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On the existence of solutions of generalized equilibrium problems with α-β-η-monotone mappings

A. Farajzadeh[a], S. Plubtieng[b,c,*], A. Hosseinpour[b,c]

[a] *Department of Mathematics, Razi University, Kermanshah 67149, Iran.*
[b] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.*
[c] *Center of Excellence in Nonlinear Analysis and Optimization, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.*

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Abstract

The present paper is concerned with the new concept of relaxed α-β-η-monotonicity and relaxed α-β-η-pseudomonotonicity in Banach space which is applied to prove the existence of solutions of generalized equilibrium problem and classic equilibrium problem. In this regard, we use the well-known KKM-theory to obtain solutions of mentioned problems.

Keywords: KKM-mappings, hemicontinuity, α-β-η-monotonicity, α-β-η-pseudomonotonicity, semicontinuous mappings, Banach space.
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1. Introduction

This work focuses on the existence of solutions of generalized equilibrium problems with the new concept of relaxed α-β-η-monotonicity. The most important application of generalized equilibrium problems is in economics [1, 3], variational inequalities [5], optimization, fixed point theory [6] and so on. Over the last few years, the concept of generalized equilibrium problems has been studied by various authors and has developed rapidly (see [2, 13, 14, 17, 18]). Onjai-uea and his colleagues in [15] presented a relaxed hybrid

*Corresponding author
Email addresses: `farajzadehali@gmail.com` (A. Farajzadeh), `Somyotp@nu.ac.th` (S. Plubtieng), `h.mathematical@gmail.com` (A. Hosseinpour)

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steepest method to find a common element for the set of fixed points of a nonexpansive mapping, the set of solutions of a variational inequality for an inverse-strongly monotone mapping and the set of solutions of generalized mixed equilibrium problems in Hilbert spaces. In 2013, Mahato and Nahak published a paper in which they obtained the existence results for mixed equilibrium problems in a reflexive Banach space [12]. Ding and his colleagues considered a collectively fixed point theorem and an equilibrium existence theorem for generalized games in product locally G-convex uniform spaces [8]. However, in recent years, the iterative algorithms of solutions for generalized equilibrium problems have been studied by several authors. For instance, a new class of generalized mixed implicit equilibrium-like problems has been introduced by Ding [7]. He used the auxiliary principle technique to obtain the solution of the mentioned problem. Zang and Deng in [19] studied the multi-valued general mixed implicit equilibrium-like problems and presented a new predictor corrector iterative algorithm by using the auxiliary principle technique. They also proved the convergence of the suggested algorithm in weaker conditions. One can refer to [4, 9, 11] for more details.

2. Preliminaries

This work has been done in real Banach space X. In this work, K is considered as a nonempty convex subset of real Banach space X. In our study, we deal with the following generalized equilibrium problem:

Find $\overline{x} \in K$ such that

$$f(\overline{x}, y) + \varphi(\overline{x}, y) - \varphi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K, \tag{2.1}$$

where $f : K \times K \longrightarrow R$ is an equilibrium function, that is, $f(x, x) = 0$, for all $x \in K$, and $\varphi : K \times K \longrightarrow R$ is a real valued function.

If $\varphi \equiv 0$, problem (2.1) reduces to the following equilibrium problem of finding $\overline{x} \in K$ such that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in K. \tag{2.2}$$

Now, we present some fundamental definitions which will be used in the rest of this paper.

Definition 2.1. A function $f : K \longrightarrow R$ is said to be hemicontinuous at $y \in K$, if and only if $\lim_{t \to 0^+} f(tx + (1 - t)y) = f(y)$, for each $x \in K$.

Note that every continuous function is hemicontinuous, but the converse is not necessarily true. Have a look at the following example.

Example 2.2. The function $f : R \times R \longrightarrow R$ defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0), \end{cases}$$

is hemicontinuous on $R \times R$, but not continuous at $(0, 0)$.

Definition 2.3. Let X be a Banach space. A single-valued mapping $f : X \longrightarrow R$ is called

1. weakly upper semicontinuous (u.s.c.) at $x_0 \in X$, if

$$f(x_0) \geq \limsup_n f(x_n)$$

for any sequence $\{x_n\}$ of X which converges to x_0 weakly;

2. weakly lower semicontinuous (l.s.c.) at $x_0 \in X$, if

$$f(x_0) \leq \liminf_n f(x_n)$$

for any sequence $\{x_n\}$ of X which converges to x_0 weakly.

