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Strong convergence of a general iterative algorithm for asymptotically nonexpansive semigroups in Banach spaces

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Abstract

In this paper, we study a general iterative process strongly converging to a common fixed point of an asymptotically nonexpansive semigroup $\{T(t) : t \in \mathbb{R}^+\}$ in the framework of reflexive and strictly convex spaces with a uniformly Gáteaux differentiable norm. The process also solves some variational inequalities. Our results generalize and extend many existing results in the research field.

Keywords: Asymptotically nonexpansive semigroups, variational inequality, strong convergence, reflexive and strictly convex Banach spaces, fixed point.
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1. Introduction

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\mathbb{R}^+$ and $\mathbb{N}$ are the set of nonnegative real numbers and positive integers, respectively. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \quad \forall x \in E.$$

Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T. If $\{x_n\}$ is a sequence in E, we use $x_n \to x$ ($x_n \rightharpoonup x$) to denote strong (weak) convergence of the sequence $\{x_n\}$ to x.

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Recall that a mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$||f(x) - f(y)|| \leq \alpha||x - y||, \quad \forall x, y \in C.$$

We use Π_C to denote the collection of mappings f verifying the above inequality. That is

$$\Pi_C = \{f : C \to C \mid f \text{ is a contraction with constant } \alpha\}.$$

Note that each $f \in \Pi_C$ has a unique fixed point in C.

A mapping $T : C \to C$ is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C.$$

$T : C \to C$ is said to be asymptotically nonexpansive (see [6]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n||x - y||, \quad \forall x, y \in C, \quad \forall n \geq 1.$$

Let H be a real Hilbert space, and assume that A is a strongly positive bounded linear operator (see [17]) on H, that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, J(x)\rangle \geq \overline{\gamma}||x||^2, \quad \forall x, y \in H.$$

Then we can construct the following variational inequality problem with viscosity. Find $x^* \in C$ such that

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf, and γ is a suitable positive constant.

Many investigations have been done on fixed point iterative algorithms (see [3–5, 9, 10, 21, 24–29, 34, 36, 40]), as it is an important subject in nonlinear operator theory in a Banach space or a Hilbert space and has application in many areas, in particular, in image recovery and signal processing (see [2, 19, 22, 35, 37, 38]). Early in 1967, Halpern [8] firstly introduced the following iteration scheme:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.1}$$

where T is a nonexpansive mapping from C into itself, u and $x_0 \in C$ are both given points, and $x_{n+1} \in C$. The author proved that if $\{\alpha_n\}$ satisfies $\alpha_n \in (0, 1), \lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges strongly to a fixed point of T. In 2004, Xu [31] studied the following iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0,$$

where $\alpha_n \in (0, 1), x_0 \in C$, T is also a nonexpansive mapping and f is a contraction mapping from C into itself, $x_{n+1} \in C$. The author obtained a strong convergence theorem under some mild restrictions on the parameters by using the so-called viscosity approximation method introduced by Moudafi [18]. Afterward, Marino and Xu [17] considered the following iterative process on the basis of Xu [31]:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0,$$

where T is a self-nonexpansive mapping on H, $\{\alpha_n\}$ satisfies certain conditions, and A is a strong positive bounded linear operator on H. They proved that the sequence defined by the above iterative process converges strongly to a fixed point of T which is a unique solution of the variational inequality $\langle (A - \gamma f)x^*, x^* - x\rangle \leq 0$, for all $x \in F(T)$.

On the other hand, in 2008, Lou et al. [15] introduced the viscosity iteration process for an asymptotically nonexpansive mapping under the framework of a uniformly convex Banach space with a uniformly Gáteaux differentiable norm as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T^n x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences satisfying certain conditions.

In fact, the Lipschitzian semigroups are closely allied to nonexpansive mappings and asymptotically nonexpansive mappings of all time.

Recall that a one-parameter family $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ is said to be a Lipschitzian semigroup on C (see [32]) if the following conditions are satisfied:

i) $T(0)x = x, \quad \forall x \in C$;
ii) $T(s + t)x = T(t)T(s)x, \quad \forall t, s \in \mathbb{R}^+, \quad \forall x \in C$;
iii) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous;
iv) there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,

$$||T(t)x - T(t)y|| \leq L_t||x - y||, \quad \forall x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called a nonexpansive semigroup if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$. Note that for asymptotically nonexpansive semigroup $\mathcal{T}$, we can always assume that the Lipschitzian constants $\{L_t\}_{t>0}$ are such that $L_t \geq 1$ for each $t > 0$, L_t is nonincreasing in t, and $\lim_{t\to\infty} L_t = 1$; otherwise we replace L_t for each $t > 0$, by $\overline{L_t} := \max\{\sup_{s\geq t} L_s, 1\}$. Moreover, if $t_n > 0$ such that $\lim_{n\to\infty} t_n = \infty$, we obtain $L_{t_n} \to 1$ as $n \to \infty$. $\mathcal{T}$ is said to have a fixed point if there exists $x_0 \in C$ such that $T(t)x_0 = x_0$, for all $t > 0$. We denote by $F(\mathcal{T})$, the set of fixed points of $\mathcal{T}$, i.e., $F(\mathcal{T}) := \bigcap_{t\in\mathbb{R}^+} F(T(t))$.

A continuous operator of the semigroup $\mathcal{T}$ is said to be uniformly asymptotically regular (in short u.a.r.) on C if for all $h \geq 0$ and any bounded subset D of C, $\lim_{t\to\infty} \sup_{x\in D} ||T(h)T(t)x - T(t)x|| = 0$ (see [11]).

In 2008, Song and Xu [23] introduced the following iteration scheme for nonexpansive semigroups:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{t_n\}$ is a sequence of nonnegative real numbers divergent to infinity. Under certain restrictions to the sequence $\{\alpha_n\}$, they proved the strong convergence of $\{x_n\}$ to a member of $F(\mathcal{T})$ in a reflexive and strictly convex Banach space with a uniformly Gáteaux differentiable norm. Afterward, Zegeye and Shahzad [39] studied the sequence generated by the following algorithm

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)T(t_n)x_n, \quad \forall n \geq 0,$$

and proved strong convergence of $\{x_n\}$ to a member of $F(\mathcal{T})$ in the same Banach space for asymptotically nonexpansive semigroups. Very recently, Yang [32] proposed a generalized algorithm as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 0,$$

where f is a contraction mapping from C into itself and A is a strong positive bounded linear operator on C. Under certain conditions, on the basis of [17] and [23], the authors established strong convergence theorem for nonexpansive semigroups by using the above scheme in the framework of reflexive, smooth, and strictly convex Banach space with a uniformly Gáteaux differentiable norm. However, in the proof of Theorem 3.5 in [32], it is obviously impossible that

$$((\overline{\gamma}\alpha_m)^2 - 2\overline{\gamma}\alpha_m)||u_m - x_n||^2 \leq (\overline{\gamma}\alpha_m^2 - 2\alpha_m)\langle A(u_m - x_n), j(u_m - x_n)\rangle$$

with a control sequence $\{\alpha_m\}$ satisfying the condition $\lim_{m\to\infty} \alpha_m = 0$ which were also occurred in [16, 20].

