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Hybrid projection algorithm concerning split equality fixed point problem for quasi-pseudo-contractive mappings with application to optimization problem

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Abstract

The purpose of this paper is by using the shrinking projection method to study the split equality fixed point problem for a class of quasi-pseudo-contractive mappings in the setting of Hilbert spaces. Under suitable conditions, some strong convergence theorems are obtained. As applications, we utilize the results presented in the paper to study the existence problem of solutions to the split equality variational inequality problem and the split equality convex minimization problem. The results presented in our paper extend and improve some recent results.

Keywords: Split equality fixed point problem, quasi-pseudo-contractive mapping, hybrid projection algorithm, strong convergence theorem.
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1. Introduction

Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator. Recall that the split feasibility problem (SFP) is formulated as to find a point $q \in H_1$ such that:

$$q \in C \quad \text{and} \quad Aq \in Q. \tag{1.1}$$

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It is easy to see that $q \in C$ solves equation (1.1) if and only if it solves the following fixed point equation

$$q = P_C(I - \gamma A^*(I - P_Q)A)q, \quad x \in C,$$

where P_C (resp. P_Q) is the (orthogonal) projection from H_1 (resp. H_2) onto C (resp. Q), $\gamma > 0$, and A^* is the adjoint of A.

In 1994, Censor and Elfving [4] first introduced the (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the (SFP) can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [3, 5, 6]. The (SFP) in an infinite dimensional real Hilbert space can be found in [7, 9, 15, 16, 18].

Recently, Moudafi and Al-Shemas [12–14] introduced the following split equality feasibility problem (SEFP):

$$\text{to find} \quad x \in C, \quad y \in Q \quad \text{such that} \quad Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.2) reduces to (1.1).

In order to solve split equality feasibility problem (1.2), Moudafi [13] proposed the following simultaneous iterative algorithm:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta B^*(Ax_{k+1} - By_k)), \end{cases}$$

and under suitable conditions he proved the weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.2) in Hilbert spaces.

In order to avoid using the projection, recently, Moudafi [14] introduced and studied the following problem: let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. If $C = F(T)$ and $Q = F(S)$, then the split feasibility problem (1.1) reduces to:

$$\text{find} \ \ q \in F(T) \ \ \text{such that} \ Aq \in F(S), \tag{1.3}$$

which is called split common fixed point problem (in short, (SCFPP)). If $C = F(T)$ and $Q = F(S)$, then the split equality feasibility problem (1.2) reduces

$$\text{to find} \ \ x \in F(T) \ \ \text{and} \ \ y \in F(S) \ \ \text{such that} \ Ax = By, \tag{1.4}$$

which is called split equality fixed point problem (in short, (SEFPP)).

Recently Moudafi [12] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{1.5}$$

He studied the weak convergence of the sequences generated by scheme (1.5) under the condition that T and S are firmly quasi-nonexpansive mappings.

In 2015, Che and Li [10] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tu_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n)Sv_n, \end{cases} \tag{1.6}$$

and proved the weak convergence of the scheme (1.6) under the condition that the operators T and S are quasi-nonexpansive mappings.

Very recently, Chang et al. [8] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T))u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)((1-\xi_n)I + \xi_n S((1-\eta_n)I + \eta_n S))v_n. \end{cases} \tag{1.7}$$

They established the weak convergence of the scheme (1.7) under the condition that the operators T and S are quasi-pseudo-contractive mappings which is more general than the classes of quasi-nonexpansive mappings, directed mappings, and demi-contractive mappings.

In 2014, He and Du [11] proposed the following iterative algorithm by shrinking projection method for finding a solution of (SCFPP) (1.3):

$$\begin{cases} y_n = (1-\alpha)x_n + \alpha T x_n, \\ z_n = \beta x_n + (1-\beta)T y_n, \\ w_n = P_C(z_n + \xi A^*(S-I)Az_n), \\ C_{n+1} = \{v \in C_n : ||w_n - v|| \le ||z_n - v|| \le ||x_n - v||\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}. \end{cases} \tag{1.8}$$

They established the strong convergence of the scheme (1.8) under the condition that the operator T is a Lipschitzian pseudocontractive mapping and S is demi-contractive mapping.

