Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5733–5740
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Positive solutions for fractional differential equation involving the Riemann-Stieltjes integral conditions with two parameters

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Communicated by Y. J. Cho

Abstract

Through the application of the upper-lower solutions method and the fixed point theorem on cone, under certain conditions, we obtain that there exist appropriate regions of parameters in which the fractional differential equation has at least one or no positive solution. In the end, an example is worked out to illustrate our main results.

Keywords: Fractional differential equation, Riemann-Stieltjes integral conditions, upper-lower solutions, the fixed point theorem.
2010 MSC: 26A33, 34B16, 34B18.

1. Introduction

Fractional order equations can describe the characteristics exhibited in numerous complex processes and systems having long-memory in time, and due to this reason a large number of classical integer-order models for complicated systems are being substituted by fractional order models, so fractional order equations have proven to be strong tools in the modeling of phenomena arising from heat conduction, chemical engineering, underground water flow, plasma physics, and also in various field of science and engineering.

The purpose of this paper is to study the following fractional differential equation involving the Riemann-Stieltjes integral conditions and two parameters.

$$\begin{cases} D_{0+}^{\alpha} x(t) + \lambda a(t) f(t, x(t)) = 0, \ 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \mu \displaystyle\int_0^1 h(x(t)) dA(t), \end{cases} \tag{$P_{\lambda,\,\mu}$}$$

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Received 2016-01-07

where $n-1<\alpha\leq n$, $n\geq 3$, D^{α}_{0+} is the standard Riemann-Liouville derivative, $\lambda,\ \mu>0$ are parameters, A is right continuous on $[0,1)$, left continuous at $t=1$, and nondecreasing on $[0,1]$, $A(0)=0$, $\int_0^1 x(t)dA(t)$ denotes the Riemann-Stieltjes integral of x with respect to A, $a:(0,1)\to[0,+\infty)$ is continuous and may be singular at $t=0,1$, and $h:[0,+\infty)\to[0,+\infty)$ and $f:[0,1]\times[0,+\infty)\to(0,+\infty)$ are continuous functions.

Due to the wide application of fractional order differential equations, there are many studies which focus on the solvability of fractional differential equations. For some recent results on this topic, see [1, 4, 6, 7, 9, 11, 12, 14, 15] and the references therein. El-Shahed [3] considered the following fractional order differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t)+\lambda a(t)f(t,u(t))=0,\ 0<t<1,\\ u(0)=u'(0)=u'(1)=0,\end{cases}$$

where $2<\alpha<3$, D^{α}_{0+} is the standard Riemann-Liouville derivative, $\lambda>0$ is a parameter, and $a:(0,1)\to[0,+\infty)$ and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ are continuous functions. Through the use of the Krasnosel'skii fixed point theorem on cone expansion and compression, the author in [3] showed the existence and non-existence of positive solutions for the above fractional boundary value problem.

With the same equation as $(P_{\lambda,\ \mu})$, $n=3$, and the boundary conditions become $x(0)=x'(0)=0,\ x(1)=\int_0^1 h(s)x(s)ds$, zhao et al. in [15] obtained the existence results of positive solutions by using the standard tools of the Krasnosel'skii fixed point theorem when the parameter λ lies in some intervals.

In this paper, we discuss the fractional differential equation $(P_{\lambda,\ \mu})$, which is involved the Riemann-Stieltjes integral conditions and two parameters λ,μ, and we find a function Υ about λ,μ, such that $(P_{\lambda,\ \mu})$ has at least one positive solution for $0<\mu\leq\Upsilon(\lambda)$ and has no positive solutions for $\mu>\Upsilon(\lambda)$.

2. Preliminaries and lemmas

Definition 2.1 ([8, 10])**.** Let $\alpha>0$ and let u be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of $[0,+\infty)$. Then for $t>0$, we call

$$I^{\alpha}_{0+}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

the Riemann-Liouville fractional integral of u of order α.