In this paper, inspired by the existing results, we propose the more generalized iterative algorithm as follows:

$$\begin{cases} x_{n+1} = \alpha_n \gamma f_n(x_n) + \beta_n x_n + \delta_n u_n + ((1-\beta_n-\delta_n)I - \alpha_n A)T(t_n)y_n, \\ \quad y_n = (1-c_n-\sigma_n)x_n + \sigma_n v_n + c_n T(t_n)x_n, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ is an asymptotically nonexpansive semigroup, $\{f_n\}_{n=1}^{\infty}$ is an infinite family of contractive mappings from C into itself, A is a strong positive bounded linear operator, and $\{u_n\}, \{v_n\}$ are two bounded sequences in C. We prove under certain appropriate assumptions on the sequences $\{\alpha_n\}, \{\gamma_n\}, \{\delta_n\}, \{c_n\}, \{\sigma_n\}$, and $\{t_n\}$, that $\{x_n\}$ defined by (1.2) converges strongly to a member of $F(\mathcal{T})$ in the framework of a reflexive and strictly convex Banach space with a uniformly Gáteaux differentiable norm and correct the mistake above. Our results generalize and extend the corresponding results given by Marino and Xu [17], Lou et al. [15], Yang [32], Song and Xu [23], Zegeye and Shahzad [39], and many others.

2. Preliminaries and lemmas

Recall that a Banach space E is said to be strictly convex if $||x|| = ||y|| = 1$, and $x \neq y$ implies $||x+y|| < 2$. In a strictly convex Banach space E, we have that if $||x|| = ||y|| = ||tx + (1-t)y||$ for $t \in (0,1)$ and $x, y \in E$, then $x = y$.

Let E be a Banach space with $\dim E \geq 2$. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{||x+y||}{2} : ||x|| = ||y|| = 1, ||x-y|| = \varepsilon\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let $S := \{x \in E : ||x|| = 1\}$ denote the unit sphere of the Banach space E. Then the Banach space E is said to be smooth provided the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t} \tag{2.1}$$

exists for each $x, y \in S$. In this case, the norm of E is said to be Gáteaux differentiable. The space E is said to have a uniformly Gáteaux differentiable norm if for each $y \in S$ the limit (2.1) is attained uniformly for $x \in S$. It is well-known that if E is uniformly convex then E is reflexive and strictly convex, and if E is smooth then any duality mapping on E is single-valued and norm-to-weak* continuous. If E has a uniformly Gáteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded sets and also E is smooth.

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, ...) \in l^\infty$. We write $\mu(a_n)$ instead of $\mu((a_0, a_1, ...))$. Recall that a Banach limit μ is a bounded functional on l^∞ such that

$$||\mu|| = \mu(1) = 1, \qquad \liminf_{n\to\infty} a_n \leq \mu(a_n) \leq \limsup_{n\to\infty} a_n, \quad \mu(a_{n+r}) = \mu(a_n)$$

for any fixed positive integer r and for all $(a_0, a_1, ...) \in l^\infty$.

Let D be a nonempty subset of C. A sequence $\{f_n\}$ of mappings of C into E is said to be stable on D (see [1]) if $\{f_n(x) : n \in \mathbb{N}\}$ is a singleton for every $x \in D$. It is clear that if $\{f_n\}$ is stable on D, then $f_n(x) = f_1(x)$ for all $n \in \mathbb{N}$ and $x \in D$.

In a smooth Banach space, we say an operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \geq \overline{\gamma}||x||^2, \quad ||aI - bA|| = \sup_{||x||\leq 1} |\langle (aI - bA)x, J(x)\rangle|,$$

where I is the identity mapping, $a \in [0,1]$, $b \in [-1,1]$, and J is normalized duality mapping.

Lemma 2.1 ([32, Lemma 2.1]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq ||A||^{-1}$, then $||1 - \rho A|| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.2 ([39, Theorem 3.1]). *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E with a uniformly Gáteaux differentiable norm. Suppose that $\{x_n\}$ is a bounded sequence in C, and $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ is an asymptotically nonexpansive semigroup on C with a sequence $\{L_t\} \subset [1, \infty)$ such that $\lim_{n\to\infty} ||x_n - T(t)x_n|| = 0$ for all $t \geq 0$. Define the set*

$$K = \{x \in C : \mu||x_n - x||^2 = \min_{y\in C} \mu||x_n - y||^2\}.$$

If $F(\mathcal{T}) \neq \emptyset$, then $K \bigcap F(\mathcal{T}) \neq \emptyset$.

Lemma 2.3 ([7, Lemma 2.1]). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gáteaux differentiable norm and let S be a directed set. let $\{x_\alpha : \alpha \in S\}$ be a bounded set of E. Let $u \in C$. Then $\mu||x_\alpha - z||^2$ attains its minimum over C at u if and only if*

$$\mu(z - u, J(x_\alpha - u)) \leq 0$$

for all $z \in C$, where J is the duality map of E.

Lemma 2.4 ([33, Lemma 2.3]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then $\lim_{n\to\infty} ||y_n - x_n|| = 0$.*

In [12, 13], by using different methods, Liu proved the following lemma, and also see [14].

Lemma 2.5 ([12, 13]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences and let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = +\infty$. If there exists a positive integer n_0 such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n, \quad n \geq n_0, \tag{2.2}$$

where $b_n = \alpha_n a_n^*$, $\lim_{n\to\infty} a_n^* = 0$, and $\sum_{n=0}^{\infty} c_n < +\infty$, then $\lim_{n\to\infty} a_n = 0$.

Corollary 2.6 ([30, Lemma 2.5]). *Let $\{\alpha_n\}$ and $\{c_n\}$ be two nonnegative real sequences and let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = +\infty$. If there exists a positive integer n_0 such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + c_n, \quad n \geq n_0, \tag{2.3}$$

where $\{\sigma_n\}$ is a real sequence with $\limsup_{n\to\infty} \sigma_n \leq 0$ and $\sum_{n=0}^{\infty} c_n < +\infty$, then $\lim_{n\to\infty} a_n = 0$.

Proof. In fact, let

$$a_n^* = \begin{cases} \sigma_n, & \sigma_n \geq 0, \\ 0, & \sigma_n < 0. \end{cases}$$

Then $a_n^* \geq 0$ $(n = 1, 2, 3...)$ and $\sigma_n \leq a_n^*$ $(n = 1, 2, 3...)$. By $\limsup_{n\to\infty} \sigma_n \leq 0$, we can easily get $\lim_{n\to\infty} a_n^* = 0$. It follows from (2.2) that (2.3) holds. Hence, by Lemma 2.5, we see that $\lim_{n\to\infty} a_n = 0$. That is, Corollary 2.6 holds. □

3. Main results

Lemma 3.1. *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with a uniformly Gáteaux differentiable norm, $C \pm C \subset C$. Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a u.a.r. nonexpansive semigroup on C with a sequence $\{L_t\} \subset [1, \infty)$ such that $F(\mathcal{T}) \neq \emptyset$, and $\{f_n\} \subset \Pi_C$ is stable on $F(\mathcal{T})$.*

Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$, $A(C) \subset C$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, and α is contraction constant of all f_n. Let $\{x_n\}$ be a sequence defined by

$$x_n = \alpha_n \gamma f_n(x_n) + (I - \alpha_n A)T(t_n)x_n, \ \forall n \geq 1 \tag{3.1}$$

such that $\{\alpha_n\}$ is a sequence in $(0,1)$, $\lim_{n\to\infty} t_n = \infty$, and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{L_{t_n}-1}{\alpha_n} = 0$. Then the sequence $\{x_n\}$ converges strongly, as $n \to \infty$, to a point x^* of $F(\mathcal{T})$ which satisfies the variational inequality:

$$\langle (A - \gamma f_1)x^*, j(x^* - p)\rangle \leq 0, \quad p \in F(\mathcal{T}), \ f_1 \in \Pi_C. \tag{3.2}$$

Proof. Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{L_{t_n}-1}{\alpha_n} = 0$, we may assume, without loss of generality, that

$$\alpha_n < \min\left\{||A||^{-1}, \ \frac{2}{\overline{\gamma} - \gamma\alpha}\right\}, \quad \frac{L_{t_n} - 1}{\alpha_n} \leq \frac{\overline{\gamma} - \gamma\alpha}{2}, \quad \forall n \geq 1.$$