Definition 2.4. A multi-valued mapping $f : K \longrightarrow 2^X$ is called a KKM-mapping, if for any $\{y_1, \ldots, y_n\} \subset K$, $\text{co}\{y_1, \ldots, y_n\} \subset \bigcup_{i=1}^n f(y_i)$, where 2^X denotes the family of all nonempty subsets of X and co denotes the convex hull.

Example 2.5. Let $K = [0, 1]$ and $X = R$. In this case, the following mapping is a KKM-mapping.

$$f : [0, 1] \longrightarrow 2^R$$
$$f(x) \longmapsto [0, x].$$

Lemma 2.6 ([10])**.** *Let K be a nonempty subset of a topological vector space X and let $f : K \longrightarrow 2^X$ be a KKM-mapping. If $f(y)$ is closed in X, for all $y \in K$ and compact for at least one $y \in K$, then*

$$\bigcap_{y \in K} f(y) \neq \emptyset.$$

In the following, let us introduce a new definition of relaxed α-β-η-monotone which is significant in our research.

Definition 2.7. The mapping $f : K \times K \longrightarrow R$ is called relaxed α-β-η-monotone, if there exist mappings $\eta : K \times K \longrightarrow X$, $\alpha : X \longrightarrow R$ and $\beta : K \times K \longrightarrow R$ such that

$$f(x, y) + f(y, x) \leq \alpha(\eta(x, y)) + \beta(x, y), \quad \forall x, y \in K,$$

and

$$\liminf_{t \to 0^+} \left[\frac{\alpha(\eta(x, y))}{t} + \frac{\beta(x, ty + (1 - t)x)}{t} \right] \leq 0.$$

Remark that, if $\alpha = 0$ and $\beta = 0$, then the definition reduces to the definition of monotonicity of f. Hence, Definition 2.7 is an extension of monotonicity.

Example 2.8. Let $\alpha(x) = -1$, $\beta = 0$ and η be an arbitrary function, hence

$$\liminf_{t \to 0^+} \left[\frac{\alpha(\eta(x, y))}{t} + \frac{\beta(x, ty + (1 - t)x)}{t} \right] = -\infty \leq 0.$$

If we choose $f(x, y) = -2$, in this case f is α-β-η-monotone with respect to Definition 2.7, but f is not α-β-monotone with respect to Definition 6 in [16].

3. Existence results for α-β-η-monotone mappings

We start this section with the following theorem which is an existence result of solution of problem (2.1).

Theorem 3.1. *Let $f : K \times K \longrightarrow R$ be relaxed α-β-η-monotone, hemicontinuous in the first argument and convex in the second argument with $f(x, x) = 0$, for all $x \in K$. Let $\varphi : K \times K \longrightarrow R$ be convex in the second argument. Then, the solution set of generalized equilibrium problem* (2.1) *is equal to the solution set of the following problem:*

Find $\overline{x} \in K$ such that

$$f(y, \overline{x}) + \varphi(\overline{x}, \overline{x}) - \varphi(\overline{x}, y) \leq \alpha(\eta(\overline{x}, y)) + \beta(\overline{x}, y), \quad \forall y \in K. \tag{3.1}$$

Proof. Let problem (2.1) have a solution, then

$$\exists\, \overline{x} \in K \text{ such that } f(\overline{x}, y) + \varphi(\overline{x}, y) - \varphi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K.$$

It follows from the α-β-η-monotonicity of f that

$$f(\overline{x}, y) + f(y, \overline{x}) \leq \alpha(\eta(\overline{x}, y)) + \beta(\overline{x}, y), \quad \forall y \in K. \tag{3.2}$$

According to problem (2.1) and equation (3.2), we get

$$\begin{aligned} f(y,\overline{x})+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},y) &\le \alpha(\eta(\overline{x},y))+\beta(\overline{x},y)-[f(\overline{x},y)+\varphi(\overline{x},y)-\varphi(\overline{x},\overline{x})] \\ &\le \alpha(\eta(\overline{x},y))+\beta(\overline{x},y), \quad \forall y\in K. \end{aligned}$$

So, $\overline{x}\in K$ is a solution of problem (3.1). Conversely, let $\overline{x}\in K$ be a solution of problem (3.1). Therefore,

$$f(y,\overline{x})+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},y)\le \alpha(\eta(\overline{x},y))+\beta(\overline{x},y), \quad \forall y\in K. \tag{3.3}$$

