* 29*).*

(2). *For* $\mu \in \bar{\mathcal{L}}_{1,12,2}^{21}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 25*).*

(3). *For* $\mu \in \bar{\mathcal{L}}_{21,1,2}^{12}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-1 homoclinic loop (Figure* 26*).*

(4). *For* $\mu \in \bar{\mathcal{L}}_{21,12,2}^{1}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-2 double homoclinic loop (Figure* 27*).*

(5). *For* $\mu \in \bar{\mathcal{L}}_{21,1,12}^{2}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 double homoclinic loop (Figure* 28*).*

(6). *For* $\mu \in \bar{\mathcal{L}}_{21,1}^{12,2}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 19*).*

(7). *For* $\mu \in \bar{\mathcal{L}}_{1,12}^{21,2}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-1 homoclinic loop (Figure* 20*).*

(8). *For* $\mu \in \bar{\mathcal{L}}_{12,2}^{21,1}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-2 homoclinic loop (Figure* 21*).*

(9). *For* $\mu \in \bar{\mathcal{L}}_{2,21}^{1,12}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 1-2 homoclinic loop (Figure* 22*).*

(10). *For* $\mu \in \bar{\mathcal{L}}_{21,12}^{1,2}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 double homoclinic loop (Figure* 23*).*

(11). *For* $\mu \in \bar{\mathcal{L}}_{1,2}^{21,12}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 2-2 double homoclinic loop (Figure* 24*).*

(12). *For* $\mu \in \bar{\mathcal{L}}_{21}^{1,12,2}$, (6.1) *has a solution* $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 15*).*

(13). *For* $\mu \in \bar{\mathcal{L}}_{1}^{21,12,2}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 16*).*

(14). *For* $\mu \in \bar{\mathcal{L}}_{12}^{21,1,2}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 17*).*

(15). *For* $\mu \in \bar{\mathcal{L}}_{2}^{21,1,12}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, *that is, system* (1.2) *has a large 2-2 homoclinic loop (Figure* 18*).*

(16). *For* $\mu \in \bar{\mathcal{R}}$, (6.1) *has a solution* $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, *that is, system* (1.2) *has a large 2-2 periodic loop (Figure* 30*).*

Proof. In this case, $1 + \alpha(\mu) < 1$, by times scale transformations $s_{21} \to (s_{21})^{\frac{1}{1+\alpha(\mu)}}$, $s_1 \to (s_1)^{\frac{1}{1+\alpha(\mu)}}$, $s_{12} \to (s_{12})^{\frac{1}{1+\alpha(\mu)}}$, $s_2 \to (s_2)^{\frac{1}{1+\alpha(\mu)}}$, (6.1) becomes

$$\begin{cases} \delta\left(-(w_1^{12})^{-1} s_{21} - (s_1)^{\frac{1}{1+\alpha(\mu)}}\right) + M_1^1 \mu + h.o.t. = 0, \\ \delta\left((w_1^{12})^{-1} s_1 + (s_{12})^{\frac{1}{1+\alpha(\mu)}}\right) + M_1^1 \mu + h.o.t. = 0, \\ \delta\left((w_2^{12})^{-1} s_{12} + (s_2)^{\frac{1}{1+\alpha(\mu)}}\right) + M_2^1 \mu + h.o.t. = 0, \\ \delta\left(-(w_2^{12})^{-1} s_2 - (s_{21})^{\frac{1}{1+\alpha(\mu)}}\right) + M_2^1 \mu + h.o.t. = 0. \end{cases} \tag{6.11}$$

Similar to that of Theorem 6.3, (6.11) has a unique solution

$$\begin{cases} s_{21} = -w_1^{12} (s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t., \\ s_1 = -w_1^{12} (s_{12})^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t., \\ s_{12} = -w_2^{12} (s_2)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t., \\ s_2 = -w_2^{12} (s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t., \end{cases} \tag{6.12}$$

in the small neighborhood of $s_{21} = s_1 = s_{12} = s_2 = 0$. Thus, we get the uniqueness and non-coexistence.

