Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5741–5755
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Geraghty and Ćirić type fixed point theorems in b-metric spaces

Rajendra Pant[a], R. Panicker[b,*]

[a]Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India.
[b]Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa.

Communicated by P. Kumam

Abstract

In this paper, we obtain some fixed point theorems for admissible mappings in b-metric spaces. Some useful examples are given to illustrate the facts. We also discuss an application to a nonlinear quadratic integral equation. Our results complement, extend and generalize a number of fixed point theorems including the well-known Geraghty [M. A. Geraghty, Proc. Amer. Math. Soc., **40** (1973), 604–608] and Ćirić [L. B. Ćirić, Proc. Amer. Math. Soc., **45** (1974), 267–273] theorems on b-metric spaces.

Keywords: Fixed point, b-metric spaces, generalized α-quasi-contraction, (α, β)- Geraghty type contractive mapping
2010 MSC: 47H10, 54H25.

1. Introduction

Geraghty [13] and Ćirić [9] obtained two important generalizations of the classical Banach contraction principle (BCP) as follows:

Theorem 1.1 ([13]). *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

where $\beta : [0,\infty) \to [0, 1)$ is a function satisfying $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then T has a unique fixed point $z \in X$.

*Corresponding author
Email addresses: pant.rajendra@gmail.com (Rajendra Pant), rpanicker@wsu.ac.za (R. Panicker)

Received 2016-05-07

Theorem 1.2 ([9]). *Let X be a T-orbitally complete metric space and $T : X \to X$ be a quasi-contraction, that is, there exists a real number $r \in [0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq r\ m(x, y),$$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. Then T has a unique fixed point $z \in X$.

As per Rhoades [18], a quasi-contraction on a metric space is the most general among contractions.

Recently, Kumam et al. [16] introduced the notion generalized quasi-contraction and obtained an interesting generalization of Theorem 1.2.

Definition 1.3. A self-mapping T of a metric space X is called a generalized quasi-contraction, if there exists a number $r \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq r\ M(x, y),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(T^2x, x), d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\}.$$

Theorem 1.4 ([16]). *Let T be a generalized quasi-contraction on a T-orbitally complete metric space X. Then T has a unique fixed point $z \in X$.*

On the other hand, Samet et al. [19] introduced the concept of α-ψ contractive type mappings as well as α-admissible mappings and established various results in complete metric spaces. Indeed, they extended and generalized many existing fixed point results in the literature. Subsequently, a number of extensions and generalizations of their results have appeared in [2, 3, 7, 8, 15, 21] and elsewhere. Motivated by Ćirić [9], Geraghty [13], Kumam et al. [16] and Samet et al. [19], in this paper we obtain some fixed point theorems for admissible mappings in b-metric spaces. Besides presenting some useful examples, we discuss an application to a nonlinear quadratic integral equation.

2. Preliminaries

For the sake of completeness, we recall some basic definitions and results.

Definition 2.1 ([9, 16]). Let X be a metric space and $T : X \to X$ be a self-mapping. For each $x \in X$ and $n \in \mathbb{N}$, define

$$O(x; n) = \{x, Tx, ..., T^n x\} \text{ and } O(x; \infty) = \{x, Tx, ..., T^n x, ...\}.$$

The set $O(x; \infty)$ is called the orbit of T and the metric space X is said be T-orbitally complete, if every Cauchy sequence in $O(x; \infty)$ is convergent in X.

Every complete metric space is T-orbitally complete for all mappings $T : X \to X$ but the converse is not true.

Example 2.2 ([16]). Let X be a metric space which is not complete and $T : X \to X$, a mapping defined by $Tx = x_0$ for all $x \in X$ and some $x_0 \in X$. Then X is a T-orbitally complete metric space but not complete.

In [10–12], Czerwik et al. introduced a wider class of metric spaces namely b-metric spaces and extended some fixed point theorems from metric spaces to these spaces. In recent years, a number of fixed point results for single-valued and multi-valued operators in b-metric spaces have been studied extensively in [4–6, 10–12, 17, 20] and elsewhere.

Definition 2.3 ([10–12]). Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a functional. Then d is called a b-metric on X, if

(1) $d(x,y) = 0$, if $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \le s[d(x,z) + d(y,z)]$, where $s \ge 1$.

The pair (X,d) is called a b-metric space or a generalized metric space.

If we take $s = 1$, we get the usual definition of a metric space. However, a b-metric on X needs not to be a metric on X. Therefore the class of b-metrics is larger than the class of metrics.