Definition 2.2 ([8, 10])**.** The Riemann-Liouville fractional derivative of order $\alpha>0$, $n-1\leq\alpha<n$, $n\in\mathbb{N}$, is defined as

$$D^{\alpha}_{0+}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $\mathbb{N}$ denotes the natural numbers set and the function $u(t)$ is n times continuously differentiable on $[0,+\infty)$.

Lemma 2.3 ([8, 10])**.** *Let $\alpha>0$, if the fractional derivatives $D^{\alpha-1}_{0+}u(t)$ and $D^{\alpha}_{0+}u(t)$ are continuous on $[0,+\infty)$, then,*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_1,c_2,\cdots,c_n\in(-\infty,+\infty)$, and n is the smallest integer greater than or equal to α.

Lemma 2.4. *Under the condition $y\in C(0,1)\cap L^1(0,1)$, the fractional boundary value problem*

$$\begin{cases} D^{\alpha}_{0+}x(t)+y(t)=0,\ 0<t<1,\ n-1<\alpha\leq n,\ n\geq 2,\\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0,\\ x(1)=\mu\displaystyle\int_0^1 h(x(t))dA(t)\end{cases}\tag{2.1}$$

has a unique integral representation

$$x(t) = \int_0^1 G(t,s)y(s)ds + \mu t^{\alpha-1}\int_0^\infty h(x(t))dA(t),$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1,\\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

Proof. By Lemma 2.3, the boundary value problem (2.1) can be written as the following equivalent integral equation

$$x(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

where $C_1, C_2, \cdots, C_n$ are constants to be determined. Considering the fact that $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, we get $C_2 = C_3 = \cdots = C_n = 0$. On the other hand, together with $x(1) = \mu\int_0^1 h(x(t))dA(t)$, we have

$$C_1 = \mu\int_0^1 h(x(t))dA(t) + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds.$$

Therefore, the unique integral representation of BVP (2.1) is

$$\begin{aligned} x(t) &= \left(\mu\int_0^1 h(x(t))dA(t) + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds\right)t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds\\ &= \int_0^1 G(t,s)y(s)ds + \mu t^{\alpha-1}\int_0^1 h(x(t))dA(t), \end{aligned}$$

where $G(t,s)$ is defined as (2.2). The proof is complete. □

Lemma 2.5 ([13])**.** *The Green function $G(t,s)$ defined as* (2.2) *in Lemma* 2.4 *has the following properties:*

(i) *$G(t,s)$ is continuous and $G(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$;*

(ii) *$\omega(t)\xi(s) \le G(t,s) \le \xi(s)$ for any $t,s \in [0,1]$, in which*

$$\omega(t) = \frac{(1-t)t^{\alpha-1}}{\alpha-1}, \quad \xi(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}.$$

Now, we establish the classical lower and upper solutions method for our problem. As usual, we say that $u(t)$ is a lower solution for $(P_{\lambda,\,\mu})$ if

$$\begin{cases} D_{0^+}^\alpha u(t) + \lambda a(t)f(t,u(t)) \ge 0, \ 0<t<1,\\ \quad u(0) \le 0, u'(0) \le 0, \cdots, u^{(n-2)}(0) \le 0,\\ \quad u(1) \le \mu\displaystyle\int_0^1 h(u(t))dA(t). \end{cases}$$

Similarly, we define the upper solution $v(t)$ of $(P_{\lambda,\,\mu})$:

$$\begin{cases} D_{0^+}^\alpha v(t) + \lambda a(t)f(t,v(t)) \le 0, \ 0<t<1,\\ \quad v(0) \ge 0, v'(0) \ge 0, \cdots, v^{(n-2)}(0) \ge 0,\\ \quad v(1) \ge \mu\displaystyle\int_0^1 h(v(t))dA(t). \end{cases}$$