For each $n \geq 1$ and $t_n \geq 0$, define a mapping $S_n : C \to E$ by

$$S_n x = \alpha_n \gamma f_n(x) + (I - \alpha_n A)T(t_n)x, \quad \forall x \in C.$$

Since $C \pm C \subset C$, it is easy to see $S_n : C \to C$. For all $x, y \in C$, by Lemma 2.1, we have

$$\begin{aligned}
||S_n x - S_n y|| =& ||\alpha_n\gamma(f_n(x) - f_n(y)) + (I - \alpha_n A)(T(t_n)x - T(t_n)y)|| \\
\leq& \alpha_n\gamma||f_n(x) - f_n(y)|| + ||I - \alpha_n A|| ||T(t_n)x - T(t_n)y|| \\
\leq& \alpha_n\gamma\alpha||x - y|| + (1 - \alpha_n\overline{\gamma})L_{t_n}||x - y|| \\
=& [1 - \alpha_n(\overline{\gamma} - \gamma\alpha) + (L_{t_n} - 1)(1 - \alpha_n\overline{\gamma})]\, ||x - y|| \\
\leq& \left[1 - \frac{\alpha_n(\overline{\gamma} - \gamma\alpha)(1 + \alpha_n\overline{\gamma})}{2}\right] ||x - y|| \\
\leq& \left[1 - \frac{\alpha_n(\overline{\gamma} - \gamma\alpha)}{2}\right] ||x - y||.
\end{aligned}$$

Thus, $S_n : C \to C$ is a contractive mapping. By the Banach contraction mapping principle, it yields a unique fixed point $x_n \in C$ such that

$$x_n = \alpha_n\gamma f_n(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1.$$

Let $p \in F(\mathcal{T})$, then

$$\begin{aligned}
||x_n - p|| &= ||\alpha_n(\gamma f_n(x_n) - Ap) + (I - \alpha_n A)(T(t_n)x_n - p)|| \\
&= ||\alpha_n(\gamma f_n(x_n) - \gamma f_n(p)) + (I - \alpha_n A)(T(t_n)x_n - p) + \alpha_n(\gamma f_n(p) - Ap)|| \\
&\leq \alpha_n\gamma\alpha||x_n - p|| + (1 - \alpha_n\overline{\gamma})L_{t_n}||x_n - p|| + \alpha_n||\gamma f_n(p) - Ap||.
\end{aligned}$$

It follows that

$$\left[(\overline{\gamma} - \gamma\alpha) - \frac{L_{t_n} - 1}{\alpha_n}(1 - \alpha_n\overline{\gamma})\right] ||x_n - p|| \leq ||\gamma f_n(p) - Ap||.$$

Since $\{f_n\}$ is stable on $F(\mathcal{T})$, that is $f_n(p) = f_1(p)$ for all $n \in \mathbb{N}$, therefore,

$$||x_n - p|| \leq \frac{2||\gamma f_1(p) - Ap||}{\overline{\gamma} - \gamma\alpha}.$$

This implies that $\{x_n\}$ is bounded, and so are $\{T(t_n)x_n\}$ and $\{f_n(x_n)\}$. Moreover, it follows from (3.1) and $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} ||x_n - T(t_n)x_n|| = \lim_{n\to\infty} \alpha_n||\gamma f_n(x_n) - AT(t_n)x_n|| = 0.$$

Since $\{T(t) : t \in \mathbb{R}^+\}$ is u.a.r. on C and $\lim_{n\to\infty} t_n = \infty$, then for any $t \geq 0$,

$$\lim_{n\to\infty} ||T(t)T(t_n)x_n - T(t_n)x_n|| \leq \lim_{n\to\infty} \sup_{x\in D} ||T(t)T(t_n)x - T(t_n)x|| = 0,$$

where D is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned}||x_n - T(t)x_n|| \leq& ||x_n - T(t_n)x_n|| + ||T(t_n)x_n - T(t)T(t_n)x_n|| + ||T(t)T(t_n)x_n - T(t)x_n||\\ \leq&(1 + L_t)||x_n - T(t_n)x_n|| + ||T(t_n)x_n - T(t)T(t_n)x_n||,\end{aligned}$$

and therefore, $||x_n - T(t)x_n|| \to 0$, as $n \to \infty$. Define the set

$$K = \{x \in C : \mu||x_n - x||^2 = \min_{y\in C} \mu||x_n - y||^2\}.$$

By Lemma 2.2 we get that there exists $x^* \in K$ such that $x^* \in K \bigcap F(\mathcal{T})$. Since $C \pm C \subset C$, we have $x^* + \gamma f_1(x^*) - Ax^* \in C$, and then it follows from Lemma 2.3 that

$$\mu\langle x^* + \gamma f_1(x^*) - Ax^* - x^*, j(x_n - x^*)\rangle \leq 0,$$

which implies that

$$\mu\langle \gamma f_1(x^*) - Ax^*, j(x_n - x^*)\rangle \leq 0. \tag{3.3}$$

Notice that

$$\begin{aligned}||x_n - x^*||^2 =&\langle \alpha_n\gamma f_n(x_n) + (I - \alpha_nA)T(t_n)x_n - x^*, j(x_n - x^*)\rangle\\ =&\langle \alpha_n(\gamma f_n(x_n) - \gamma f_n(x^*)) + (I - \alpha_nA)(T(t_n)x_n - x^*) + \alpha_n(\gamma f_n(x^*) - Ax^*), j(x_n - x^*)\rangle\\ \leq&\alpha_n\gamma\alpha||x_n - x^*||^2 + (1 - \alpha_n\overline{\gamma})L_{t_n}||x_n - x^*||^2 + \alpha_n\langle \gamma f_n(x^*) - Ax^*, j(x_n - x^*)\rangle.\end{aligned}$$

Since $\{f_n\}$ is stable on $F(\mathcal{T})$, that is $f_n(x^*) = f_1(x^*)$ for all $n \in \mathbb{N}$, we derive that

$$[\alpha_n(\overline{\gamma} - \gamma\alpha) - (L_{t_n} - 1)(1 - \alpha_n\overline{\gamma})]||x_n - x^*||^2 \leq \alpha_n\langle \gamma f_1(x^*) - Ax^*, j(x_n - x^*)\rangle.$$

Therefore,

$$||x_n - x^*||^2 \leq \frac{2}{\overline{\gamma} - \gamma\alpha}\langle \gamma f_1(x^*) - Ax^*, j(x_n - x^*)\rangle.$$

This together with (3.3) implies that

$$\mu||x_n - x^*||^2 \leq \frac{2}{\overline{\gamma} - \gamma\alpha}\mu\langle \gamma f_1(x^*) - Ax^*, j(x_n - x^*)\rangle \leq 0.$$

Hence, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. Again, since

$$x_n = \alpha_n\gamma f_n(x_n) + (I - \alpha_nA)T(t_n)x_n,$$

we derive that

$$(A - \gamma f_n)x_n = -\frac{1}{\alpha_n}(I - \alpha_nA)(I - T(t_n))x_n.$$

Notice that

$$\begin{aligned}\langle (I - T(t_n))x_n - (I - T(x_n))p, j(x_n - p)\rangle =&||x_n - p||^2 - \langle T(t_n)x_n - T(x_n)p, j(x_n - p)\rangle\\ \geq&||x_n - p||^2 - L_{t_n}||x_n - p||^2\\ =&-(L_{t_n} - 1)||x_n - p||^2,\end{aligned}$$

it follows that, for all $p \in F(\mathcal{T})$,

$$\begin{aligned}\langle (A-\gamma f_{n_k})x_{n_k}, j(x_{n_k}-p)\rangle &= -\frac{1}{\alpha_{n_k}}\langle (I-\alpha_{n_k}A)(I-T(t_{n_k}))x_{n_k}, j(x_{n_k}-p)\rangle \\ &= -\frac{1}{\alpha_{n_k}}\langle (I-T(t_{n_k}))x_{n_k}-(I-T(t_{n_k}))p, j(x_{n_k}-p)\rangle \\ &\quad + \langle A(I-T(t_{n_k}))x_{n_k}, j(x_{n_k}-p)\rangle \\ &\le \frac{L_{t_{n_k}}-1}{\alpha_{n_k}}||x_{n_k}-p||^2 + \langle A(I-T(t_{n_k}))x_{n_k}, j(x_{n_k}-p)\rangle.\end{aligned} \tag{3.4}$$