The following examples are some known b-metric spaces.

Example 2.4. Let $X = \{x_1, x_2, x_3\}$ and $d : X \times X \to [0,\infty)$ be a function such that

$$d(x_1,x_2) = a > 2, \quad d(x_1,x_3) = d(x_2,x_3) = 1, \quad d(x_n,x_n) = 0,$$
$$d(x_n,x_k) = d(x_k,x_n), \quad d(x_n,x_k) \le \frac{a}{2}[d(x_n,x_i) + d(x_i,x_k)], \quad n,k,i \in \{1,2,3\}.$$

Then (X,d) is a b-metric space.

Example 2.5 ([5])**.** Let $\mathbb{R}$ be the set of reals and $\ell_p(\mathbb{R}) = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$ with $0 < p < 1$. The functional $d : \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R}) \to \mathbb{R}$ defined by

$$d(x,y) := \left(\sum_{k=1}^{\infty} |x_n - y_n|^p \right)^{1/p}, \text{ for all } x = \{x_n\}, \quad y = \{y_n\} \in \ell_p(\mathbb{R}),$$

is a b-metric on $\ell_p(\mathbb{R})$ with coefficient $s = 2^{1/p} > 1$.

Notice that the above result holds for the general case $\ell_p(X)$ with $0 < p < 1$, where X is a Banach space.

Definition 2.6. Let X be a b-metric space and $\{x_n\}$ a sequence in X. Then

(a) the sequence $\{x_n\}$ is convergent, if there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n,z) = 0$;

(b) the sequence $\{x_n\}$ is Cauchy, if $\lim_{n,m\to\infty} d(x_n,x_m) = 0$;

(c) X is complete, if every Cauchy sequence in X is convergent.

Remark 2.7. Also note that,

(d) every convergent sequence $\{x_n\}$ in X has a unique limit;

(e) every convergent sequence $\{x_n\}$ in X is Cauchy.

In general, a b-metric needs not to be a continuous functional.

Example 2.8 ([17])**.** Let $X = \mathbb{N} \cup \{\infty\}$ and $d : X \times X \to [0,\infty)$ be defined by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd (and } m \ne n) \text{ or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X,d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(x_n,\infty) = \lim_{n\to\infty} d(2n,\infty) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

but $\lim_{n\to\infty} d(x_n,1) = 2 \ne 5 = d(\infty,1)$.

Definition 2.9 ([19]). Let $\alpha : X \times X \to [0, \infty)$ be a functional. A mapping $T : X \to X$ is said to be α-admissible, if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Definition 2.10 ([14]). The mapping $T : X \to X$ is said to be triangular α-admissible, if for all $x, y, z \in X$,

(i) $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;

(ii) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$.

3. Generalized α-quasi contraction

In this section, we obtain a Ćirić type result for admissible mappings. Now onwards, $\mathbb{N}$ denotes the set of natural numbers and X a b-metric space (X, d), where d is continuous.

Definition 3.1. Let X be a b-metric space. A mapping $T : X \to X$ is said to be generalized α-quasi contraction, if there exists a functional $\alpha : X \times X \to [0, \infty)$ and $q < \frac{1}{s^2}$ such that

$$\alpha(x, y)d(Tx, Ty) \leq qM(x, y).$$

Our main result of this section is prefaced by the following lemmas.

Lemma 3.2 ([14]). *Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Lemma 3.3. *Let X be a b-metric space and $T : X \to X$ be a generalized α-quasi contraction satisfying the following conditions:*

(A) *T is triangular α-admissible;*

(B) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Then for all positive integers $i, j \in \{1, 2, \cdots, n\}, (i < j)$

$$d(T^i x_0, T^j x_0) \leq q.\delta[O(x_0, n)].$$

Proof. By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Since T is triangular α-admissible, from Lemma 3.2 it follows that

$$\alpha(T^i x_0, T^j x_0) = \alpha(x_i, x_j) \geq 1, \quad \text{for } i, j \in \mathbb{N} \cup \{0\} \text{ with } i < j.$$