(1). If (6.12) has a solution $s_{21} = s_1 = s_{12} = s_2 = 0$, then (6.11) is turned to

$$\begin{cases} M_1^1 \mu + h.o.t. = 0, \\ M_2^1 \mu + h.o.t. = 0. \end{cases} \tag{6.13}$$

Thus, if $\operatorname{rank}\{M_1^1, M_2^1\} = 2$, then, there exists an $(l-2)$-dimensional surface $\bar{\mathcal{L}}_{21,1,12,2}$ defined by (6.13) which has normal plane $\operatorname{span}\{M_1^1, M_2^1\}$ at $\mu = 0$, such that, for $\mu \in \bar{\mathcal{L}}_{21,1,12,2}$, system (1.2) has a unique double homoclinic loop $\Gamma(\mu) = \Gamma_1(\mu) \cup \Gamma_2(\mu)$ in the small neighborhood of $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, the double homoclinic loops are preserved.

Furthermore, if $M_1^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $\bar{\mathcal{L}}_1 = \{\mu : \ M_1^1\mu + h.o.t. = 0\}$ which has normal vector M_1^1 at $\mu = 0$, such that, for $\mu \in \bar{\mathcal{L}}_1$, system (1.2) has a unique small 1-homoclinic loop $\Gamma_1(\mu)$ in the small neighborhood of Γ_1.

Similarly, if $M_2^1 \neq 0$, then, there exists an $(l-1)$-dimensional surface $\bar{\mathcal{L}}_2 = \{\mu : \ M_2^1\mu + h.o.t. = 0\}$ which has normal vector M_2^1 at $\mu = 0$, such that, for $\mu \in \bar{\mathcal{L}}_2$, system (1.2) has a unique small 1-homoclinic loop $\Gamma_2(\mu)$ in the small neighborhood of Γ_2.

(2). If (6.12) has a solution $s_{21} > 0$, $s_1 = s_{12} = s_2 = 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t., \\ \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ \delta^{-1} M_2^1 \mu + h.o.t. = 0, \\ -(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}_{1,12,2}^{21} = \{\mu : \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. > 0, \delta^{-1} M_2^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_{1,12,2}^{21}$ of large 1-1 homoclinic loop as follows.

$$\begin{cases} \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ \delta^{-1} M_2^1 \mu + h.o.t. = 0, \\ -(\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

(3). If (6.12) has a solution $s_{12} > 0$, $s_{21} = s_1 = s_2 = 0$, then (6.11) is turned to

$$\begin{cases} \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t., \\ \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}_{21,1,2}^{12} = \{\mu : \delta^{-1} M_1^1 \mu + h.o.t. < 0, \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_{21,1,2}^{12}$ of large 1-1 homoclinic loop as follows.

$$\begin{cases} (-\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

(4). If (6.12) has a solution $s_1 > 0$, $s_{21} = s_{12} = s_2 = 0$, then (6.11) is turned to

$$\begin{cases} -(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ s_1 = -\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t., \\ \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}_{21,12,2}^{1} = \{\mu : \delta^{-1} M_1^1 \mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_{21,12,2}^{1}$ of large 1-2 double homoclinic loop as follows.

$$\begin{cases} -(-\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} M_1^1 \mu + h.o.t. = 0, \\ \delta^{-1} M_2^1 \mu + h.o.t. = 0. \end{cases}$$

(5). If (6.12) has a solution $s_2 > 0$, $s_{21} = s_1 = s_{12} = 0$, then (6.11) is turned to

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (s_2)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ s_2 = \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^2_{21,1,12} = \{\mu : \delta^{-1}M_2^1\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^2_{21,1,12}$ of large 2-1 double homoclinic loop as follows.

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

(6). If (6.12) has a solution $s_{12} > 0$, $s_2 > 0$, $s_{21} = s_1 = 0$, then (6.11) is turned to

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_{12} = -w_2^{12}(s_2)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \\ s_2 = \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^{12,2}_{21,1} = \{\mu : \delta^{-1}M_1^1\mu + h.o.t. < 0\} \cap \{\mu : -w_2^{12}(\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^{12,2}_{21,1}$ of large 2-1 homoclinic loop as follows.