On the other hand, as $\{f_n\}$ is stable on $F(\mathcal{T})$, that is, $f_n(x^*)=f_1(x^*)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle (A-\gamma f_1)x^*, j(x^*-p)\rangle =&\langle (A-\gamma f_1)x^*, j(x^*-p)-j(x_{n_k}-p)\rangle \\ &+ \langle (A-\gamma f_{n_k})x^*-(A-\gamma f_{n_k})x_{n_k}, j(x_{n_k}-p)\rangle \\ &+ \langle (A-\gamma f_{n_k})x_{n_k}, j(x_{n_k}-p)\rangle.\end{aligned} \tag{3.5}$$

Substituting (3.4) into (3.5) and letting $k \to \infty$, we have

$$\langle (A-\gamma f_1)x^*, j(x^*-p)\rangle \le 0, \tag{3.6}$$

that is, $x^* \in F(\mathcal{T})$ is a solution of (3.2).

Let $\{x_{n_i}\} \subset \{x_n\}$ be another subsequence such that $x_{n_i} \to p \in F(\mathcal{T})$ as $i \to \infty$. Then from (3.6) we get

$$\langle (A-\gamma f_1)p, j(p-x^*)\rangle \le 0. \tag{3.7}$$

Adding up (3.6) and (3.7), we have that

$$\begin{aligned}0 \ge&\langle (A-\gamma f_1)x^*-(A-\gamma f_1)p, j(x^*-q)\rangle \\ =&\langle A(x^*-p), j(x^*-p)\rangle - \gamma\langle f_1(x^*)-f_1(p), j(x^*-p)\rangle \\ \ge&\overline{\gamma}||x^*-p||^2 - \gamma||f_1(x^*)-f_1(p)||||x^*-p|| \\ \ge&(\overline{\gamma}-\gamma\alpha)||x^*-p||^2.\end{aligned}$$

Hence $p=x^*$. The proof is completed. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with a uniformly Gáteaux differentiable norm, $C \pm C \subset C$. Let $\mathcal{T}=\{T(t): t \in \mathbb{R}^+\}$ be a u.a.r. nonexpansive semigroup on C with a sequence $\{L_t\} \subset [1,\infty)$ such that $F(\mathcal{T}) \ne \emptyset$, and $\{f_n\} \subset \Pi_C$ is stable on $F(\mathcal{T})$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$, $A(C) \subset C$. Assume that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_{n+1} = \alpha_n\gamma f_n(x_n) + \beta_n x_n + \delta_n u_n + ((1-\beta_n-\delta_n)I-\alpha_n A)T(t_n)y_n, \\ \quad y_n = (1-c_n-\sigma_n)x_n + \sigma_n v_n + c_n T(t_n)x_n, \quad \forall n \ge 1, \end{cases} \tag{3.8}$$

satisfying

(1) $\alpha_n \in (0,1)$, $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{L_{t_n}-1}{\alpha_n} = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(2) $\beta_n \in (0,1)$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(3) $\delta_n, \sigma_n \in [0,1]$, $\sum_{n=1}^\infty \delta_n < \infty$, $\sum_{n=1}^\infty \sigma_n < \infty$;
(4) $h, t_n \ge 0, t_{n+1} = t_n + h$, $\lim_{n\to\infty} t_n = \infty$;
(5) $c_n \in [0,1]$, $\lim_{n\to\infty}|c_{n+1}-c_n| = 0$, $\limsup_{n\to\infty} c_n < 1$.

Suppose $\{u_n\}$ and $\{v_n\}$ are bounded in C, then as $n \to \infty$, the sequence $\{x_n\}$ defined by (3.8) *converges strongly to some common fixed point x^* of $F(\mathcal{T})$ which is the unique solution in $F(\mathcal{T})$ to the variational inequality* (3.2).

Proof. By the conditions (1) and (2), we may assume, with no loss of generality, that $\alpha_n \le (1-\beta_n-\delta_n)||A||^{-1}$ and $\frac{L_{t_n}-1}{\alpha_n} \le \frac{\overline{\gamma}-\gamma\alpha}{6}$ for all $n \ge 1$. Since A is a linear bounded self-adjoint operator on E, then $||A|| = \sup\{|\langle Ax, J(x)\rangle| : x \in E, ||x|| = 1\}$. When $||x||$=1, as

$$\begin{aligned}\langle((1-\beta_n-\delta_n)I-\alpha_n A)x, J(x)\rangle =& 1-\beta_n-\delta_n-\alpha_n\langle Ax, J(x)\rangle\\ \ge& 1-\beta_n-\delta_n-\alpha_n||A||\\ \ge& 0,\end{aligned}$$

we have

$$\begin{aligned}||(1-\beta_n-\delta_n)I-\alpha_n A|| =& \sup\{\langle((1-\beta_n-\delta_n)I-\alpha_n A)x, J(x)\rangle : x \in E, ||x|| = 1\}\\ =& \sup\{1-\beta_n-\delta_n-\alpha_n\langle Ax, J(x)\rangle : x \in E, ||x|| = 1\}\\ \le& 1-\beta_n-\delta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Taking a point $p \in F(\mathcal{T})$, from (3.8), we obtain

$$\begin{aligned}||y_n-p|| &= ||(1-c_n-\sigma_n)(x_n-p)+\sigma_n(v_n-p)+c_n(T(t_n)x_n-p)||\\ &\le (1-c_n-\sigma_n)||x_n-p||+\sigma_n||v_n-p||+c_nL_{t_n}||x_n-p||\\ &\le [1+c_n(L_{t_n}-1)]||x_n-p||+\sigma_n||v_n-p||.\end{aligned}\tag{3.9}$$

By condition (1), there exists $n_0 \in \mathbb{N}$ such that

$$\overline{\gamma}-\gamma\alpha-\frac{2(L_{t_n}-1)}{\alpha_n}-\frac{(L_{t_n}-1)^2}{\alpha_n^2} \ge \frac{\overline{\gamma}-\gamma\alpha}{2}, \quad n \ge n_0.\tag{3.10}$$