Let $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Then $T^{i-1}x_0, T^i x_0, T^{j-1}x_0, T^j x_0 \in O(x_0, n)$. Since T is a generalized α-quasi contraction, we have

$$\begin{aligned}
d(T^i x_0, T^j x_0) &= d(TT^{i-1}x_0, TT^{j-1}x_0) \\
&\leq \alpha(T^{i-1}x_0, T^{j-1}x_0) d(TT^{i-1}x_0, TT^{j-1}x_0) \\
&\leq q.\max\{d(T^{i-1}x_0, T^{j-1}x_0), d(T^{i-1}x_0, TT^{i-1}x_0), d(T^{j-1}x_0, TT^{j-1}x_0), \\
&\quad d(T^{i-1}x_0, TT^{j-1}x_0), d(T^{j-1}x_0, TT^{i-1}x_0), d(T^2T^{i-1}x_0, T^{i-1}x_0), \\
&\quad d(T^2T^{i-1}x_0, TT^{i-1}x_0), d(T^2T^{i-1}x_0, T^{j-1}x_0), d(T^2T^{i-1}x_0, TT^{j-1}x_0)\} \\
&= q.\max\{d(T^{i-1}x_0, T^{j-1}x_0), d(T^{i-1}x_0, T^i x_0), d(T^{j-1}x_0, T^j x_0), d(T^{i-1}x_0, T^j x_0), \\
&\quad d(T^{j-1}x_0, T^i x_0), d(T^{i+1}x_0, T^{i-1}x_0), d(T^{i+1}x_0, T^i x_0), d(T^{i+1}x_0, T^{j-1}x_0), \\
&\quad d(T^{i+1}x_0, T^j x_0)\} \\
&\leq q.\delta[O(x_0, n)].
\end{aligned}$$

This proves the lemma. □

Remark 3.4. It follows from the above lemma that if T is a generalized α-quasi contraction and $x_0 \in X$, then for every positive integer n, there exists a positive integer $k \leq n$ such that

$$d(x_0, T^k x_0) = \delta[O(x_0, n)].$$

Theorem 3.5. *Let X be a T-orbitally complete b-metric space (with constant $s \geq 1$) and $T : X \to X$ a generalized α-quasi contraction satisfying conditions* (A) *and* (B) *of Lemma* 3.3. *Then T has a fixed point in X.*

Proof. By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. We show that the sequence $\{T^n x_0\}$ is a Cauchy sequence. By the triangle inequality and Lemma 3.3 and Remark 3.4, we have

$$\begin{aligned} d(x_0, T^k x_0) &\leq s[d(x_0, Tx_0) + d(Tx_0, T^k x_0)] \\ &\leq s[d(x_0, Tx_0) + q.\delta[O(x_0, n)]] \\ &= s[d(x_0, Tx_0) + q.d(x_0, T^k x_0)]. \end{aligned}$$

Therefore,

$$\delta[O(x_0, n)] = d(x_0, T^k x_0) \leq \frac{s}{1 - qs} d(x_0, Tx_0).$$

Let n and m be positive integers with $n < m$. Since T is a generalized α-quasi contraction, it follows from Lemma 3.3 that

$$\begin{aligned} d(T^n x_0, T^m x_0) &= d(TT^{n-1} x_0, TT^{m-1} x_0) \\ &\leq \alpha(T^{n-1} x_0, T^{m-1} x_0) d(TT^{n-1} x_0, TT^{m-1} x_0) \\ &\leq q. \max\{d(T^{n-1} x_0, T^{m-1} x_0), d(T^{n-1} x_0, TT^{n-1} x_0), d(T^{m-1} x_0, T^m x_0), \\ &\quad d(T^{n-1} x_0, T^m x_0), d(T^{m-1} x_0, TT^{n-1} x_0), d(T^2 T^{n-1} x_0, T^{n-1} x_0), \\ &\quad d(T^2 T^{n-1} x_0, TT^{n-1} x_0), d(T^2 T^{n-1} x_0, T^{m-1} x_0), d(T^2 T^{n-1} x_0, T^m x_0)\} \\ &= q. \max\{d(T^{n-1} x_0, T^{m-n} T^{n-1} x_0), d(T^{n-1} x_0, TT^{n-1} x_0), \\ &\quad d(T^{m-n} T^{n-1} x_0, T^{m-n+1} T^{n-1} x_0), d(T^{n-1} x_0, T^{m-n+1} T^{n-1} x_0), \\ &\quad d(T^{m-n} T^{n-1} x_0, TT^{n-1} x_0), d(T^2 T^{n-1} x_0, T^{n-1} x_0), d(T^2 T^{n-1} x_0, TT^{n-1} x_0), \\ &\quad d(T^2 T^{n-1} x_0, T^{m-n} T^{n-1} x_0), d(T^2 T^{n-1} x_0, T^{m-n+1} T^{n-1} x_0)\}. \end{aligned}$$