In this paper, the space $X = C[0,1]$ will be used in the study of $(P_{\lambda,\,\mu})$. Clearly, $(X, \|\cdot\|)$ is a Banach space if it is endowed with the norm: $\|x\| = \max_{0\le t\le 1}|x(t)|$. Let

$$K = \{x \in X : x(t) \ge \varpi(t)\|x\|, \ t \in [0,1]\},$$

where $\varpi(t) = \min\{\omega(t), \ t^{\alpha-1}\}$. It is easy to see that K is a positive cone in X. In what follows, we list the conditions to be used later:

(H_1) $f : [0,1] \times [0,+\infty) \to (0,+\infty)$, $h : [0,+\infty) \to [0,+\infty)$ are continuous and f, h are nondecreasing with respect to u, that is,

$$f(t,u_1) \le f(t,u_2),\ h(u_1) \le h(u_2),\ u_1 \le u_2,\ t \in [0,1].$$

(H_2) $a : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous, $a(t) \not\equiv 0$, $t \in (0,1)$, and

$$0 < \int_0^1 \xi(s)a(s)ds < +\infty.$$

(H_3)

$$\lim_{u\to+\infty} \min_{t\in[a,b]\subset[0,1]} \frac{f(t,u)}{u} = +\infty, \quad \lim_{u\to+\infty} \frac{h(u)}{u} = +\infty.$$

Under conditions (H_1) and (H_2), define a nonlinear integral operator $T : X \to X$ by

$$Tx(t) = \lambda \int_0^1 G(t,s)a(s)f(s,x(s))ds + \mu t^{\alpha-1} \int_0^1 h(x(t))dA(t),\ t \in [0,1]. \tag{2.3}$$

Obviously, we know that $x \in C[0,1]$ is a solution of $(P_{\lambda,\ \mu})$ if and only if $x \in C[0,1]$ is a fixed point of T in K defined as (2.3).

In this paper, the following fixed point lemma is crucial in order to obtain the main results of $(P_{\lambda,\ \mu})$.

Lemma 2.6 ([5]). *Let P be a positive cone in a Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and $A : P \cap \overline{\Omega}_2 \backslash \Omega_1 \to P$ is a completely continuous operator. If the following conditions are satisfied:*

$\|Ax\| \le \|x\|,\ \forall x \in P \cap \partial\Omega_1, \quad \|Ax\| \ge \|x\|,\ \forall x \in P \cap \partial\Omega_2,$

or

$\|Ax\| \ge \|x\|,\ \forall x \in P \cap \partial\Omega_1, \quad \|Ax\| \le \|x\|,\ \forall x \in P \cap \partial\Omega_2,$

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main results

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. Then $T : X \to X$ is a completely continuous operator and $T(K) \subset K$.*

Proof. According to Arzela-Ascoli theorem, we can see that $T : X \to X$ is completely continuous, and we only prove $T(K) \subseteq K$. For any $x \in K$, $t \in [0,1]$, by (H_1), (H_2), and Lemma 2.5, we have

$$\begin{aligned}\|Tx(t)\| &= \max_{t\in[0,1]} |Tx(t)| \\ &= \max_{t\in[0,1]} \left|\lambda \int_0^1 G(t,s)a(s)f(s,x(s))ds + \mu t^{\alpha-1} \int_0^1 h(x(t))dA(t)\right| \\ &\le \lambda \int_0^1 \xi(s)a(s)f(s,x(s))ds + \mu \int_0^1 h(x(t))dA(t) < +\infty.\end{aligned}$$

On the other hand, by (H_2) and Lemma 2.5, for any $x \in K$, $t \in [0,1]$, we also have

$$\begin{aligned}Tx(t) =&\lambda \int_0^1 G(t,s)a(s)f(s,x(s))ds + \mu t^{\alpha-1} \int_0^1 h(x(t))dA(t) \\ \ge&\lambda \int_0^1 \omega(t)\xi(s)a(s)f(s,x(s))ds + \mu t^{\alpha-1} \int_0^1 h(x(t))dA(t) \\ \ge&\min\left\{\omega(t),\ t^{\alpha-1}\right\} \left(\lambda \int_0^1 \xi(s)a(s)f(s,x(s))ds + \mu \int_0^1 h(x(t))dA(t)\right).\end{aligned}$$