Let $y\in K$ and t be an arbitrary element of $[0,1]$. Obviously, $x_t=ty+(1-t)\overline{x}\in K$. Hence, from (3.3), we obtain

$$f(x_t,\overline{x})+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},x_t)\le \alpha(\eta(\overline{x},x_t))+\beta(\overline{x},x_t), \quad \forall t\in(0,1]. \tag{3.4}$$

Since f is convex in the second variable, we get

$$0=f(x_t,x_t)\le tf(x_t,y)+(1-t)f(x_t,\overline{x}), \tag{3.5}$$

and from the convexity φ in the second argument, we also have

$$\varphi(\overline{x},x_t)\le t\varphi(\overline{x},y)+(1-t)\varphi(\overline{x},\overline{x}). \tag{3.6}$$

It follows from (3.4)-(3.6) that

$$\begin{aligned} t[f(x_t,\overline{x})-f(x_t,y)+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},y)] &\le f(x_t,\overline{x})+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},x_t) \\ &\le \alpha(\eta(\overline{x},x_t))+\beta(\overline{x},x_t), \end{aligned}$$

which implies that

$$f(x_t,\overline{x})-f(x_t,y)+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},y)\le \frac{\alpha(\eta(\overline{x},x_t))}{t}+\frac{\beta(\overline{x},x_t)}{t}.$$

According to hemicontinuity of f in the first argument and the definition of relaxed α-β-η-monotone of f, by taking $t\to 0^+$, we have

$$f(\overline{x},\overline{x})-f(\overline{x},y)+\varphi(\overline{x},\overline{x})-\varphi(\overline{x},y)\le 0, \quad \forall y\in K,$$

and so, note $f(\overline{x},\overline{x})=0$,

$$f(\overline{x},y)+\varphi(\overline{x},y)-\varphi(\overline{x},\overline{x})\ge 0, \quad \forall y\in K.$$

Hence, $\overline{x}\in K$ is a solution of problem (2.1) which completes the proof. □

In what follows, we demonstrate that problem (2.1) admits a solution. This topic stated in the next theorem is the most important issue in our work.

Theorem 3.2. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Let $f:K\times K\longrightarrow R$ be relaxed α-β-η-monotone, hemicontinuous in the first argument, convex in the second argument with $f(x,x)=0$, $\varphi:K\times K\longrightarrow R$ be convex in the second variable, $\alpha:K\longrightarrow R$ be weakly upper semi-continuous and $\beta:K\times K\longrightarrow R$ be weakly upper semi-continuous in the second argument. Then, problem* (2.1) *admits a solution.*

Proof. Let $F:K\longrightarrow 2^X$ be a multi-valued mapping defined by

$$F(y)=\{x\in K \mid f(x,y)+\varphi(x,y)-\varphi(x,x)\ge 0\}.$$

Obviously, $\overline{x}\in K$ is a solution of equation (2.1), if and only if $\overline{x}\in\bigcap_{y\in K}F(y)$. We are going to show that $\bigcap_{y\in K}F(y)\neq\emptyset$. We claim that F is a KKM-mapping. Suppose to the contrary that F is not a KKM-mapping. So there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $\mathrm{co}\{x_1, \ldots, x_n\} \not\subseteq \bigcup_{i=1}^n F(x_i)$. Therefore, there exists $x_0 \in \mathrm{co}\{x_1, \ldots, x_n\}$ where for all $i \in \{1, \ldots, n\}$, $x_0 \notin F(x_i)$. Hence, for $i = 1, 2, \ldots, n$, we have

$$f(x_0, x_i) + \varphi(x_0, x_i) - \varphi(x_0, x_0) < 0. \tag{3.7}$$

Thus, there exist $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\Sigma_{i=1}^n \lambda_i = 1$ such that $x_0 = \sum_{i=1}^n \lambda_i x_i$. By multiplying both sides of relation (3.7) by λ_i and adding them, we obtain

$$\sum_{i=1}^{n} \lambda_i [f(x_0, x_i) + \varphi(x_0, x_i) - \varphi(x_0, x_0)] < 0.$$

This and our assumptions on f and φ lead us to the contradiction $0 < 0$. Hence, the multi-valued mapping F is a KKM mapping.