Since

$$O(T^{n-1} x_0, m - n + 1) = \{T^{n-1} x_0, TT^{n-1} x_0, T^2 T^{n-1} x_0, \cdots, T^{m-n} T^{n-1} x_0, T^{m-n+1} T^{n-1} x_0\},$$

the above inequality reduces to

$$d(T^n x_0, T^m x_0) \leq q.\delta[O(T^{n-1} x_0, m - n + 1)]. \tag{3.1}$$

By Remark 3.4, there exists an integer k_1, $1 \leq k_1 \leq m - n + 1$ such that

$$\delta[O(T^{n-1} x_0, m - n + 1)] = d(T^{n-1} x_0, T^{k_1} T^{n-1} x_0). \tag{3.2}$$

Again, by Lemma 3.3, we have

$$\begin{aligned} d(T^{n-1} x_0, T^{k_1} T^{n-1} x_0) &= d(TT^{n-2} x_0, T^{k_1+1} T^{n-2} x_0) \\ &\leq q.\delta[O(T^{n-2} x_0, k_1 + 1)] \\ &\leq q.\delta[O(T^{n-2} x_0, m - n + 2)]. \end{aligned}$$

Then (3.2) becomes

$$\delta[O(T^{n-1}x_0, m-n+1)] \leq q.\delta[O(T^{n-2}x_0, m-n+2)]. \tag{3.3}$$

Therefore, from (3.1) and (3.3), we get

$$\begin{aligned} d(T^n x_0, T^m x_0) &\leq q.\delta[O(T^{n-1}x_0, m-n+1)] \\ &\leq q^2.\delta[O(T^{n-2}x_0, m-n+2)] \\ &\vdots \\ &\leq q^n.\delta[O(x_0, m)] \\ &\leq \frac{q^n s}{1-qs} d(x_0, Tx_0). \end{aligned}$$

Since $\lim\limits_{n\to\infty} q^n = 0$, the sequence $\{T^n x_0\}$ is Cauchy in X. Since X is T-orbitally complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} T^n x_0 = u.$$

By the triangular inequality, we get

$$\begin{aligned} d(u, Tu) &\leq s[d(u, T^{n+1}x_0) + d(Tu, T^{n+1}x_0)] \\ &= s[d(u, T^{n+1}x_0) + d(Tu, TT^n x_0)] \\ &\leq s[d(u, T^{n+1}x_0) + \alpha(u, T^n x_0) d(Tu, TT^n x_0)] \\ &\leq s[d(u, T^{n+1}x_0) + q \max\{d(T^n x_0, u), d(T^n x_0, TT^n x_0), d(u, Tu), d(T^n x_0, Tu), \\ &\quad d(u, TT^n x_0), d(T^2T^n x_0, T^n x_0), d(T^2T^n x_0, TT^n x_0), d(T^2T^n x_0, u), d(T^2T^n x_0, Tu)\}] \\ &= s[d(u, T^{n+1}x_0) + q \max\{d(T^n x_0, u), d(T^n x_0, T^{n+1}x_0), d(u, Tu), d(T^n x_0, Tu), \\ &\quad d(u, T^{n+1}x_0), d(T^{n+2}x_0, T^n x_0), d(T^{n+2}x_0, T^{n+1}x_0), d(T^{n+2}x_0, u), d(T^{n+2}x_0, Tu)\}] \\ &\leq s[d(u, T^{n+1}x_0) + q \max\{d(T^n x_0, u), s[d(T^n x_0, u) + d(u, T^{n+1}x_0)], d(u, Tu), \\ &\quad s[d(T^n x_0, u) + d(u, Tu)], d(u, T^{n+1}x_0), s[d(T^{n+2}x_0, u) + d(u, T^n x_0)], \\ &\quad s[d(T^{n+2}x_0, u) + d(u, T^{n+1}x_0)], d(T^{n+2}x_0, u), s[d(T^{n+2}x_0, u) + d(u, Tu)\}]. \end{aligned}$$

By letting $n \to \infty$, we get

$$\begin{aligned} d(u, Tu) &\leq qs \max\{d(u, Tu), sd(u, Tu)\} \\ &= qs^2 d(u, Tu). \end{aligned}$$

Since $q < \dfrac{1}{s^2}$, we get $d(u, Tu) = 0$. Hence u is a fixed point of T. □

Corollary 3.6 ([21])**.** *Let (X, d) be a complete b-metric space (with constant $s \geq 1$), $\alpha : X \times X \to [0, \infty)$ a functional and $T : X \to X$ be an α-quasi-contraction, that is,*

$$\alpha(x, y) d(Tx, Ty) \leq q m(x, y)$$

for all $x, y \in X$, where $0 \leq q < 1$ and

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Suppose that the following conditions hold:

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

If we set $q < \dfrac{1}{s^2 + s}$, then T has a fixed point in X.