Motived by above results, the purpose of this paper is by using the shrinking projection method to study the split equality fixed point problem for a class of quasi-pseudo-contractive mappings in the setting of Hilbert spaces. Under suitable conditions, some strong convergence theorems are obtained. As applications, we utilize the results presented in the paper to study the existence problem of solutions to the split equality variational inequality problem and the split equality convex minimization problem. The results presented in our paper extend and improve some recent results.

2. Preliminaries

In this section, we collect some definitions, notations and conclusions, which will be needed in proving our main results.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$. Let C be a nonempty closed convex subset of H and $T : C \to C$ be a nonlinear mapping. As well-known, the following inequalities hold.

(i) $||x + y||^2 \le ||y||^2 + 2\langle x, x + y\rangle$ for all $x, y \in H$;
(ii) $||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle$ for all $x, y \in H$;
(iii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x - y||^2$ for all $x, y \in H$ and $\alpha \in [0, 1]$.

For each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$||x - P_C x|| \le ||x - y||, \ \ \forall y \in C.$$

The mapping P_C is called the metric projection from H onto C. It is well-known that P_C has the following properties:

(i) $\langle x - y, P_C x - P_C y\rangle \ge ||P_C x - P_C y||^2$ for every $x, y \in H$.
(ii) For $x \in H$ and $z \in C$, $z = P_C x$ if and only if $\langle x - z, z - y\rangle \ge 0$ for all $y \in C$.
(iii) For $x \in H$ and $y \in C$,

$$||y - P_C x||^2 + ||x - P_C x||^2 \le ||x - y||^2. \tag{2.1}$$

Definition 2.1. An operator $T : C \to C$ is said to be

(i) Nonexpansive if $||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C$.

(ii) Lipschitzian if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \quad \forall x, y \in H.$$

(iii) Quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - x^*|| \leq ||x - x^*||, \ \ \forall x \in C \text{ and } x^* \in F(T).$$

(iv) Firmly nonexpansive if

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2, \quad \forall x, y \in C.$$

(v) Firmly quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 - ||(I - T)x||^2, \quad \forall x \in C \text{ and } x^* \in F(T).$$

(vi) Demi-contractive if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + k||Tx - x||^2, \ \ \forall x \in C \text{ and } x^* \in F(T).$$

Definition 2.2. An operator $T : C \to C$ is said to be

(1) Pseudo-contractive if

$$\langle Tx - Ty, \ x - y \rangle \leq ||x - y||^2, \ \ \forall x, y \in C.$$

It is well-known that T is pseudo-contractive if and only if

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2, \quad \forall x, y \in C.$$

(2) Quasi-pseudo-contractive if $F(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||Tx - x||^2, \quad \forall x \in C \text{ and } x^* \in F(T).$$

It is obvious that the class of quasi-pseudo-contractive mappings includes the class of demi-contractive mappings as its special case.

(3) Demiclosed at 0 if for any sequence $\{x_n\} \subset C$ which converges weakly to x and $||x_n - T(x_n)|| \to 0$, then $T(x) = x$.

Lemma 2.3 ([17]). *Let H be a real Hilbert space and $T : H \to H$ be a L-Lipschitzian mapping with $L \geq 1$. Denote by*

$$K := (1 - \xi)I + \xi T((1 - \eta)I + \eta T).$$

If $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conclusions hold.

(1) *$F(T) = Fix(T((1 - \eta)I + \eta T)) = F(K)$;*

(2) *if T is demiclosed at 0, then K is also demiclosed at 0;*

(3) *in addition, if $T : H \to H$ is quasi-pseudo-contractive, then the mapping K is quasi-nonexpansive, that is,*

$$||Kx - u^*|| \leq ||x - u^*||, \ \ \forall x \in H \ \ \text{and } u^* \in F(T) = F(K).$$

3. Main results

Throughout this section, we assume that

(1) H_1, H_2, and H_3 are three real Hilbert spaces and C, Q are bounded, closed and convex subsets of H_1, H_2, respectively. $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators with adjoints A^* and B^*, respectively.

(2) $T : C \to C$ and $S : Q \to Q$ are two L-Lipschitzian and quasi-pseudo-contractive mappings with $L \geq 1$.