We define the multi-valued mapping $G : K \longrightarrow 2^X$ by

$$G(y) = \{x \in K : f(y, x) + \varphi(x, x) - \varphi(x, y) \leq \alpha(\eta(x, y)) + \beta(x, y)\}.$$

It is clear that $F(y)$ is a subset of $G(y)$, for all $y \in K$. Because, let y be an arbitrary element of K and $\overline{x} \in F(y)$, then

$$f(\overline{x}, y) + \varphi(\overline{x}, y) - \varphi(\overline{x}, \overline{x}) \geq 0.$$

The relaxed α-β-η-monotoneicity of f implies that

$$\begin{aligned} f(y, \overline{x}) + \varphi(\overline{x}, \overline{x}) - \varphi(\overline{x}, y) &\leq \alpha(\eta(\overline{x}, y)) + \beta(\overline{x}, y) - [f(\overline{x}, y) + \varphi(\overline{x}, y) - \varphi(\overline{x}, \overline{x})] \\ &\leq \alpha(\eta(\overline{x}, y)) + \beta(\overline{x}, y), \end{aligned}$$

and so $\overline{x} \in G(y)$. Then, $F(y) \subset G(y)$. Since F is a KKM-mapping and $F(y) \subset G(y)$, then G is a KKM-mapping. According to the conditions on the mappings, it is easy to verify that $G(y)$ is weakly closed, for all $y \in K$. Since K is a bounded, closed and convex subset of the reflexive Banach space X, then it is weakly compact and consequently $G(y)$ is weakly compact in K, for all $y \in K$. Consequently, it follows from Lemma 2.6 that $\bigcap_{y \in K} G(y) \neq \emptyset$, and from Theorem 3.1 that $\bigcap_{y \in K} F(y) = \bigcap_{y \in k} G(y)$. Thus, $\bigcap_{y \in K} F(y) \neq \emptyset$. Hence, there exists $\overline{x} \in K$ such that

$$f(\overline{x}, y) + \varphi(\overline{x}, y) - \varphi(\overline{x}, \overline{x}) \geq 0, \quad \forall y \in K.$$

So, the solution set of problem (2.1) is nonempty. This completes the proof. □

Example 3.3. Let $K = [0, 1]$, $\alpha(x) = -x$, $\beta(x, y) = 0$ and $\eta(x, y) = (x + y)(x - y)^2$. If we choose $f(x, y) = x(y^2 - x^2)$ and $\varphi(x, y) = x^2 + y^2$, then all assumptions of Theorem 3.2 hold. Therefore, problem (2.1) is solvable. It is easy to see that $\overline{x} = 0$ is the only solution of problem (2.1).

4. Existence results for α-β-η-pseudomonotone mappings

In this section, we introduce the concept of relaxed $\alpha - \beta - \eta-$pseudomonotonicity and discuss the existence solution of equilibrium problems (2.1) and (2.2) using this concept.

Definition 4.1. A mapping $f : K \times K \longrightarrow R$ is called relaxed α-β-η-pseudomonotone, if there exist functions $\eta : K \times K \longrightarrow X$, $\alpha : X \longrightarrow R$ and $\beta : K \times K \longrightarrow R$ such that for any $x, y \in K$, we have

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq \alpha(\eta(y, x)) + \beta(y, x),$$

where

$$\liminf_{t \to 0^+} \left[\frac{\alpha(\eta(x, y))}{t} + \frac{\beta(x, ty + (1 - t)x)}{t} \right] \leq 0.$$

If we take $\alpha = \beta = 0$, then the definition of relaxed α-β-η-pseudomonotonicity collapses to the usual definition of pseudomonotonicity. Moreover, note that each relaxed α-β-η-monotone mapping is relaxed α-β-η-pseudomonotone mapping. The following example shows that the inverse is not always true.

Example 4.2. Consider $X = R$, $K = [0, 1]$ and $f(x, y) = x - y$. We choose $\alpha(x) = -x$, $\beta(x, y) = 0$ and $\eta(x, y) = |x - y|$. If $f(x, y) \geq 0$, then $x - y \geq 0$. Hence, $f(y, x) = y - x \leq -|x - y| = \alpha(\eta(y, x)) + \beta(y, x)$ and

$$\liminf_{t\to 0^+} \left[\frac{\alpha(\eta(x,y))}{t} + \frac{\beta(x, ty + (1-t)x)}{t} \right] = -\infty \leq 0.$$

Therefore, f is relaxed α-β-η-pseudomonotone. Whereas, f is not relaxed α-β-η-monotone.