It then follows from the definition of $\{x_n\}$, (3.9) and (3.10) that

$$\begin{aligned}||x_{n+1}-p|| =& ||\alpha_n(\gamma f_n(x_n)-Ap)+\beta_n(x_n-p)+\delta_n(u_n-p)\\ &+[(1-\beta_n-\delta_n)I-\alpha_n A](T(t_n)y_n-p)||\\ =& ||\alpha_n(\gamma f_n(x_n)-\gamma f_n(p))+\beta_n(x_n-p)+\delta_n(u_n-p)\\ &+[(1-\beta_n-\delta_n)I-\alpha_n A](T(t_n)y_n-p)+\alpha_n(\gamma f_n(p)-Ap)||\\ \le& \alpha_n\gamma\alpha||x_n-p||+\beta_n||x_n-p||+\delta_n||u_n-p||\\ &+(1-\beta_n-\delta_n-\alpha_n\overline{\gamma})L_{t_n}||y_n-p||+\alpha_n||\gamma f_n(p)-Ap||\\ \le& \alpha_n\gamma\alpha||x_n-p||+\beta_n||x_n-p||+\alpha_n||\gamma f_n(p)-Ap||\\ &+(1-\beta_n-\delta_n-\alpha_n\overline{\gamma})[1+c_n(L_{t_n}-1)]L_{t_n}||x_n-p||\\ &+(1-\beta_n-\delta_n-\alpha_n\overline{\gamma})\sigma_nL_{t_n}||v_n-p||+\delta_n||u_n-p||\\ =& \alpha_n\gamma\alpha||x_n-p||+\beta_n||x_n-p||+\alpha_n||\gamma f_n(p)-Ap||\\ &+(1-\beta_n-\delta_n-\alpha_n\overline{\gamma})[1+c_n(L_{t_n}-1)][1+(L_{t_n}-1)]||x_n-p||\\ &+(1-\beta_n-\delta_n-\alpha_n\overline{\gamma})\sigma_nL_{t_n}||v_n-p||+\delta_n||u_n-p||\\ \le& [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]||x_n-p||+[(c_n+1)(L_{t_n}-1)\\ &+c_n(L_{t_n}-1)^2]||x_n-p||+\alpha_n||\gamma f_n(p)-Ap||+\sigma_nL_{t_n}||v_n-p||+\delta_n||u_n-p||\\ \le& [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]||x_n-p||+[2(L_{t_n}-1)\\ &+(L_{t_n}-1)^2]||x_n-p||+\alpha_n||\gamma f_n(p)-Ap||+\sigma_nL_{t_n}||v_n-p||+\delta_n||u_n-p||\\ \le& \left[1-\alpha_n\left(\overline{\gamma}-\gamma\alpha-\frac{2(L_{t_n}-1)}{\alpha_n}-\frac{(L_{t_n}-1)^2}{\alpha_n^2}\right)\right]||x_n-p||\end{aligned}$$

$$
\begin{aligned}
&+\alpha_n||\gamma f_n(p)-Ap||+\sigma_n L_{t_n}||v_n-p||+\delta_n||u_n-p||\\
\leq&\left[1-\frac{\alpha_n(\overline{\gamma}-\gamma\alpha)}{2}\right]||x_n-p||+\frac{\alpha_n(\overline{\gamma}-\gamma\alpha)}{2}\left\|\frac{2(\gamma f_1(p)-Ap)}{\overline{\gamma}-\gamma\alpha}\right\|+\sigma_n L_{t_n}||v_n-p||+\delta_n||u_n-p||\\
\leq&\max\left\{||x_n-p||,\left\|\frac{2(\gamma f_1(p)-Ap)}{\overline{\gamma}-\gamma\alpha}\right\|\right\}+\sigma_n L_{t_1}||v_n-p||+\delta_n||u_n-p||,\qquad n\geq n_0.
\end{aligned}
$$

By the induction, we have

$$
||x_{n+1}-p||\leq\max\left\{||x_{n_0}-p||,\left\|\frac{2(f_1(p)-Ap)}{\overline{\gamma}-\gamma\alpha}\right\|\right\}+\left(\sum_{n=1}^{\infty}\delta_n L_{t_1}+\sum_{n=1}^{\infty}\sigma_n\right)M,\quad\forall n\geq 1,
$$

where $M=\max_{n\in\mathbb{N}}\{||u_n-p||,||v_n-p||\}$. Hence $\{x_n\}$ is bounded, and so are $\{f_n(x_n)\},\{T(t_n)x_n\}$, $\{y_n\}$, $\{T(t_n)y_n\}$. Now we claim that

$$
||x_{n+1}-x_n||\to 0\ \text{ as }\ n\to\infty.
$$

Putting $\{l_n\}$ as a sequence that

$$
l_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n},\quad\forall n\geq 1,\tag{3.11}
$$

then, we get

$$
\begin{aligned}
l_{n+1}-l_n=&\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\
=&\frac{\alpha_{n+1}\gamma f_{n+1}(x_{n+1})+\delta_{n+1}u_{n+1}+((1-\beta_{n+1}-\delta_{n+1})I-\alpha_{n+1}A)T(t_{n+1})y_{n+1}}{1-\beta_{n+1}}\\
&-\frac{\alpha_n\gamma f_n(x_n)+\delta_n u_n+((1-\beta_n-\delta_n)I-\alpha_n A)T(t_n)y_n}{1-\beta_n}\\
=&\frac{\alpha_{n+1}(\gamma f_{n+1}(x_{n+1})-AT(t_{n+1})y_{n+1})}{1-\beta_{n+1}}+\frac{\delta_{n+1}(u_{n+1}-T(t_{n+1})y_{n+1})}{1-\beta_{n+1}}\\
&-\frac{\alpha_n(\gamma f_n(x_n)-AT(t_n)y_n)}{1-\beta_n}-\frac{\delta_n(u_n-T(t_n)y_n)}{1-\beta_n}+T(t_{n+1})y_{n+1}-T(t_n)y_n,
\end{aligned}\tag{3.12}
$$

and notice that

$$
\begin{aligned}
y_{n+1}-y_n=&(1-c_{n+1}-\sigma_{n+1})x_{n+1}+\sigma_{n+1}v_{n+1}+c_{n+1}T(t_{n+1})x_{n+1}-(1-c_n-\sigma_n)x_n\\
&-\sigma_n v_n-c_nT(t_n)x_n\\
=&(1-c_{n+1})x_{n+1}-(1-c_n)x_n+c_{n+1}T(t_{n+1})x_{n+1}-c_nT(t_n)x_n\\
&+\sigma_{n+1}(v_{n+1}-x_{n+1})+\sigma_n(x_n-v_n)\\
=&(1-c_{n+1})(x_{n+1}-x_n)+(c_n-c_{n+1})x_n+c_{n+1}(T(t_{n+1})x_{n+1}-T(t_n)x_n)\\
&+(c_{n+1}-c_n)T(t_n)x_n+\sigma_{n+1}(v_{n+1}-x_{n+1})+\sigma_n(x_n-v_n)\\
=&(1-c_{n+1})(x_{n+1}-x_n)+(c_n-c_{n+1})x_n+c_{n+1}(T(t_{n+1})x_{n+1}-T(t_{n+1})x_n)\\
&+c_{n+1}(T(t_{n+1})x_n-T(t_n)x_n)+(c_{n+1}-c_n)T(t_n)x_n\\
&+\sigma_{n+1}(v_{n+1}-x_{n+1})+\sigma_n(x_n-v_n).
\end{aligned}\tag{3.13}
$$