When $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain the following results:

Corollary 3.7. *Theorem* 1.4.

Corollary 3.8. *Theorem* 1.2.

The following example shows the generality of Theorem 3.5 over 1.4.

Example 3.9. Let $X = [0, 4]$ be endowed with the b-metric $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x-y|^2$. Define $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{x}{4} & \text{if } x \in [0, 1], \\ 4 & \text{if } x \in (1, 4], \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } (x, y) \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then (X, d) is a T-orbitally complete b-metric space with $s = 2$.

If $x, y \in [0, 1]$, then

$$\begin{aligned} \alpha(x, y)d(Tx, Ty) &= 2\left|\frac{x}{4} - \frac{y}{4}\right|^2 \\ &= \frac{1}{8}|x - y|^2 = qd(x, y) \leq qM(x, y), \end{aligned}$$

where $q = \dfrac{1}{8} < \dfrac{1}{4} = \dfrac{1}{s^2}$. If $x \in [0, 1]$ and $y \in (1, 4]$, then $\alpha(x, y)d(Tx, Ty) = 0 \leq qM(x, y)$. Now, if $x = 0$ and $y = 4$, then $d(T0, T4) = 16 = M(0, 4)$. Hence $d(T0, T4) > qM(0, 4)$ for any $q < 1$. Therefore, the contractive condition of Theorem 1.4 is not satisfied. Since $\alpha(x, y)d(Tx, Ty) = 0 \leq qM(x, y)$, the mapping T is a generalized α-quasi-contraction. Further, it is easy to check that T is triangular α-admissible. Therefore, the mapping T satisfies all the conditions of Theorem 3.5 and $x = 0$ and $x = 4$ are the fixed points of T.

4. Geraghty type contractive mapping

In this section, we present some Geraghty type results for admissible mappings.

Definition 4.1 ([7])**.** Let X be a b-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. The mapping T is said to be an (α, β)-admissible mapping, if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ for all $x, y \in X$.

Definition 4.2 ([7])**.** Let $\alpha, \beta : X \times X \to [0, \infty)$. A b-metric space X is (α, β)-regular, if $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$, $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all n and there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k+1}) \geq 1$, $\beta(x_{n_k}, x_{n_k+1}) \geq 1$ for all $k \in \mathbb{N}$. Also $\alpha(x, Tx) \geq 1$, $\beta(x, Tx) \geq 1$.

We need the following class of functions to prove certain results of this section:

1. Θ is a family of functions $\theta : [0, \infty) \to [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ implies $t_n \to 0$;
2. Ψ is a family of functions $\psi : [0, \infty) \to [0, \infty)$ such that ψ is continuous, strictly increasing and $\psi(0) = 0$.

Definition 4.3. Let X be a b-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. A mapping T is said to be (α, β)-Geraghty type contractive mapping, if there exists $\theta \in \Theta$ such that for all $x, y \in X$, the following condition holds:

$$\alpha(x, Tx)\beta(y, Ty)\psi(s^3 d(Tx, Ty)) \leq \theta(\psi(N(x, y)))\psi(N(x, y)), \tag{4.1}$$

where $N(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \dfrac{d(x, Ty) + d(y, Tx)}{2s}\right\}$ and $\psi \in \Psi$.

Theorem 4.4. *Let (X, d) be a complete b-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. Suppose the following conditions hold:*

(A) *T is an (α, β)-admissible mapping;*

(B) *T is an (α, β)-Geraghty type contractive mapping;*

(C) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*

(D) *either T is continuous or X is (α, β)-regular.*

Then T has a unique fixed point.