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ \left(-w_2^{12}(\delta^{-1}w_2^{12}M_2^1\mu)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{cases}$$

(7). If (6.12) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} = 0$, $s_2 > 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (s_2)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ s_2 = -w_2^{12}(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^{21,2}_{1,12} = \{\mu : \delta^{-1}M_2^1\mu + h.o.t. < 0\} \cap \{\mu : \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^{21,2}_{1,12}$ of large 2-1 homoclinic loop as follows.

$$\begin{cases} \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ \left(-w_2^{12}(\delta^{-1}w_1^{12}M_1^1\mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

(8). If (6.12) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 = 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = -w_1^{12}(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ s_1 = -\delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ -(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^{21,1}_{12,2} = \{\mu : \delta^{-1}M^1_2\mu + h.o.t. > 0\} \cap \{\mu : -w^{12}_1(-\delta^{-1}w^{12}_1 M^1_1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. > 0\} \cap \{\mu : -\delta^{-1}w^{12}_1 M^1_1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^{21,1}_{12,2}$ of large 1-2 homoclinic loop as follows.

$$\begin{cases} \delta^{-1}M^1_2\mu + h.o.t. = 0, \\ \left(-w^{12}_1(-\delta^{-1}w^{12}_1 M^1_1\mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w^{12}_1 M^1_1\mu\right)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(9). If (6.12) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, then (6.11) is turned to

$$\begin{cases} -(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_1 = -w^{12}_1(s_{12})^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t., \\ s_{12} = -\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t., \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^{1,12}_{21,2} = \{\mu : \delta^{-1}M^1_1\mu + h.o.t. > 0\} \cap \{\mu : -\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^{1,12}_{21,2}$ of large 1-2 homoclinic loop as follows.

$$\begin{cases} \left(-w^{12}_1(-\delta^{-1}w^{12}_2 M^1_2\mu)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w^{12}_1 M^1_1\mu\right)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(10). If (6.12) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, then (6.11) is turned to

$$\begin{cases} -(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_1 = -\delta^{-1}w^{12}_1 M^1_1\mu + h.o.t., \\ (s_2)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0, \\ s_2 = \delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^{1,2}_{21,12} = \{\mu : \delta^{-1}M^1_1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}M^1_2\mu + h.o.t. < 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^{1,2}_{21,12}$ of large 2-2 double homoclinic loop as follows.

$$\begin{cases} -(-\delta^{-1}w^{12}_1 M^1_1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ (\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(11). If (6.12) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 = 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = \delta^{-1}w^{12}_1 M^1_1\mu + h.o.t., \\ (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ s_{12} = -\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t., \\ -(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

Thus, in the region $\bar{\mathcal{R}}^{21,12}_{1,2} = \{\mu : \delta^{-1}M^1_1\mu + h.o.t. < 0\} \cap \{\mu : \delta^{-1}M^1_2\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}^{21,12}_{1,2}$ of large 2-2 double homoclinic loop as follows.

$$\begin{cases} (-\delta^{-1}w^{12}_2 M^1_2\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_1\mu + h.o.t. = 0, \\ -(\delta^{-1}w^{12}_1 M^1_1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M^1_2\mu + h.o.t. = 0. \end{cases}$$

(12). If (6.12) has a solution $s_{21} = 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, then (6.11) is turned to

$$\begin{cases} -(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_1 = -w_1^{12}(s_{12})^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ s_{12} = -w_2^{12}(s_2)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \\ s_2 = \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.. \end{cases} \tag{6.14}$$

Thus, in the region $\bar{\mathcal{R}}_{21}^{1,12,2} = \{\mu : \delta^{-1}M_1^1\mu + h.o.t. > 0\} \cap \{\mu : \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\} \cap \{\mu : -w_2^{12}(\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_{21}^{1,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned} &-\left(-w_1^{12}\left(-w_2^{12}(\delta^{-1}w_2^{12}M_2^1\mu)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_1^{12}M_1^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} \\ &\quad + \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{aligned}$$

(13). If (6.12) has a solution $s_{21} > 0$, $s_1 = 0$, $s_{12} > 0$, $s_2 > 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ s_{12} = -w_2^{12}(s_2)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \\ s_2 = -w_2^{12}(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.. \end{cases} \tag{6.15}$$