Our object is to solve the following split equality fixed point problem:

$$\text{to find } x^* \in F(T),\ \ y^* \in F(S) \text{ such that } Ax^* = By^*. \tag{3.1}$$

In the sequel we use Γ to denote the set of solutions of (3.1), that is,

$$\Gamma = \{(x^*, y^*) \in F(T) \times F(S) \text{ such that } Ax^* = By^*\},$$

and assume that $\Gamma \neq \emptyset$.

Now, we present our algorithm for finding $(x^*, y^*) \in \Gamma$.

Algorithm 3.1 (Initialization)**.** Choose $\{\alpha_n\} \subset (0,1)$. Take arbitrary $x_1 \in C = C_1,\ y_1 \in Q = Q_1$.

Iterative steps:

$$\begin{cases} (a) & u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ (b) & w_n = \alpha_n x_n + (1-\alpha_n)((1-\xi)I + \xi T((1-\eta)I + \eta T))u_n, \\ (c) & v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ (d) & z_n = \alpha_n y_n + (1-\alpha_n)((1-\xi)I + \xi S((1-\eta)I + \eta S))v_n, \\ (e) & C_{n+1} \times Q_{n+1} = \{(p,q) \in C_n \times Q_n : \|w_n - p\|^2 + \|z_n - q\|^2 \le \|x_n - p\|^2 + \|y_n - q\|^2\}, \\ (f) & x_{n+1} = P_{C_{n+1}} x_1, \\ (g) & y_{n+1} = P_{Q_{n+1}} y_1. \end{cases} \tag{3.2}$$

Theorem 3.2. *Let $H_1,\ H_2,\ H_3,\ C,\ Q,\ A,\ B,\ S,\ T,\ \Gamma,\ \{x_n\},\ \{\alpha_n\}$, and $\{y_n\}$ be the same as above. If T and S are demiclosed at 0 and the following conditions are satisfied:*

(i) $\gamma_n \in (0,\ \min(\frac{1}{\|A\|^2},\ \frac{1}{\|B\|^2})),\ \ \forall n \ge 1$;

(ii) $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}},\ \forall n \ge 1$;

(iii) $\limsup_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Then there exists $(x^*, y^*) \in \Gamma$ such that the sequence $\{(x_n,\ y_n)\}$ generated by (3.2) converges strongly to (x^*, y^*).

Proof. From the constructions of $C_n \times Q_n$, we know that C_n and Q_n are closed and convex for all $n \ge 1$. Now we split the proof into four steps.

Step 1. We prove that $\Gamma \subset C_n \times Q_n$ for all $n \ge 1$.

In fact, it is obvious that $\Gamma \subset C_1 \times Q_1$. Suppose that $\Gamma \subset C_n \times Q_n$ for some $n \ge 1$, we prove that $\Gamma \subset C_{n+1} \times Q_{n+1}$.

For any given $(p,\ q) \in \Gamma \subset C_n \times Q_n$, then $p \in F(T),\ q \in F(S)$ and $Ap = Bq$. From equation (3.2) (a), we have

$$\begin{aligned} \|u_n - p\|^2 =& \|x_n - \gamma_n A^*(Ax_n - By_n) - p\|^2 \\ =& \|x_n - p\|^2 + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 - 2\gamma_n \langle x_n - p,\ A^*(Ax_n - By_n)\rangle \\ \le& \|x_n - p\|^2 + \gamma_n^2 \|A\|^2 \|Ax_n - By_n\|^2 - 2\gamma_n \langle Ax_n - Ap,\ Ax_n - By_n\rangle. \end{aligned} \tag{3.3}$$

Similarly, from (3.2) (c), we have

$$\|v_n - q\|^2 \le \|y_n - q\|^2 + \gamma_n^2 \|B\|^2 \|Ax_n - By_n\|^2 + 2\gamma_n \langle By_n - Bq,\ Ax_n - By_n\rangle. \tag{3.4}$$

Put

$$\begin{aligned} K :=& (1-\xi)I + \xi T((1-\eta)I + \eta T), \\ G :=& (1-\xi)I + \xi S((1-\eta)I + \eta S). \end{aligned}$$

By the assumptions of Theorem 3.2, condition (ii), and Lemma 2.3, we know that the mappings K and G have the following properties:

(1) both K and G are quasi-nonexpansive;

(2) $F(K) = F(T)$ and $F(G) = F(S)$;

(3) K and G demiclosed at 0.