Substituting (3.13) into (3.12), we have

$$
\begin{aligned}
||l_{n+1}-l_n||\leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}||\gamma f_{n+1}(x_{n+1})-AT(t_{n+1})y_{n+1})||+\frac{\delta_n}{1-\beta_n}||u_n-T(t_n)y_n||\\
&+\frac{\alpha_n}{1-\beta_n}||\gamma f_n(x_n)-AT(t_n)y_n||+\frac{\delta_{n+1}}{1-\beta_{n+1}}||u_{n+1}-T(t_{n+1})y_{n+1}||\\
&+||T(t_{n+1})y_{n+1}-T(t_n)y_n||
\end{aligned}
$$

$$
\begin{aligned}
&\le ||T(t_{n+1})y_{n+1} - T(t_{n+1})y_n|| + ||T(t_{n+1})y_n - T(t_n)y_n|| + F_n \\
&\le L_{t_{n+1}}||y_{n+1} - y_n|| + ||T(t_n + h)y_n - T(t_n)y_n|| + F_n \\
&= ||y_{n+1} - y_n|| + (L_{t_{n+1}} - 1)||y_{n+1} - y_n|| + ||T(h)T(t_n)y_n - T(t_n)y_n|| + F_n \\
&\le (1 - c_{n+1})||x_{n+1} - x_n|| + |c_n - c_{n+1}|||x_n|| + |c_{n+1} - c_n|||T(t_n)x_n|| \\
&\quad + c_{n+1}||T(t_{n+1})x_n - T(t_n)x_n|| + c_{n+1}||T(t_{n+1})x_{n+1} - T(t_{n+1})x_n|| \\
&\quad + \sigma_{n+1}||v_{n+1} - x_{n+1}|| + \sigma_n||x_n - v_n|| + (L_{t_{n+1}} - 1)||y_{n+1} - y_n|| \\
&\quad + ||T(h)T(t_n)y_n - T(t_n)y_n|| + F_n \\
&\le (1 - c_{n+1})||x_{n+1} - x_n|| + |c_n - c_{n+1}|||x_n|| + c_{n+1}L_{t_{n+1}}||x_{n+1} - x_n|| \\
&\quad + c_{n+1}||T(h)T(t_n)x_n - T(t_n)x_n|| + |c_{n+1} - c_n|||T(t_n)x_n|| \\
&\quad + \sigma_{n+1}||v_{n+1} - x_{n+1}|| + \sigma_n||x_n - v_n|| + (L_{t_{n+1}} - 1)||y_{n+1} - y_n|| \\
&\quad + ||T(h)T(t_n)y_n - T(t_n)y_n|| + F_n \\
&\le ||x_{n+1} - x_n|| + c_{n+1}||T(h)T(t_n)x_n - T(t_n)x_n|| \\
&\quad + ||T(h)T(t_n)y_n - T(t_n)y_n|| + c_{n+1}(L_{t_{n+1}} - 1)||x_{n+1} - x_n|| \\
&\quad + (L_{t_{n+1}} - 1)||y_{n+1} - y_n|| + |c_{n+1} - c_n|(||x_n|| + ||T(t_n)x_n||) \\
&\quad + \sigma_{n+1}||v_{n+1} - x_{n+1}|| + \sigma_n||x_n - v_n|| + F_n \\
&\le ||x_{n+1} - x_n|| + c_{n+1}||T(h)T(t_n)x_n - T(t_n)x_n|| \\
&\quad + ||T(h)T(t_n)y_n - T(t_n)y_n|| + F_n + G_n,
\end{aligned}
$$

where

$$
\begin{aligned}
F_n =& \frac{\alpha_{n+1}}{1 - \beta_{n+1}}||\gamma f_{n+1}(x_{n+1}) - AT(t_{n+1})y_{n+1})|| + \frac{\delta_{n+1}}{1 - \beta_{n+1}}||u_{n+1} - T(t_{n+1})y_{n+1}|| \\
&+ \frac{\alpha_n}{1 - \beta_n}||\gamma f_n(x_n) - AT(t_n)y_n|| + \frac{\delta_n}{1 - \beta_n}||u_n - T(t_n)y_n||, \\
G_n =& c_{n+1}(L_{t_{n+1}} - 1)||x_{n+1} - x_n|| + (L_{t_{n+1}} - 1)||y_{n+1} - y_n|| + \sigma_n||x_n - v_n|| \\
&+ |c_{n+1} - c_n|(||x_n|| + ||T(t_n)x_n||) + \sigma_{n+1}||v_{n+1} - x_{n+1}||.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
||l_{n+1} - l_n|| - ||x_{n+1} - x_n|| \le & c_{n+1}||T(h)T(t_n)x_n - T(t_n)x_n|| \\
& + ||T(h)T(t_n)y_n - T(t_n)y_n|| + F_n + G_n.
\end{aligned} \tag{3.14}
$$

Since $\{T(t) : t \in \mathbb{R}^+\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, it follows that

$$\lim_{n\to\infty} ||T(h)T(t_n)x_n - T(t_n)x_n|| \le \lim_{t\to\infty} \sup_{x\in B} ||T(h)T(t)x - T(t)x|| = 0,$$

$$\lim_{n\to\infty} ||T(h)T(t_n)y_n - T(t_n)y_n|| \le \lim_{t\to\infty} \sup_{x\in B} ||T(h)T(t)x - T(t)x|| = 0,$$

where B is any bounded set containing $\{x_n\}$. Moreover, since $\{x_n\}$, $\{y_n\}$, $\{T(t_n)x_n\}$, $\{T(t_n)y_n\}$, $\{f_n(x_n)\}$, $\{u_n\}$, $\{v_n\}$ are bounded, by conditions (1), (2), (3), (5), (3.14) implies that

$$\limsup_{n\to\infty}(||l_{n+1} - l_n|| - ||x_{n+1} - x_n||) \le 0.$$

Hence by Lemma 2.4, we have $\lim_{n\to\infty} ||l_n - x_n|| = 0$. Consequently, it follows from (3.11) that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty}(1 - \beta_n)||l_n - x_n|| = 0$. Again since

$$
\begin{aligned}
||y_n - x_n|| &= ||(1 - c_n - \sigma_n)x_n + \sigma_n v_n + c_n T(t_n)x_n - x_n|| \\
&\le \sigma_n||v_n - x_n|| + c_n||T(t_n)x_n - x_n||,
\end{aligned}
$$

we have

$$\begin{aligned}
||x_n - T(t_n)x_n|| \leq& ||x_n - x_{n+1}|| + ||x_{n+1} - T(t_n)x_n||\\
=& ||x_n - x_{n+1}|| + ||\alpha_n\gamma f_n(x_n) + \beta_n x_n + \delta_n u_n + ((1-\beta_n-\delta_n)I - \alpha_n A)T(t_n)y_n - T(t_n)x_n||\\
=& ||x_n - x_{n+1}|| + ||\alpha_n(\gamma f_n(x_n) - AT(t_n)x_n) + \beta_n(x_n - T(t_n)x_n)\\
&+ \delta_n(u_n - T(t_n)x_n) + (1-\beta_n-\delta_n)I - \alpha_n A)(T(t_n)y_n - T(t_n)x_n)||\\
\leq& ||x_n - x_{n+1}|| + \alpha_n||\gamma f_n(x_n) - AT(t_n)x_n|| + \beta_n||x_n - T(t_n)x_n||\\
&+ \delta_n||u_n - T(t_n)x_n|| + (1-\beta_n-\delta_n-\alpha_n\overline{\gamma})L_{t_n}||y_n - x_n||\\
\leq& ||x_n - x_{n+1}|| + \alpha_n||\gamma f_n(x_n) - AT(t_n)x_n|| + \beta_n||x_n - T(t_n)x_n||\\
&+ \delta_n||u_n - T(t_n)x_n|| + (1-\beta_n-\delta_n-\alpha_n\overline{\gamma})L_{t_n}(\sigma_n||v_n - x_n|| + c_n||T(t_n)x_n - x_n||)\\
\leq& [\beta_n + (1-\beta_n)c_nL_{t_n}]||T(t_n)x_n - x_n|| + ||x_n - x_{n+1}||\\
&+ \alpha_n||\gamma f_n(x_n) - AT(t_n)x_n|| + \delta_n||u_n - T(t_n)x_n|| + \sigma_nL_{t_n}||v_n - x_n||,
\end{aligned}$$

it then follows that

$$\begin{aligned}
(1-\beta_n)(1-c_nL_{t_n})||T(t_n)x_n - x_n|| \leq& ||x_n - x_{n+1}|| + \alpha_n||\gamma f_n(x_n) - AT(t_n)x_n||\\
&+ \delta_n||u_n - T(t_n)x_n|| + \sigma_nL_{t_n}||v_n - x_n||.
\end{aligned}$$