Thus, in the region $\bar{\mathcal{R}}_{1}^{21,12,2} = \{\mu : \delta^{-1}M_1^1\mu + h.o.t. < 0\} \cap \{\mu : -w_2^{12}(\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\} \cap \{\mu : -w_2^{12}[(-w_2^{12}(\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.]^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_{1}^{21,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned} &\left(-w_2^{12}\left(-w_2^{12}(\delta^{-1}w_1^{12}M_1^1\mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} \\ &\quad + \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{aligned}$$

(14). If (6.12) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} = 0$, $s_2 > 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = -w_1^{12}(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ s_1 = -\delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ (s_2)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0, \\ s_2 = -w_2^{12}(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.. \end{cases} \tag{6.16}$$

Thus, in the region $\bar{\mathcal{R}}_{12}^{21,1,2} = \{\mu : \delta^{-1}M_2^1\mu + h.o.t. < 0\} \cap \{\mu : -\delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\} \cap \{\mu : -w_1^{12}(-\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_{1}^{21,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned} &\left(-w_2^{12}\left(-w_1^{12}(-\delta^{-1}w_1^{12}M_1^1\mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_2^{12}M_2^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} \\ &\quad + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{aligned}$$

(15). If (6.12) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 = 0$, then (6.11) is turned to

$$\begin{cases} s_{21} = -w_1^{12}(s_1)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ s_1 = -w_1^{12}(s_{12})^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_1^{12}M_1^1\mu + h.o.t., \\ s_{12} = -\delta^{-1}w_2^{12}M_2^1\mu + h.o.t., \\ -(s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{6.17}$$

Thus, in the region $\bar{\mathcal{R}}_2^{21,1,12} = \{\mu : \delta^{-1}M_2^1\mu + h.o.t. > 0\} \cap \{\mu : -w_1^{12}(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\} \cap \{\mu : -w_1^{12}[-w_1^{12}(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_1^{12}M_1^1\mu + h.o.t.]^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu + h.o.t. > 0\}$, we get the equation of bifurcation surface $\bar{\mathcal{L}}_1^{21,12,2}$ of large 2-2 homoclinic loop as follows.

$$\begin{aligned} &-\left(-w_1^{12}\left(-w_1^{12}(-\delta^{-1}w_2^{12}M_2^1\mu)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_1^{12}M_1^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}w_1^{12}M_1^1\mu\right)^{\frac{1}{1+\alpha(\mu)}} \\ &\quad + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{aligned}$$

(16). If (6.12) has a solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$, then, differentiating (6.12), and denoting by $(s_i)_\mu$ as the gradient of $s_i(\mu)$ with respect to μ, we get

$$\begin{cases} (s_{21})_\mu = -(w_1^{12})\dfrac{1}{1+\alpha(\mu)}(s_1)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_1)_\mu + \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_1)_\mu = -(w_1^{12})\dfrac{1}{1+\alpha(\mu)}(s_{12})^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{12})_\mu - \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_2^{12})\dfrac{1}{1+\alpha(\mu)}(s_2)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_2)_\mu - \delta^{-1}w_2^{12}M_2^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12})\dfrac{1}{1+\alpha(\mu)}(s_{21})^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{21})_\mu + \delta^{-1}w_2^{12}M_2^1 + h.o.t.. \end{cases} \tag{6.18}$$

(i). If μ is situated in the neighborhood of $\bar{\mathcal{L}}_{21}^{1,21,12}$, then, substituting (6.14) into (6.18), we get