Theorem 4.3. *Let $f : K \times K \longrightarrow R$ be generalized relaxed α-β-η-pseudomonotone, hemicontinuous in the first argument and convex in the second argument with $f(x, x) = 0$, for all $x \in K$. Then, generalized equilibrium problem (2.2) is equivalent to the following problem:*

Find $\overline{x} \in K$ such that

$$f(y, \overline{x}) \leq \alpha(\eta(y, \overline{x})) + \beta(y, \overline{x}), \quad \forall y \in K. \tag{4.1}$$

Proof. Let $\overline{x} \in K$ be a solution of problem (2.2), that is

$$f(\overline{x}, y) \geq 0, \quad \forall y \in K.$$

So, by the relaxed α-β-η-pseudomonotonicity of f, we get

$$f(y, \overline{x}) \leq \alpha(\eta(y, \overline{x})) + \beta(y, \overline{x}), \quad \forall y \in K.$$

Hence, $\overline{x} \in K$ is a solution of problem defined by (4.1).

Conversely, assume that $\overline{x} \in K$ is a solution of (4.1). Then, for any $y \in K$, let $x_t = ty + (1 - t)\overline{x}$, $t \in (0, 1]$. Obviously, $x_t \in K$, and it follows that

$$f(x_t, \overline{x}) \leq \alpha(\eta(x_t, \overline{x})) + \beta(x_t, \overline{x}). \tag{4.2}$$

Since f is convex in the second argument, we obtain

$$0 = f(x_t, x_t) \leq tf(x_t, y) + (1 - t)f(x_t, \overline{x}). \tag{4.3}$$

Equations (4.2) and (4.3) imply that

$$f(x_t, \overline{x}) - f(x_t, y) \leq \frac{\alpha(\eta(x_t, \overline{x}))}{t} + \frac{\beta(x_t, \overline{x})}{t}, \quad \forall y \in K.$$

Hemicontinuity of f in the first argument and the definition of relaxed α-β-η-monotone of f, by taking $t \to 0^+$ imply that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in K.$$

Hence, $\overline{x} \in K$ is a solution of problem (2.2), and it completes the proof. □

Theorem 4.4. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Let $f : K \times K \longrightarrow R$ be relaxed α-β-η-pseudomonotone, hemicontinuous in the first argument, convex in the second argument with $f(x, x) = 0$. Moreover, $\alpha : K \longrightarrow R$ is weakly upper semicontinuous and $\beta : K \times K \longrightarrow R$ is weakly upper semicontinuous in the second argument. Then, problem (2.2) admits a solution.*

Proof. Let $F : K \longrightarrow 2^X$ be defined by

$$F(y) = \{x \in K \mid f(x, y) \geq 0\}.$$

It is clear that $\overline{x} \in K$ is a solution of problem (2.2), if and only if $\overline{x} \in \bigcap_{y\in K} F(y)$. Hence, we prove that $\bigcap_{y\in K} F(y) \neq \emptyset$.

It is easy to see that F is a KKM-mapping. Because, otherwise, there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $\mathrm{co}\{x_1, \ldots, x_n\} \not\subseteq \bigcup_{i=1}^n F(x_i)$. This means that there exists $x_0 \in \mathrm{co}\{x_1, \ldots, x_n\}$ such that $f(x_0, x_i) < 0$, for $i = 1, \ldots, n$. Thus, there exist $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\Sigma_{i=1}^n \lambda_i = 1$ such that $x_0 = \sum_{i=1}^n \lambda_i x_i$. Hence,

$$\sum_{i=1}^n \lambda_i f(x_0, x_i) < 0.$$

According to the convexity of f in the second variable, we reach the contradiction $0 < 0$. Hence, F is a KKM-mapping.

Define the set-valued mapping $G : K \longrightarrow 2^X$ by

$$G(y) = \{x \in K \mid f(y, x) \leq \alpha(\eta(y, x)) + \beta(y, x)\}.$$

The relaxed α-β-η-pseudomonotonicity of f implies that $F(y) \subseteq G(y)$, for all $y \in K$. Hence, G is also a KKM-mapping.