Hence from (3.2) (b) we have that

$$\begin{aligned} ||w_n - p||^2 &= ||\alpha_n x_n + (1-\alpha_n)((1-\xi)I + \xi T((1-\eta)I + \eta T))u_n - p)||^2 \\ &= ||\alpha_n(x_n - p) + (1-\alpha_n)(Ku_n - p)||^2 \\ &= \alpha_n||x_n - p||^2 + (1-\alpha_n)||Ku_n - p||^2 - \alpha_n(1-\alpha_n)||Ku_n - x_n||^2 \\ &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||u_n - p||^2 - \alpha_n(1-\alpha_n)||Ku_n - x_n||^2. \end{aligned} \tag{3.5}$$

Similarly, from (3.2) (c) we have

$$||z_n - q||^2 \le \alpha_n||y_n - q||^2 + (1-\alpha_n)||v_n - q||^2 - \alpha_n(1-\alpha_n)||Gv_n - y_n||^2. \tag{3.6}$$

Adding up (3.5) and (3.6) and by virtue of (3.3) and (3.4), we have that

$$\begin{aligned} &||w_n - p||^2 + ||z_n - q||^2 \\ &\le \alpha_n||x_n - p||^2 + \alpha_n||y_n - q||^2 + (1-\alpha_n)||u_n - p||^2 + (1-\alpha_n)||v_n - q||^2 \\ &\quad - \alpha_n(1-\alpha_n)||Ku_n - x_n||^2 - \alpha_n(1-\alpha_n)||Gv_n - y_n||^2 \\ &\le \alpha_n||x_n - p||^2 + \alpha_n||y_n - q||^2 \\ &\quad + (1-\alpha_n)\{||x_n - p||^2 + \gamma_n^2||A||^2||Ax_n - By_n||^2 - 2\gamma_n\langle Ax_n - Ap,\ Ax_n - By_n\rangle\} \\ &\quad + (1-\alpha_n)\{||y_n - p||^2 + \gamma_n^2||B||^2||Ax_n - By_n||^2 + 2\gamma_n\langle By_n - Bq,\ Ax_n - By_n\rangle\} \\ &\quad - \alpha_n(1-\alpha_n)||Ku_n - x_n||^2 - \alpha_n(1-\alpha_n)||Gv_n - y_n||^2 \\ &= ||x_n - p||^2 + ||y_n - q||^2 + \gamma_n^2(1-\alpha_n)\{||A||^2 + ||B||^2\}||Ax_n - By_n||^2 \\ &\quad - (1-\alpha_n)2\gamma_n\{\langle Ax_n - Ap,\ Ax_n - By_n\rangle - \langle By_n - Bq,\ Ax_n - By_n\rangle\} \\ &\quad - \alpha_n(1-\alpha_n)\{||Ku_n - x_n||^2 + ||Gv_n - y_n||^2\} \\ &= ||x_n - p||^2 + ||y_n - q||^2 + \gamma_n^2(1-\alpha_n)\{||A||^2 + ||B||^2\}||Ax_n - By_n||^2 \\ &\quad - (1-\alpha_n)2\gamma_n||Ax_n - By_n||^2 - \alpha_n(1-\alpha_n)\{||Ku_n - x_n||^2 + ||Gv_n - y_n||^2\} \\ &= ||x_n - p||^2 + ||y_n - q||^2 - (1-\alpha_n)\gamma_n(2 - \gamma_n(||A||^2 + ||B||^2))||Ax_n - By_n||^2 \\ &\quad - \alpha_n(1-\alpha_n)\{||Ku_n - x_n||^2 + ||Gv_n - y_n||^2\}. \end{aligned} \tag{3.7}$$

Since $\gamma_n \in (0, \min(\frac{1}{||A||^2},\ \frac{1}{||B||^2}))$, $\gamma_n||A||^2 < 1$, and $\gamma_n||B||^2 < 1$, we have

$$0 < \gamma_n(||A||^2 + ||B||^2) < 2.$$

This implies that $\gamma_n(2 - \gamma_n(||A||^2 + ||B||^2)) > 0$. Therefore, (3.7) can be written as

$$\begin{aligned} ||w_n - p||^2 + ||z_n - q||^2 \le &||x_n - p||^2 + ||y_n - q||^2 \\ &- (1-\alpha_n)\gamma_n(2 - \gamma_n(||A||^2 + ||B||^2))||Ax_n - By_n||^2 \\ &- \alpha_n(1-\alpha_n)\{||Ku_n - x_n||^2 + ||Gv_n - y_n||^2\} \\ \le &||x_n - p||^2 + ||y_n - q||^2. \end{aligned} \tag{3.8}$$

This implies that $(p,\ q) \in C_{n+1} \times Q_{n+1}$. Thus we have $\Gamma \subset C_n \times Q_n$ for all $n \in \mathbb{N}$.