$$\begin{cases} (s_{21})_\mu = -(w_1^{12})\dfrac{1}{1+\alpha(\mu)}(\delta^{-1}M_1^1\mu + h.o.t.)^{-\alpha(\mu)}(s_1)_\mu + \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_1)_\mu = -(w_1^{12})\dfrac{1}{1+\alpha(\mu)}\left(-w_2^{12}(\delta^{-1}w_2^{12}M_2^1)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1}w_2^{12}M_2^1\mu + h.o.t.\right)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_{12})_\mu \\ \qquad - \delta^{-1}w_1^{12}M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_2^{12})\dfrac{1}{1+\alpha(\mu)}(\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}}(s_2)_\mu - \delta^{-1}w_2^{12}M_2^1 + h.o.t., \\ (s_2)_\mu = \delta^{-1}w_2^{12}M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{21})_\mu = \delta^{-1}w_1^{12}M_1^1 + O((\delta^{-1}M_1^1\mu + h.o.t.)^{-\alpha(\mu)}) + h.o.t.$, this means that $s_{21} = s_{21}(\mu)$ increases along the direction $w_1^{12}M_1^1$ in the small neighborhood of $\bar{\mathcal{L}}_{21}^{1,21,12}$.

(ii). If μ is situated in the neighborhood of $\bar{\mathcal{L}}_1^{21,12,2}$, then, substituting (6.15) into (6.18), we get

$$\begin{cases} (s_{21})_\mu = \delta^{-1} w_1^{12} M_1^1 + h.o.t., \\ (s_1)_\mu = -(w_1^{12}) \dfrac{1}{1+\alpha(\mu)} (-\delta^{-1} M_1^1 \mu + h.o.t.)^{-\alpha(\mu)} (s_{12})_\mu - \delta^{-1} w_1^{12} M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_2^{12}) \dfrac{1}{1+\alpha(\mu)} \left(-w_2^{12} (\delta^{-1} w_1^{12} M_1^1 \mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} w_2^{12} M_2^1 \mu + h.o.t. \right)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}} (s_2)_\mu \\ \qquad - \delta^{-1} w_2^{12} M_2^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12}) \dfrac{1}{1+\alpha(\mu)} (\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}} (s_{21})_\mu + \delta^{-1} w_2^{12} M_2^1 + h.o.t.. \end{cases}$$

So, $(s_1)_\mu = -\delta^{-1} w_1^{12} M_1^1 + O((-\delta^{-1} M_1^1 \mu + h.o.t.)^{-\alpha(\mu)}) + h.o.t.$, this means that $s_1 = s_1(\mu)$ increases along the direction $-w_1^{12} M_1^1$ in the small neighborhood of $\bar{\mathcal{L}}_1^{21,12,2}$.

(iii). If μ is situated in the neighborhood of $\bar{\mathcal{L}}_{12}^{21,1,2}$, then, substituting (6.16) into (6.18), we get

$$\begin{cases} (s_{21})_\mu = -(w_1^{12}) \dfrac{1}{1+\alpha(\mu)} (-\delta^{-1} w_1^{12} M_1^1 \mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}} (s_1)_\mu + \delta^{-1} w_1^{12} M_1^1 + h.o.t., \\ (s_1)_\mu = -\delta^{-1} w_1^{12} M_1^1 + h.o.t., \\ (s_{12})_\mu = -(w_2^{12}) \dfrac{1}{1+\alpha(\mu)} (-\delta^{-1} M_2^1 \mu + h.o.t.)^{-\alpha(\mu)} (s_2)_\mu - \delta^{-1} w_2^{12} M_2^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12}) \dfrac{1}{1+\alpha(\mu)} \left(-w_1^{12} (-\delta^{-1} w_1^{12} M_1^1 \mu)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. \right)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}} (s_{21})_\mu \\ \qquad + \delta^{-1} w_2^{12} M_2^1 + h.o.t.. \end{cases}$$

So, $(s_{12})_\mu = -\delta^{-1} w_2^{12} M_2^1 + O((-\delta^{-1} M_2^1 \mu + h.o.t.)^{-\alpha(\mu)}) + h.o.t.$, this means that $s_{12} = s_{12}(\mu)$ increases along the direction $-w_2^{12} M_2^1$ in the small neighborhood of $\bar{\mathcal{L}}_{12}^{21,1,2}$.