Then $Tx(t) \ge \varpi(t)\|Tx\|$, which implies $TK \subseteq K$. The proof is complete. $\square$

Theorem 3.2. *Assume that* (H_1)-(H_3) *hold. Then there exists a constant* $C_I > 0$ *such that for all possible positive solutions* $x(t)$ *of* $(P_{\lambda,\ \mu})$, *one has* $\|x\| \le C_I$, *where* $\lambda,\ \mu \in I$, *and* I *is a compact subset of* $(0, +\infty)$.

Proof. Suppose on the contrary that there exists a sequence $\{x_n\}$ of positive solutions of $(P_{\lambda_n,\ \mu_n})$ and $\{\lambda_n\},\ \{\mu_n\} \in I$ for all $n \in \mathbb{N}$, such that $\lim_{n\to+\infty} \|x_n\| = +\infty$. By Theorem 3.1, we have $x_n \in K$, thus $x_n(t) \ge \varpi(t)\|x_n\|$, $t \in [0, 1]$. Since I is compact, the sequences $\{\lambda_n\},\ \{\mu_n\}$ have a convergent subsequence which we denote without loss of generality still by $\{\lambda_n\},\ \{\mu_n\}$ such that

$$\lim_{n\to+\infty} \lambda_n = \lambda^*,\ \lim_{n\to+\infty} \mu_n = \mu^*.$$

From the assumption, we know $\lambda^* > 0$, $\mu^* > 0$, and we have $\lambda_n \ge \frac{\lambda^*}{2} > 0$, $\mu_n \ge \frac{\mu^*}{2} > 0$ for sufficiently large n. By (H_3), choose L, l such that

$$\lambda^* \varpi^2 L \int_a^b \xi(s)a(s)ds > 1,\ \mu^* \varpi^2 l \int_a^b dA(t) > 1,$$

where $\varpi = \min_{t\in[a,b]} \varpi(t)$. Then there exists $R > 0$, such that

$$f(t, u) \ge Lu,\ h(u) \ge lu,\ u \ge R,\ t \in [a, b].$$

Since $\lim_{n\to+\infty} \|x_n\| = +\infty$, we have $\|x_n\| \ge \frac{R}{\varpi} \ge R$, for sufficiently large n. Therefore for any $x_n \in K$, we know $x_n(t) \ge \varpi\|x_n\| \ge R,\ t \in [a, b]$. So

$$f(t, x_n(t)) \ge Lx_n(t) \ge L\varpi\|x_n\|,\ h(x_n(t)) \ge lx_n(t) \ge \varpi\|x_n\|,\ t \in [a, b].$$

Thus, we have

$$\begin{aligned}
x_n(t) =& \lambda_n \int_0^1 G(t, s)a(s)f(s, x_n(s))ds + \mu_n t^{\alpha-1} \int_0^1 h(x_n(t))dA(t)\\
\ge& \frac{\lambda^*}{2} \int_a^b \omega(t)\xi(s)a(s)Lx_n(s)ds + \frac{\mu^* t^{\alpha-1}}{2} \int_a^b lx_n(t)dA(t)\\
\ge& \frac{1}{2}\lambda^* \varpi^2 L\|x_n\| \int_a^b \xi(s)a(s)ds + \frac{1}{2}\mu^* \varpi^2 l\|x_n\| \int_a^b dA(t)\\
>& \|x_n\|.
\end{aligned}$$

This is a contradiction, so we get that all the positive solutions $x(t)$ of $(P_{\lambda,\ \mu})$ satisfying $\|x\| \le C_I$. The proof is complete. □

Theorem 3.3. *Let* $u(t)$, $v(t)$ *be lower and upper solutions of* $(P_{\lambda,\ \mu})$, *respectively, such that* $0 \le u(t) \le v(t)$. *Then* $(P_{\lambda,\ \mu})$ *has a solution* $x(t)$ *satisfying* $u(t) \le x(t) \le v(t)$ *for* $t \in [0, 1]$.