Step 2. Next we prove that $\{x_n\}$ and $\{y_n\}$ both are Cauchy sequences in C and Q, respectively, and $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$ for some $(x^*, y^*) \in C \times Q$.

In fact, since $\Gamma \subset C_n \times Q_n$, from (3.2) (f), (g) we have

$$||x_{n+1} - x_1|| \leq ||p - x_1||, \quad ||y_{n+1} - y_1|| \leq ||q - y_1|| \text{ for all } (p, q) \in \Gamma \tag{3.9}$$

and

$$||x_n - x_1|| \leq ||x_{n+1} - x_1||, \quad ||y_n - y_1|| \leq ||y_{n+1} - y_1|| \text{ for all } n \in \mathbb{N}. \tag{3.10}$$

It follows from (3.9) and (3.10) that $\{x_n\}, \{y_n\}$ both are bounded and $\{||x_n - x_1||\}, \{||y_n - y_1||\}$ are nondecreasing in $[0, \infty)$. Therefore the limits

$$\lim_{n\to\infty} ||x_n - x_1|| \text{ and } \lim_{n\to\infty} ||y_n - y_1||$$

exist. For any $m, n \in \mathbb{N}$ with $m > n$, it follows from

$$x_m = P_{C_m} x_1 \in C_n, \quad y_m = P_{Q_m} y_1 \in Q_n,$$

and (2.1) that

$$||x_m - x_n||^2 + ||x_1 - x_n||^2 = ||x_m - P_{C_n} x_1||^2 + ||x_1 - P_{C_n} x_1||^2 \leq ||x_m - x_1||^2$$

and

$$||y_m - y_n||^2 + ||y_1 - y_n||^2 = ||y_m - P_{Q_n} y_1||^2 + ||y_1 - P_{Q_n} y_1||^2 \leq ||y_m - y_1||^2.$$

These imply that

$$\lim_{m,n\to\infty} ||x_n - x_m|| = 0, \quad \lim_{m,n\to\infty} ||y_n - y_m|| = 0.$$

These show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in C and Q, respectively. By the completeness of C and Q, there exist $x^* \in C$ and $y^* \in Q$ such that

$$x_n \to x^* \text{ and } y_n \to y^* \text{ as } n \to \infty. \tag{3.11}$$

Step 3. Now we prove that

$$\lim_{n\to\infty} ||Ku_n - u_n|| = \lim_{n\to\infty} ||Gv_n - v_n|| = 0.$$

In fact, it follows from (3.2) that

$$x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n \text{ and } y_{n+1} = P_{Q_{n+1}} y_1 \in Q_{n+1} \subset Q_n, \quad \forall n \geq 1.$$

Therefore by virtue of (3.2) (e) for any $n \in \mathbb{N}$, we have

$$||w_n - x_{n+1}||^2 + ||z_n - y_{n+1}||^2 \leq ||x_n - x_{n+1}||^2 + ||y_n - y_{n+1}||^2.$$

This together with (3.11) shows that

$$\begin{aligned} ||w_n - x_n||^2 + ||z_n - y_n||^2 \leq & (||w_n - x_{n+1}|| + ||x_{n+1} - x_n||)^2 + (||z_n - y_{n+1}|| + ||y_{n+1} - y_n||)^2 \\ = & ||w_n - x_{n+1}||^2 + ||z_n - y_{n+1}||^2 + ||x_{n+1} - x_n||^2 + ||y_{n+1} - y_n||^2 \\ & + 2||w_n - x_{n+1}|| \cdot ||x_{n+1} - x_n|| + 2||z_n - y_{n+1}|| \cdot ||y_{n+1} - y_n|| \\ \leq & 2||x_{n+1} - x_n||^2 + 2||y_{n+1} - y_n||^2 + 2||w_n - x_{n+1}|| \cdot ||x_{n+1} - x_n|| \\ & + 2||z_n - y_{n+1}|| \cdot ||y_{n+1} - y_n|| \to 0 \text{ (as } n \to \infty). \end{aligned}$$