By the hypothesis on the mappings, the values of the multi-valued mapping G are weakly closed and since K is a closed bounded subset of the reflexive Banach space X, then $G(y)$ is weakly compact, for all $y \in K$. Hence, the multi-valued mapping G satisfies all assumptions of Lemma 2.6 and then $\bigcap G(y)$ is nonempty and hence by Theorem 4.3, $\bigcap F(y)$ is nonempty. Consequently, there exists $\overline{x} \in K$ such that $f(\overline{x}, y) \geq 0$, for all $y \in K$ which completes the proof. □

Example 4.5. Let $K = [0, \frac{3}{2}]$, $\alpha(x) = -x$, $\beta = 0$ and $\eta(x, y) = |x - y|$. If we choose $f(x, y) = (x - y)\cos(y)$, then all assumptions of Theorem 4.4 hold. Therefore, problem (2.2) admits a solution. It is easy to see that $x = \frac{3}{2}$ is a solution of this problem.

5. Conclusion

To sum up, we have introduced a new concept of relaxed α-β-η-monotonicity and have applied the well-known KKM-theory to obtain some existence results for solutions of generalized equilibrium problems. Moreover, we have proven the existence of solutions of equilibrium problems by using the new concept of relaxed α-β-η-pseudomonotonicity and KKM-theory.

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References

[1] C. L. Ballard, D. Fullerton, J. B. Shoven, J. Whalley, *A general equilibrium model for tax policy evaluation*, University of Chicago Press, Chicago, (1985). 1

[2] M. Binachi, S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl., **90** (1996), 31–43. 1

[3] G. Bonanno, *General equilibrium theory with imperfect competition*, J. Econ. Surv., **4** (1990), 297–328. 1

[4] L.-C. Ceng, C.-M. Chen, C.-F. Wen, C. T. Pang, *Relaxed iterative algorithms for generalized mixed equilibrium problems with constraints of variational inequalities and variational inclusions*, Abstr. Appl. Anal., **2014** (2014), 25 pages. 1

[5] O. Chadli, Z. Chbani, H. Riahi, *Equilibrium problem with generalized monotone bifunctions and applications to variational inequalities*, J. Optim. Theory Appl., **105** (2000), 299–323. 1

[6] Y. J. Cho, N. Petrot, *An optimization problem related to generalized equilibrium and fixed point problems with applications*, Fixed Point Theory, **11** (2010), 237–250. 1

[7] X.-P. Ding, *Iterative algorithm of solutions for generalized mixed implicit equilibrium-like problems*, Appl. Math. Comput., **162** (2005), 799–809. 1

[8] X.-P. Ding, J.-C. Yao, L.-J. Lin, *Solutions of system of generalized vector quasi-equilibrium problems in locally G-convex uniform spaces*, J. Math. Anal. Appl., **298** (2004), 398–410. 1

[9] J. N. Ezeora, Y. Shehu, *An iterative method for mixed point problems of nonexpansive and monotone mappings and generalized equilibrium problems*, Common. Math. Anal., **12** (2012), 76–95. 1

[10] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142** (1961), 305–310. 2.6

[11] Y. F. Ke, C. F. Ma, *A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem*, Fixed Point Theory Appl., **2013** (2013), 21 pages. 1

[12] N. K. Mahato, C. Nahak, *Mixed equilibrium problems with relaxed α-monotone mapping in Banach spaces*, Rend. Circ. Mat. Palermo, **62** (2013), 207–213. 1

[13] N. K. Mahato, C. Nahak, *Equilibrium problems with generalized relaxed monotonicities in Banach spaces*, Opsearch, **51** (2014), 257–269. 1

[14] M. A. Noor, *Auxiliary principle technique for equilibrium problems*, J. Optim. Theory Appl., **122** (2004), 371–386. 1

[15] N. Onjai-uea, C. Jaiboon, P. Kumam, *A relaxed hybrid steepest descent method for common solutions of generalized mixed equilibrium problems and fixed point problems*, Fixed Point Theory Appl., **2011** (2011), 20 pages. 1

[16] H. A. Rizvi, A. Kılıçman, R. Ahmad, *Generalized equilibrium problem with mixed relaxed monotonicity*, Sci. World J., **2014** (2014), 4 pages. 2.8

[17] W. Takahashi, K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Nonlinear Anal., **70** (2009), 45–57. 1

[18] S. H. Wang, G. Marino, F. H. Wang, *Strong convergence theorems for a generalized equilibrium problem with a relaxed monotone mapping and a countable family of nonexpansive mappings in a Hilbert space*, Fixed Point Theory Appl., **2010** (2010), 22 pages. 1

[19] X.-Y. Zang, L. Deng, *Iterative algorithm of solutions for multivalued general mixed implicit equilibrium-like problems*, Appl. Math. Mech. (English Ed.), **29** (2008), 477–484. 1