Proof. Let

$$\begin{aligned}
&\widetilde{T}x(t) = \lambda \int_0^1 G(t, s)a(s)f(s, \zeta(s, x))ds + \mu t^{\alpha-1} \int_0^1 h(\zeta(t, x))dA(t),\\
&\zeta(t, x) = \max\{u(t),\ \min\{x(t), v(t)\}\}.
\end{aligned}$$

It is easy to prove that the operator $\widetilde{T}$ has the following properties.

(1) $\widetilde{T}$ is a completely continuous operator.
(2) If $x \in K$ is a fixed point of $\widetilde{T}$, then $x \in K$ is a fixed point of T with $u(t) \le x(t) \le v(t)$ for $t \in [0, 1]$.
(3) If $x = \eta\widetilde{T}x$ with $0 \le \eta \le 1$, then $\|x\| \le C_I$, where C_I does not depend on η and $x \in K$.

Therefore, by using the topological degree of Leray-Schauder (see [2, Corollary 8.1, page. 61]), we obtain a fixed point of the operator T. The proof is complete. □

Theorem 3.4. *Assume that* (H_1)-(H_3) *hold. If* $(P_{\lambda_1,\ \mu_1})$ *has a positive solution, then* $(P_{\lambda_2,\ \mu_2})$ *has a positive solution for all* $0 < \lambda_2 < \lambda_1$, *and* $0 < \mu_2 < \mu_1$.

Proof. Let $x_1(t)$ be the solution of $(P_{\lambda_1,\ \mu_1})$, then $x_1(t)$ be the upper solution of $(P_{\lambda_2,\ \mu_2})$ with $0 < \lambda_2 < \lambda_1, 0 < \mu_2 < \mu_1$. Since $f(t,u) > 0$, $t \in [0,1]$, $x = 0$ is not a solution of $(P_{\lambda_2,\ \mu_2})$, but it is the lower solution of $(P_{\lambda_2,\ \mu_2})$. Therefore, by Theorem 3.3, we obtain the result. The proof is complete. □

Theorem 3.5. *Assume that* (H_1)-(H_3) *hold. Then* $(P_{\lambda,\ \mu})$ *has a positive solution for sufficiently small* $\lambda > 0,\ \mu > 0$.

Proof. Define

$$M(r_1) = \max_{\substack{x\in K\\ \|x\|=r_1}} \left\{ \int_0^1 \xi(s)a(s)f(s,r_1)ds,\ \int_0^1 h(r_1)dA(t) \right\}.$$

For any fixed $r_1 > 0$, let $\Omega_1 = \{x \in X : \|x\| < r_1\}$, $\sigma = \frac{r_1}{2M(r_1)}$. For $\lambda \le \sigma$, $\mu \le \sigma$, $x \in \partial\Omega_1 \cap K$, we have

$$\begin{aligned} Tx(t) &= \max_{t\in[0,1]} \left|\lambda \int_0^1 G(t,s)a(s)f(s,x(s))ds + \mu t^{\alpha-1} \int_0^1 h(x(t))dA(t)\right| \\ &\le \lambda \int_0^1 \xi(s)a(s)f(s,x(s))ds + \mu \int_0^1 h(x(t))dA(t) \\ &\le \lambda M(r_1) + \mu M(r_1) \le r_1 = \|x\|. \end{aligned}$$