Therefore we have

$$\lim_{n\to\infty} ||w_n - x_n|| = 0, \quad \lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{3.12}$$

On the other hand, from (3.8) we obtain

$$\begin{aligned}&(1-\alpha_n)\gamma_n(2-\gamma_n(||A||^2+||B||^2))||Ax_n-By_n||^2\\&\quad+\alpha_n(1-\alpha_n)\{||Ku_n-x_n||^2+||Gv_n-y_n||^2\}\\&\le ||x_n-p||^2-||w_n-p||^2+||y_n-q||^2-||z_n-q||^2\\&=(||x_n-p||+||w_n-p||)\cdot(||x_n-p||-||w_n-p||)\\&\quad+(||y_n-q||+||z_n-q||)\cdot(||y_n-q||-||z_n-q||)\\&\le(||x_n-p||+||w_n-p||)\cdot||x_n-w_n||+(||y_n-q||+||z_n-q||)\cdot||y_n-z_n||.\end{aligned}\tag{3.13}$$

Letting $n \to \infty$ and taking the limit in (3.13), from (3.12) we get

$$||Ax_n - By_n|| \to 0;\ \ ||Ku_n - x_n|| \to 0;\ \ ||Gv_n - y_n|| \to 0. \tag{3.14}$$

It follows from (3.14) and (3.2) that

$$\lim_{n\to\infty} ||u_n - x_n|| \to 0 \ \text{ and } \ \lim_{n\to\infty} ||v_n - y_n|| \to 0. \tag{3.15}$$

This together with (3.14) shows that

$$\begin{cases}||Ku_n - u_n|| \le ||Ku_n - x_n|| + ||x_n - u_n|| \to 0;\\ ||Gv_n - v_n|| \le ||Gv_n - y_n|| + ||y_n - v_n|| \to 0.\end{cases}\tag{3.16}$$

Step 4. We prove that $(x^*, y^*) \in \Gamma$.

In fact, it follows from (3.11) and (3.15) that

$$u_n \to x^* \text{ and } v_n \to y^*. \tag{3.17}$$

By (3.16), (3.17), and the demiclosed property of K and G, we have $Kx^* = x^*$ and $Gy^* = y^*$. These imply that $x^* \in F(T)$ and $y^* \in F(S)$.

Finally, we prove that $Ax^* = By^*$. In fact, since $Ax_n - By_n \to Ax^* - By^*$, by (3.14), we have

$$||Ax^* - By^*|| = \lim_{n\to\infty} ||Ax_n - By_n|| = 0.$$

Thus $Ax^* = By^*$. This completes the proof of Theorem 3.2. □

4. Applications

4.1. Application to the split equality variational inequality problem

Throughout this section, we assume that H_1, H_2, and H_3 are three real Hilbert spaces. C and Q both are nonempty and closed convex subsets of H_1 and H_2, respectively and assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operator and A^* and B^* are the adjoints of A and B, respectively.

Let $M : C \to H_1$ be a mapping. *The variational inequality problem for mapping* M is to find a point $x^* \in C$ such that

$$\langle Mx^*,\ z - x^*\rangle \ge 0, \quad \forall z \in C. \tag{4.1}$$

We will denote the solution set of (4.1) by $VI(M, C)$.

A mapping $M : C \to H_1$ is said to be *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My,\ x - y\rangle \ge \alpha||Mx - My||^2, \quad \forall x, y \in C.$$

It is easy to see that if M is α-inverse-strongly monotone, then M is $\frac{1}{\alpha}$-Lipschitzian.

Setting $h(x, y) = \langle Mx,\ y - x\rangle : C \times C \to \mathbb{R}$, it is easy to show that h is an *equilibrium function*, i.e., it satisfies the following conditions (A1)-(A4):

(A1) $h(x, x) = 0$ for all $x \in C$;

(A2) h is monotone, i.e., $h(x, y) + h(y, x) \leq 0$ for all $x, y \in C$;

(A3) $\limsup_{t\downarrow 0} h(tz + (1 - t)x, y) \leq h(x, y)$ for all $x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto h(x, y)$ is convex and lower semi-continuous.