By the conditions (2) and (5), it is easy to see that there exists $N \geq 0$, we have

$$(1-\beta_n)(1-c_nL_{t_n}) \geq c > 0,\ n \geq N,$$

where c is a constant. It follows that

$$\lim_{n\to\infty} ||T(t_n)x_n - x_n|| = 0,$$

and hence for any $t \geq 0$,

$$\begin{aligned}
||x_n - T(t)x_n|| \leq& ||x_n - T(t_n)x_n|| + ||T(t_n)x_n - T(t)T(t_n)x_n|| + ||T(t)T(t_n)x_n - T(t)x_n||\\
\leq& ||x_n - T(t_n)x_n|| + ||T(t_n)x_n - T(t)T(t_n)x_n|| + L_t||x_n - T(t_n)x_n|| \to 0,\ n \to \infty,
\end{aligned}$$

that is $||x_n - T(t)x_n|| \to 0, n \to \infty$. For each $m \geq 1$, let $z_m \in C$ be the unique fixed point of the contraction mapping

$$S_m x = \alpha_m\gamma f_m(x) + (I - \alpha_m A)T(t_m)x,$$

where t_m and α_m satisfy the conditions of Lemma 3.1. Then it follows from Lemma 3.1 that $\lim_{m\to\infty} z_m = x^*$. Since

$$\begin{aligned}
||z_m - x_{n+1}||^2 =& \langle \alpha_m\gamma f_m(z_m) + (I - \alpha_m A)T(t_m)z_m - x_{n+1}, j(z_m - x_{n+1})\rangle\\
=& \langle \alpha_m(\gamma f_m(z_m) - Az_m) + \alpha_m(Az_m - AT(t_m)z_m)\\
&+ (T(t_m)z_m - T(t_m)x_{n+1}) + (T(t_m)x_{n+1} - x_{n+1}), j(z_m - x_{n+1})\rangle\\
\leq& \alpha_m\langle \gamma f_m(z_m) - Az_m, j(z_m - x_{n+1})\rangle + L_{t_m}||z_m - x_{n+1}||^2\\
&+ \alpha_m||A||||z_m - T(t_m)z_m||||z_m - x_{n+1}|| + ||T(t_m)x_{n+1} - x_{n+1}||||z_m - x_{n+1}||,
\end{aligned}$$

we have

$$\begin{aligned}
\langle \gamma f_m(z_m) - Az_m, j(x_{n+1} - z_m)\rangle \leq& ||A||||z_m - T(t_m)z_m||||z_m - x_{n+1}||\\
&+ \frac{1}{\alpha_m}||T(t_m)x_{n+1} - x_{n+1}||||z_m - x_{n+1}||\\
&+ \frac{L_{t_m}-1}{\alpha_m}||z_m - x_{n+1}||^2\\
\leq& \frac{L_{t_m}-1}{\alpha_m}M^2 + ||A||||z_m - T(t_m)z_m||M\\
&+ \frac{1}{\alpha_m}||T(t_m)x_{n+1} - x_{n+1}||M,
\end{aligned} \tag{3.15}$$

where $M > 0$ is a constant such that $M \geq ||z_m - x_{n+1}||$. Therefore, firstly, taking upper limit as $n \to \infty$, and then as $m \to \infty$ in (3.15), we obtain that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle \gamma f_m(z_m) - Az_m, j(x_{n+1} - z_m)\rangle \leq 0. \tag{3.16}$$

On the other hand, since $\lim_{m\to\infty} z_m = x^*$ due to the fact that the duality map J is single-valued and norm topology to weak* topology uniformly continuous on bounded sets of E, we obtain $\lim_{m\to\infty}(x_{n+1} - z_m) = x_{n+1} - x^*$, thus

$$\langle \gamma f_1(x^*) - Ax^*, j(x_{n+1} - z_m)\rangle \to \langle \gamma f_1(x^*) - Ax^*, j(x_{n+1} - x^*)\rangle \text{ uniformly for } n, \text{ as } m \to \infty.$$

Therefore

$$\lim_{m\to\infty} H(x_n, z_m) = 0 \text{ uniformly for } n,$$

where $H(x_n, z_m) = \langle \gamma f_1(x^*) - Ax^*, j(x_{n+1} - x^*) - j(x_{n+1} - z_m)\rangle$. Moreover, by $f_m(x^*) = f_1(x^*)$ for all $m \in \mathbb{N}$, we have

$$\begin{aligned}\langle \gamma f_1(x^*) - Ax^*, j(x_{n+1} - x^*)\rangle =& \langle \gamma f_1(x^*) - Ax^*, j(x_{n+1} - x^*) - j(x_{n+1} - z_m)\rangle \\ &+ \langle \gamma f_m(x^*) - \gamma f_m(z_m), j(x_{n+1} - z_m)\rangle + \langle \gamma f_m(z_m) - Az_m, j(x_{n+1} - z_m)\rangle \\ &+ \langle Az_m - Ax^*, j(x_{n+1} - z_m)\rangle \\ \leq& \langle \gamma f_1(x^*) - Ax^*, j(x_{n+1} - x^*) - j(x_{n+1} - z_m)\rangle \\ &+ \langle \gamma f_m(z_m) - Az_m, j(x_{n+1} - z_m)\rangle + \gamma\alpha ||z_m - x^*|| ||z_m - x_{n+1}|| \\ &+ ||A|| ||z_m - x^*|| ||z_m - x_{n+1}||.\end{aligned}$$

Now we prove

$$\limsup_{m\to\infty} \limsup_{n\to\infty} H(x_n, z_m) = 0.$$

Since $\limsup_{m\to\infty} \limsup_{n\to\infty} H(x_n, z_m)$ exists, we can assume that there exist $\{x_{n_k}\} \subset \{x_n\}, \{z_{m_j}\} \subset \{z_m\}$ such that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} H(x_n, z_m) = \lim_{j\to\infty} \lim_{k\to\infty} H(x_{n_k}, z_{m_j})$$

and we can define

$$\lim_{k\to\infty} H(x_{n_k}, z_{m_j}) = W_j.$$

Since

$$\lim_{j\to\infty} H(x_{n_k}, z_{m_j}) = 0, \text{ uniformly for } k$$

there exists $J \in \mathbb{N}$, when $j > J$, we have

$$|H(x_{n_k}, z_{m_j})| < \varepsilon, \text{ uniformly for } k,$$

which means

$$|W_j| \leq \varepsilon, \quad k \to \infty.$$

Therefore

$$\limsup_{m\to\infty} \limsup_{n\to\infty} H(x_n, z_m) = \lim_{j\to\infty} W_j = 0. \tag{3.17}$$

Combining (3.16) and (3.17), we get

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle (\gamma f_1 - A)x^*, j(x_{n+1} - x^*)\rangle \leq 0.$$

Thus

$$\limsup_{n\to\infty} \langle (\gamma f_1 - A)x^*, j(x_{n+1} - x^*)\rangle \leq 0. \tag{3.18}$$