Thus

$$\|Tx\| \le \|x\| \text{ for any } x \in \partial\Omega_1 \cap K.$$

On the other hand, by the inequality in (H_3), choose $\overline{L}, \bar{l}$ such that

$$\lambda\varpi^2\overline{L} \int_a^b \xi(s)a(s)ds \ge \frac{1}{2},\ \mu\varpi^2\bar{l} \int_a^b dA(t) \ge \frac{1}{2},$$

then there exists $r_0 > 0$, such that

$$f(t,u) \ge \overline{L}u,\ h(u) \ge \bar{l}u,\ u \ge r_0,\ t \in [a,b].$$

Let $r_2 > \max\{r_1,\ \frac{r_0}{\varpi}\}$, where ϖ is defined in Section 2, $\Omega_2 = \{x \in X : \|x\| < r_2\}$, for any $x \in \partial\Omega_2 \cap K$, we have

$$x(t) \ge \varpi(t)\|x\| \ge \varpi\|x\| \ge r_0,\ t \in [a,b].$$

Hence, we conclude that

$$\begin{aligned} Tx(t) =& \lambda \int_0^1 G(t,s)a(s)f(s,x(s))ds + \mu t^{\alpha-1} \int_0^1 h(x(t))dA(t) \\ \ge& \lambda \int_a^b \omega(t)\xi(s)a(s)\overline{L}x(s)ds + \mu t^{\alpha-1} \int_a^b \bar{l}x(t)dA(t) \\ \ge& \lambda\varpi^2\overline{L}\|x\| \int_a^b \xi(s)a(s)ds + \mu\varpi^2\bar{l}\|x\| \int_a^b dA(t) \ge \|x\|. \end{aligned}$$

Thus

$$\|Tx\| \ge \|x\| \text{ for any } x \in \partial\Omega_2 \cap K.$$

It follows from the above discussion, Lemma 2.6 and Theorem 3.1, we know that T has a fixed point x in $(\partial\Omega_2 \cap K)\backslash(\overline{\partial\Omega_1 \cap K})$. The proof is complete. □

Theorem 3.6. *Assume that* (H_1)-(H_3) *hold. Then* $(P_{\lambda,\ \mu})$ *has no positive solution for sufficiently large* $\lambda > 0,\ \mu > 0$.

Proof. Suppose on the contrary that there exist sufficiently large $\lambda_n > 0,\ \mu_n > 0$ where $(P_{\lambda_n,\ \mu_n})$ has a positive solution x_n. By the similar proof as Theorem 3.2, for sufficiently large $\lambda_n > 0,\ \mu_n > 0$, by ($H_3$), we obtain $x_n(t) > \|x_n\|$, $t \in [0, 1]$, this is a contradiction, so we know that $P_{\lambda,\ \mu}$ has no positive solution for sufficiently large $\lambda > 0,\ \mu > 0$. The proof is complete. □

From the previous Theorems 3.1 to 3.6, define the set

$$\overline{\lambda} = \sup\{\lambda > 0 :\ \text{such that } (P_{\lambda,\ \mu}) \text{ has a positive solution for some } \ \mu > 0\}.$$

Through the Theorem 3.6, we know $0 < \overline{\lambda} < +\infty$, and through the Theorems 3.4 and 3.5, we know for any $\lambda \in (0, \overline{\lambda})$, there exists $\mu > 0$, such that $(P_{\lambda,\ \mu})$ has a positive solution. Suppose $(P_{\lambda_n,\ \mu_n})$ has positive solution and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n,\ \lim_{n\to+\infty} \lambda_n = \overline{\lambda}.$$

By Theorem 3.2, we know the positive solution x_n of $(P_{\lambda_n,\ \mu_n})$ is bounded. Again by the completely continuity of the operator T, there exists $\mu > 0$, such that $(P_{\overline{\lambda},\ \mu})$ has a positive solution. Now define the function $\Upsilon : (0, \overline{\lambda}] \to (0, +\infty)$ by

$$\Upsilon(\lambda) = \sup\{\mu > 0 :\ (P_{\lambda,\ \mu}) \text{ has a positive solution}\}.$$

By Theorem 3.4, the function Υ is continuous and nonincreasing. We thus claim that $\Upsilon(\lambda)$ is attained. In fact, it suffices to use Theorem 3.5 and the compactness of the operator T. Finally, it follows from the definition of the function Υ that $(P_{\lambda,\ \mu})$ has at least one positive solution for $0 < \mu \le \Upsilon(\lambda)$ and has no positive solutions for $\mu > \Upsilon(\lambda)$.