(iv). If μ is situated in the neighborhood of $\bar{\mathcal{L}}_2^{21,1,12}$, then, substituting (6.17) into (6.18), we get

$$\begin{cases} (s_{21})_\mu = -(w_1^{12}) \dfrac{1}{1+\alpha(\mu)} \left(-w_1^{12} (-\delta^{-1} w_2^{12} M_2^1 \mu)^{\frac{1}{1+\alpha(\mu)}} - \delta^{-1} w_1^{12} M_1^1 \mu + h.o.t. \right)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}} (s_1)_\mu \\ \qquad + \delta^{-1} w_1^{12} M_1^1 + h.o.t., \\ (s_1)_\mu = -(w_1^{12}) \dfrac{1}{1+\alpha(\mu)} (-\delta^{-1} w_2^{12} M_2^1 \mu + h.o.t.)^{\frac{-\alpha(\mu)}{1+\alpha(\mu)}} (s_{12})_\mu - \delta^{-1} w_1^{12} M_1^1 + h.o.t., \\ (s_{12})_\mu = -\delta^{-1} w_2^{12} M_2^1 + h.o.t., \\ (s_2)_\mu = -(w_2^{12}) \dfrac{1}{1+\alpha(\mu)} (\delta^{-1} M_2^1 \mu + h.o.t.)^{-\alpha(\mu)} (s_{21})_\mu + \delta^{-1} w_2^{12} M_2^1 + h.o.t.. \end{cases}$$

So, $(s_2)_\mu = \delta^{-1} w_2^{12} M_2^1 + O((\delta^{-1} M_2^1 \mu + h.o.t.)^{-\alpha(\mu)}) + h.o.t.$, this means that $s_2 = s_2(\mu)$ increases along the direction $w_2^{12} M_2^1$ in the small neighborhood of $\bar{\mathcal{L}}_2^{21,1,2}$.

Denote by $\bar{\mathcal{R}}$, the region which is bounded by $\bar{\mathcal{L}}_{21}^{1,12,2}$, $\bar{\mathcal{L}}_1^{21,12,2}$, $\bar{\mathcal{L}}_{12}^{21,1,2}$, $\bar{\mathcal{L}}_2^{21,1,2}$, and, the vectors $w_1^{12} M_1^1$, $-w_1^{12} M_1^1$, $-w_2^{12} M_2^1$, and $w_2^{12} M_2^1$ point into it from $\bar{\mathcal{L}}_{21}^{1,12,2}$, $\bar{\mathcal{L}}_1^{21,12,2}$, $\bar{\mathcal{L}}_{12}^{21,1,2}$, and $\bar{\mathcal{L}}_2^{21,1,2}$, respectively.

By the discussion of above, we get (6.12) has solution $s_{21} > 0$, $s_1 > 0$, $s_{12} > 0$, $s_2 > 0$ for $\mu \in \bar{\mathcal{R}}$, that is, system (1.2) has a large 2-2 periodic loop.

About the bifurcation diagram, see Figure 34, where $\bar{\mathcal{L}}_{21,12}^{1,2} = \bar{\mathcal{L}}_{21,12,2}^{1} \cap \bar{\mathcal{L}}_{21,1,12}^{2}$, $\bar{\mathcal{L}}_{1,2}^{21,12} = \bar{\mathcal{L}}_{1,12,2}^{21} \cap \bar{\mathcal{L}}_{21,1,2}^{12}$. $\square$

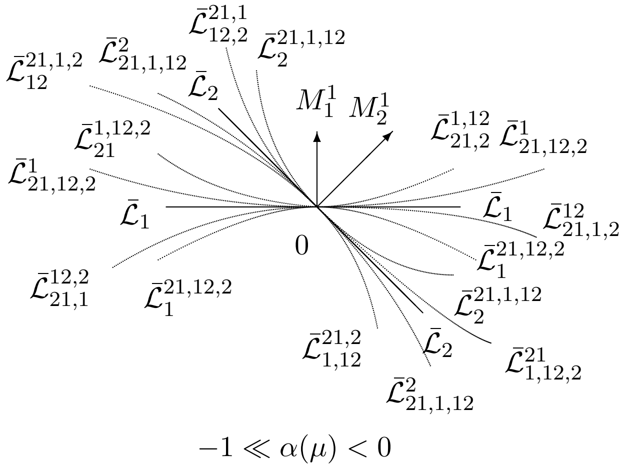


Figure 34

Acknowledgment

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