For given $\lambda > 0$ and $x \in H$, the *resolvent of the equilibrium function* h is the operator $R_{\lambda,h} : H \to C$ defined by

$$R_{\lambda,h}(x) := \{z \in C : h(z, y) + \frac{1}{\lambda}\langle y - z,\ z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Proposition 4.1 ([1])**.** *The resolvent operator $R_{\lambda,h}$ of the equilibrium function h has the following properties:*

(1) *$R_{\lambda,h}$ is single-valued;*
(2) *$F(R_{\lambda,h}) = VI(M, C)$, where $VI(M, C)$ is the solution set of variational inequality* (4.1) *which is a nonempty closed and convex subset of C;*
(3) *$R_{\lambda,h}$ is a firmly nonexpansive mapping. Therefore $R_{\lambda,h}$ is demi-closed at* 0.

Let $T : C \to H_1$ and $S : Q \to H_2$ be two α-inverse-strongly monotone mappings. The "so-called" split equality variational inequality problem with respect to T and S is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} (a) & \langle Tx^*,\ u - x^*\rangle \geq 0, \quad \forall u \in C, \\ (b) & \langle Sy^*,\ v - y^*\rangle \geq 0, \quad \forall v \in Q, \\ (c) & \qquad Ax^* = By^*. \end{cases} \tag{4.2}$$

In the sequel we use Θ to denote the solution set of split equality variational inequality problem (4.2), i.e.,

$$\Theta = \{(x^*, y^*) \in VI(T, C) \times VI(S, Q) : Ax^* = By^*\},$$

where $VI(T, C)$ (resp. $VI(S, Q)$) is the solution set of variational inequality (4.2) (a) (resp. (4.2) (b)).

Denote by $f(x, y) = \langle Tx,\ y - x\rangle : C \times C \to \mathbb{R}$ and $g(u, v) = \langle Su,\ v - u\rangle : Q \times Q \to \mathbb{R}$. For given $\lambda > 0$, $x \in H_1$ and $u \in H_2$, let $R_{\lambda,f}(x)$ and $R_{\lambda,g}(u)$ be the resolvent operator of the equilibrium function f and g, respectively which are defined by

$$R_{\lambda,f}(x) := \{z \in C : f(z, y) + \frac{1}{\lambda}\langle y - z,\ z - x\rangle \geq 0, \quad \forall y \in C\}$$

and

$$R_{\lambda,g}(u) := \{z \in Q : g(z, v) + \frac{1}{\lambda}\langle v - z,\ z - u\rangle \geq 0, \quad \forall v \in Q\}.$$

It follows from Proposition 4.1 that

$$F(R_{\lambda,f}) = VI(T, C) \neq \emptyset; \quad F(R_{\lambda,g}) = VI(S, Q) \neq \emptyset,$$

and so $R_{\lambda,f}$ and $R_{\lambda,g}$ both are quasi-pseudocontractive and 1-Lipschitzian. Therefore the split equality variational inequality problem with respect to T and S (4.2) is equivalent to the following split equality fixed point problem:

$$\text{to find } x^* \in F(R_{\lambda,f}), \ \ y^* \in F(R_{\lambda,g}) \ \text{ such that } \ Ax^* = By^*.$$

Since $R_{\lambda,f}$ and $R_{\lambda,g}$ are firmly nonexpansive with $F(R_{\lambda,f}) \neq \emptyset$ and $F(R_{\lambda,g}) \neq \emptyset$, the following theorem can be obtained from Theorem 3.2 immediately.

Theorem 4.2. *Let H_1, H_2, H_3, C, Q, A, B, T, S, $R_{\lambda,f}$, $R_{\lambda,g}$, Θ be the same as above and assume that $\Theta \neq \emptyset$. For given $x_1 \in C = C_1$, $y_1 \in Q = Q_1$, let $\{(x_n, y_n)\}$ be the sequence generated by*

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ w_n = R_{\lambda,f}(u_n), \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ z_n = R_{\lambda,g}(v_n), \\ C_{n+1} \times Q_{n+1} = \{(p,q) \in C_n \times Q_n : \|w_n - p\|^2 + \|z_n - q\|^2 \le \|x_n - p\|^2 + \|y_n - q\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \\ y_{n+1} = P_{Q_{n+1}} y_1. \end{cases} \tag{4.3}$$

If $\gamma_n \in (0,\ \min(\frac{1}{\|A\|^2},\ \frac{1}{\|B\|^2}))$ for all $n \ge 1$, then the sequence $\{(x_n,\ y_n)\}$ generated by (4.3) converges strongly to a solution of split equality variational inequality problem (4.2).