Now, it follows form (3.9) that

$$||y_n - x^*|| \le [1 + c_n(L_{t_n} - 1)]||x_n - x^*|| + \sigma_n||v_n - x^*||,$$

which together with the iterative process (3.8) implies the following estimates

$$\begin{aligned}||x_{n+1} - x^*||^2 =&\langle \alpha_n(\gamma f_n(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1 - \beta_n - \delta_n)I \\ &- \alpha_n A)(T(t_n)y_n - x^*) + \delta_n(u_n - x^*), j(x_{n+1} - x^*)\rangle \\ =&\langle \alpha_n(\gamma f_n(x_n) - \gamma f_n(x^*)) + \beta_n(x_n - x^*) + ((1 - \beta_n - \delta_n)I \\ &- \alpha_n A)(T(t_n)y_n - x^*) + \delta_n(u_n - x^*) + \alpha_n(\gamma f_n(x^*) - Ax^*), j(x_{n+1} - x^*)\rangle \\ \le&\alpha_n\gamma\alpha||x_n - x^*||||x_{n+1} - x^*|| + \beta_n||x_n - x^*||||x_{n+1} - x^*|| \\ &+ \delta_n||u_n - x^*||||x_{n+1} - x^*|| + \alpha_n\langle(\gamma f_1(x^*) - Ax^*), j(x_{n+1} - x^*)\rangle \\ &+ (1 - \beta_n - \delta_n - \alpha_n\overline{\gamma})L_{t_n}||y_n - x^*||||x_{n+1} - x^*|| \\ \le&\alpha_n\gamma\alpha||x_n - x^*||||x_{n+1} - x^*|| + \beta_n||x_n - x^*||||x_{n+1} - x^*|| \\ &+ \delta_n||u_n - x^*||||x_{n+1} - x^*|| + \alpha_n\langle(\gamma f_1(x^*) - Ax^*), j(x_{n+1} - x^*)\rangle \\ &+ (1 - \beta_n - \delta_n - \alpha_n\overline{\gamma})\sigma_n L_{t_n}||v_n - x^*||||x_{n+1} - x^*|| \\ &+ (1 - \beta_n - \delta_n - \alpha_n\overline{\gamma})[1 + c_n(L_{t_n} - 1)][1 \\ &+ (L_{t_n} - 1)]||x_n - x^*||||x_{n+1} - x^*|| \\ \le&[1 - \alpha_n(\overline{\gamma} - \gamma\alpha)]||x_n - x^*||||x_{n+1} - x^*|| + (L_{t_n} - 1)(1 \\ &+ c_n L_{t_n})||x_n - x^*||||x_{n+1} - x^*|| + \alpha_n\langle(\gamma f_1(x^*) - Ax^*), j(x_{n+1} - x^*)\rangle \\ &+ \delta_n||u_n - x^*||||x_{n+1} - x^*|| + \sigma_n L_{t_n}||v_n - x^*||||x_{n+1} - x^*|| \\ \le&[1 - \alpha_n(\overline{\gamma} - \gamma\alpha)]\frac{||x_n - x^*||^2 + ||x_{n+1} - x^*||^2}{2} \\ &+ (L_{t_n} - 1)(1 + c_n L_{t_n})||x_n - x^*||||x_{n+1} - x^*|| + \alpha_n\langle(\gamma f_1(x^*) - Ax^*), j(x_{n+1} - x^*)\rangle \\ &+ \delta_n||u_n - x^*||||x_{n+1} - x^*|| + \sigma_n L_{t_n}||v_n - x^*||||x_{n+1} - x^*||,\end{aligned}$$

and thus

$$\begin{aligned}||x_{n+1} - x^*||^2 \le&[1 - \alpha_n(\overline{\gamma} - \gamma\alpha)]||x_n - x^*||^2 \\ &+ 2\alpha_n\left[(L_{t_n} - 1)\alpha_n^{-1}(1 + L_{t_1})||x_n - x^*||||x_{n+1} - x^*|| \right. \\ &\left. + \langle(\gamma f_1(x^*) - Ax^*), j(x_{n+1} - x^*)\rangle\right] + 2\delta_n M' + 2\sigma_n L_{t_1} M',\end{aligned}$$

where $M' = \max_n\{||u_n - x^*||||x_{n+1} - x^*||, ||v_n - x^*||||x_{n+1} - x^*||\} \ge 0$. Consequently, by Corollary 2.6 and (3.18), we obtain that

$$\lim_{n\to\infty} x_n = x^*.$$

The proof is completed. □

Remark 3.3.

(i) Theorem 3.2 extends Theorem 3.4 of Marino and Xu [17] from a real Hilbert space to a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and from nonexpansive mappings to asymptotically nonexpansive semigroups.

(ii) Theorem 3.2 extends Theorem 4.2 of Song and Xu [23] from nonexpansive semigroups to asymptotically nonexpansive semigroups.

(iii) Taking $T(t_1) = T, h = t_1, \delta_n = c_n = \sigma_n \equiv 0, A = I, \gamma = 1,$ and $f_n \equiv f_1$ in Theorem 3.2 and then $C \pm C \subset C$ is not necessary. We get Theorem 2.2 of Lou et al. [15] and generalize it from a uniformly convex Banach space to a reflexive and strictly convex Banach space.

(iv) Taking $\delta_n = c_n = \sigma_n \equiv 0, A = I, \gamma = 1$, and $f_n \equiv u$ in Theorem 3.2 and then $C \pm C \subset C$ is not necessary. We get Theorem 3.3 of Zegeye and Shahzad [39].

(v) Our results completely generalize the results of Yang [32].

Corollary 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a uniformly Gáteaux differentiable norm, $C \pm C \subset C$. Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a u.a.r. nonexpansive semigroup on C with a sequence $\{L_t\} \subset [1, \infty)$ such that $F(\mathcal{T}) \neq \emptyset$, and $\{f_n\} \subset \Pi_C$ is stable on $F(\mathcal{T})$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$, $A(C) \subset C$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence defined by (3.8) satisfying*

(1) $\alpha_n \in (0, 1),\ \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{L_{t_n}-1}{\alpha_n} = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\beta_n \in (0, 1),\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(3) $\delta_n, \sigma_n \in [0, 1],\ \sum_{n=1}^{\infty} \delta_n < \infty,\ \sum_{n=1}^{\infty} \sigma_n < \infty$;

(4) $h, t_n \geq 0,\ t_{n+1} = t_n + h,\ \lim_{n\to\infty} t_n = \infty$;

(5) $c_n \in [0, 1],\ \lim_{n\to\infty} |c_{n+1} - c_n| = 0,\ \limsup_{n\to\infty} c_n < 1$.

Suppose $\{u_n\}$ and $\{v_n\}$ are bounded in C, then as $n \to \infty$, the sequence $\{x_n\}$ converges strongly to some common fixed point x^* of $F(\mathcal{T})$ which is the unique solution in $F(\mathcal{T})$ to the variational inequality (3.2).

Corollary 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H, $C \pm C \subset C$. Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a u.a.r. nonexpansive semigroup on C with a sequence $\{L_t\} \subset [1, \infty)$ such that $F(\mathcal{T}) \neq \emptyset$, and $\{f_n\} \subset \Pi_C$ is stable on $F(\mathcal{T})$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$, $A(C) \subset C$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence defined by (3.8) satisfying*

(1) $\alpha_n \in (0, 1),\ \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{L_{t_n}-1}{\alpha_n} = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\beta_n \in (0, 1),\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(3) $\delta_n, \sigma_n \in [0, 1],\ \sum_{n=1}^{\infty} \delta_n < \infty,\ \sum_{n=1}^{\infty} \sigma_n < \infty$;

(4) $h, t_n \geq 0,\ t_{n+1} = t_n + h,\ \lim_{n\to\infty} t_n = \infty$;

(5) $c_n \in [0, 1],\ \lim_{n\to\infty} |c_{n+1} - c_n| = 0,\ \limsup_{n\to\infty} c_n < 1$.

Suppose $\{u_n\}$ and $\{v_n\}$ are bounded in C, then as $n \to \infty$, the sequence $\{x_n\}$ converges strongly to some common fixed point x^* of $F(\mathcal{T})$ which is the unique solution in $F(\mathcal{T})$ to the variational inequality (3.2).

Remark 3.6. Since every nonexpansive semigroup is asymptotically nonexpansive semigroup, our theorems hold for the case when $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ is simply nonexpansive semigroup.

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