4. Example

Consider the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\frac{7}{2}} x(t) + \dfrac{\lambda}{\sqrt{t}}(x+1)^2 = 0,\ 0 < t < 1, \\ x(0) = x'(0) = 0,\ x(1) = \mu \displaystyle\int_0^1 (x(t)+2)^3 dt. \end{cases}$$

Obviously, $\alpha = \frac{7}{2}$, $A(t) = t$, $a(t) = \frac{1}{\sqrt{t}}$, $f(t, u) = (u+1)^2$, $h(u) = (u+2)^3$. $\xi(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)} = \frac{s(1-s)^{\frac{5}{2}}}{\Gamma(\frac{5}{2})}$. It is easy to see that (H_1)-(H_3) hold, so we can obtain all the results of Theorems 3.1 to 3.6 and the conclusions of our paper.

Acknowledgment

The author thanks the anonymous referees cordially for their valuable suggestions on this paper. The work was partly supported by the National Natural Science Foundation of China (11371221, 11271175, 11571296), the Natural Science Foundation of Shandong Province of China (ZR2016AP04, ZR2014AL004), a Project of Shandong Province Higher Educational Science and Technology Program (J16LI03, J14LI08), the Applied Mathematics Enhancement Program of Linyi University and the Science Research Foundation for Doctoral Authorities of Linyi University (LYDX2016BS080).

References

[1] I. J. Cabrera, J. Harjani, K. B. Sadarangani, *Positive and nondecreasing solutions to an m-point boundary value problem for nonlinear fractional differential equation*, Abstr. Appl. Anal., **2012** (2012), 15 pages. 1
[2] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, (1985). 3
[3] M. El-Shahed, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Abstr. Appl. Anal., **2007** (2007), 8 pages. 1

[4] C. S. Goodrich, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett., **23** (2010), 1050–1055 1

[5] D. J. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Notes and Reports in Mathematics in Science and Engineering, Academic Press, Inc., Boston, MA, (1988). 2.6

[6] L. M. Guo, L. S. Liu, Y. H. Wu, *Uniqueness of iterative positive solutions for the singular fractional differential equations with integral boundary conditions*, Bound. Value Probl., **2016** (2016), 20 pages. 1

[7] T. Jankowski, *Boundary problems for fractional differential equations*, Appl. Math. Lett., **28** (2014), 14–19. 1

[8] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, (1993). 2.1, 2.2, 2.3

[9] N. Nyamoradi, *Existence of solutions for multi point boundary value problems for fractional differential equations*, Arab J. Math. Sci., **18** (2012), 165–175. 1

[10] I. Podlubny, *Fractional differential equations*, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, (1999). 2.1, 2.2, 2.3

[11] F. L. Wang, Y. K. An, *On positive solutions for nonhomogeneous m-point boundary value problems with two parameters*, Bound. Value Probl., **2012** (2012), 11 pages. 1

[12] Y. Wang, L. S. Liu, X. U. Zhang, Y. H. Wu, *Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection*, Appl. Math. Comput., **258** (2015), 312–324. 1

[13] C. J. Yuan, *Multiple positive solutions for* $(n-1,1)$*-type semipositone conjugate boundary value problems of nonlinear fractional differential equations*, Electron. J. Qual. Theory Differ. Equ., **2012** (2010), 12 pages. 2.5

[14] X. U. Zhang, L. S. Liu, Y. H. Wu, B. Wiwatanapataphee, *The spectral analysis for a singular fractional differential equation with a signed measure*, Appl. Math. Comput., **257** (2015), 252–263. 1

[15] X. K. Zhao, C. W. Chai, W. G. Ge, *Existence and nonexistence results for a class of fractional boundary value problems*, J. Appl. Math. Comput., **41** (2013), 17–31. 1