4.2. Application to the split equality convex minimization problem

Let C be a nonempty closed convex subset of H_1 and Q be a nonempty closed convex subset of H_2. Let $\phi : C \to \mathbb{R}$ and $\psi : Q \to \mathbb{R}$ be two proper and convex and lower semi-continuous functions and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operator with its adjoint A^* and B^*, respectively.

The "so-called" split equality convex minimization problem for ϕ and ψ is to find $x^* \in C,\ y^* \in Q$ such that

$$\phi(x^*) = \min_{x \in C} \phi(x), \quad \psi(y^*) = \min_{x \in Q} \psi(y), \text{ and } Ax^* = By^*. \tag{4.4}$$

In the sequel, we denote by Ω the solution set of the split equality convex minimization problem (4.4), i.e.,

$$\Omega = \{(x^*, y^*) \in C \times Q : \phi(x^*) = \min_{x \in C} \phi(x), \quad \psi(y^*) = \min_{x \in Q} \psi(y), \quad \text{and} \quad Ax^* = By^*\}.$$

Let $j(x,y) := \phi(y) - \psi(x) : C \times C \to \mathbb{R}$ and $k(u,v) := \phi(v) - \psi(u) : Q \times Q \to \mathbb{R}$. It is easy to know that j and k both are equilibrium functions satisfying the conditions (A1)-(A4).

For given $\lambda > 0$, $x \in H_1$ and $u \in H_2$, we define the resolvent operators of j and k as follows:

$$R_{\lambda,j}(x) := \{z \in C : j(z,y) + \frac{1}{\lambda}\langle y - z,\ z - x\rangle \ge 0, \quad \forall y \in C\}$$

and

$$R_{\lambda,k}(u) := \{z \in Q : k(z,v) + \frac{1}{\lambda}\langle v - z,\ z - u\rangle \ge 0, \quad \forall v \in Q\}.$$

By the same argument as given in Subsection 4.1, we know that

$$F(R_{\lambda,j}) = \{x^* \in C : \phi(x^*) = \min_{x \in C} \phi(x)\}, \quad F(R_{\lambda,k}) = \{y^* \in Q : \psi(y^*) = \min_{x \in Q} \psi(y)\}.$$

Therefore the split equality convex minimization problem for ϕ and ψ is equivalent to the following split equality fixed point problem:

$$\text{to find } x^* \in F(R_{\lambda,j}), \quad y^* \in F(R_{\lambda,k}) \ \text{ such that } \ Ax^* = By^*.$$

Since $R_{\lambda,j}$ and $R_{\lambda,k}$ both are firmly nonexpansive with $F(R_{\lambda,f}) \neq \emptyset$ and $F(R_{\lambda,g}) \neq \emptyset$, the following theorem can be obtained from Theorem 3.2 immediately.

Theorem 4.3. *Let H_1, H_2, H_3, C, Q, A, B, ϕ, ψ, $R_{\lambda,j}$, $R_{\lambda,k}$, Ω be the same as above and assume that $\Omega \neq \emptyset$. For given $x_1 \in C = C_1$, $y_1 \in Q = Q_1$, let $\{(x_n, y_n)\}$ be the sequence generated by*

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ w_n = R_{\lambda,j}(u_n), \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ z_n = R_{\lambda,k}(v_n), \\ C_{n+1} \times Q_{n+1} = \{(p, q) \in C_n \times Q_n : ||w_n - p||^2 + ||z_n - q||^2 \leq ||x_n - p||^2 + ||y_n - q||^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \\ y_{n+1} = P_{Q_{n+1}} y_1. \end{cases} \tag{4.5}$$

If $\gamma_n \in (0,\ \min(\frac{1}{||A||^2},\ \frac{1}{||B||^2}))$ for all $n \geq 1$, then the sequence $\{(x_n,\ y_n)\}$ generated by (4.5) *converges strongly to a solution of split equality convex minimization problem problem* (4.4).

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