Proof. By assumption, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for $n \in \mathbb{N}$. It is obvious that if $x_{n_k} = x_{n_k+1}$ for some $n_k \in \mathbb{N}$, then x_{n_k} is a fixed point of T and we are done. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is (α, β)-admissible, so

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.$$

By continuing this manner, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$. Similarly $\beta(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$. From (4.1), we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\
&\leq \psi(s^3 d(Tx_n, Tx_{n+1})) \\
&\leq \alpha(x_n, Tx_n)\beta(x_{n+1}, Tx_{n+1})\psi(s^3 d(Tx_n, Tx_{n+1})) \\
&\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})),
\end{aligned}$$

where

$$\begin{aligned}
N(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2s}\right\} \\
&= \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s}\right\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.
\end{aligned}$$

Now, if $N(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$, then

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \\
&= \theta(\psi(N(x_n, x_{n+1})))\psi(d(x_{n+1}, x_{n+2})) \\
&< \psi(d(x_{n+1}, x_{n+2})),
\end{aligned}$$

a contradiction. Therefore $N(x_n, x_{n+1}) = d(x_n, x_{n+1})$. Now

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \\
&= \theta(\psi(N(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})) \\
&< \psi(d(x_n, x_{n+1})).
\end{aligned} \tag{4.2}$$

Since ψ is a strictly increasing mapping, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

From (4.2), we get

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(N(x_n, x_{n+1}))} \leq \theta(\psi(N(x_n, x_{n+1}))) < 1. \tag{4.3}$$

By letting $n \to \infty$ in (4.3), we have $1 \leq \lim_{n\to\infty} \theta(\psi(N(x_n, x_{n+1}))) < 1$.

That is, $\lim_{n\to\infty} \theta(\psi(N(x_n, x_{n+1}))) = 1$ and $\theta \in \Theta$ implies $\lim_{n\to\infty} \psi(N(x_n, x_{n+1})) = 0$ which yields that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.4}$$

We show that $\{x_n\}$ is a Cauchy sequence in X. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and the subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \epsilon, \tag{4.5}$$

and n_k is the smallest number such that (4.5) holds. From (4.5) we get

$$d(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{4.6}$$

By using triangle inequality, (4.5) and (4.6) we have

$$\begin{aligned}\epsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq s[d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k})] \\ &< s[d(x_{n_k}, x_{n_k-1}) + \epsilon].\end{aligned} \tag{4.7}$$

By taking the upper limit as $k \to \infty$ in (4.7) and using (4.4), we get

$$\epsilon \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) < s\epsilon. \tag{4.8}$$

From the triangle inequality, we have

$$d(x_{n_k}, x_{m_k}) \leq s[d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k})], \tag{4.9}$$

and

$$d(x_{n_k+1}, x_{m_k}) \leq s[d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x_{m_k})]. \tag{4.10}$$

By taking the upper limit as $k \to \infty$ in (4.9) and applying (4.4), (4.8) becomes

$$\epsilon \leq s\left(\limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k})\right),$$

and taking the upper limit as $k \to \infty$ in (4.10) gives

$$\limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k}) \leq s.s\epsilon = s^2\epsilon.$$

Thus

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k}) \leq s^2\epsilon. \tag{4.11}$$

Similarly, we get

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k+1}) \leq s^2\epsilon. \tag{4.12}$$

By triangular inequality, we have

$$d(x_{n_k+1}, x_{m_k}) \leq s[d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})]. \tag{4.13}$$

By taking the upper limit as $k \to \infty$ in (4.13), from (4.4) and (4.11) we obtain that

$$\frac{\epsilon}{s} \leq s \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}).$$

That is,

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}). \tag{4.14}$$

Again, by following the above process, we get

$$\limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) \leq s^3\epsilon. \tag{4.15}$$

From (4.14) and (4.15), we get

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) \leq s^3\epsilon.$$

Since X is (α, β)-regular, by (4.1) we have

$$\begin{aligned}\psi\left(s^3 d(x_{n_k+1}, x_{m_k+1})\right) &= \psi\left(s^3 d(Tx_{n_k}, Tx_{m_k})\right)\\ &\leq \alpha(x_{n_k}, Tx_{n_k})\beta(x_{m_k}, Tx_{m_k})\psi\left(s^3 d(Tx_{n_k}, Tx_{m_k})\right)\\ &\leq \theta\left(\psi(N(x_{n_k}, x_{m_k}))\right)\psi(N(x_{n_k}, x_{m_k})),\end{aligned}$$

where

$$\begin{aligned}N(x_{n_k}, x_{m_k}) &= \max\left\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{m_k}), \frac{d(x_{n_k}, Tx_{m_k}) + d(x_{m_k}, Tx_{n_k})}{2s}\right\}\\ &= \max\left\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2s}\right\}.\end{aligned}$$

By taking limit supremum as $k \to \infty$ in the above equation and using (4.4), (4.8), (4.11) and (4.12), we obtain

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \limsup_{k\to\infty} N(x_{n_k}, x_{m_k}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Similarly, we can show that

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \liminf_{k\to\infty} N(x_{n_k}, x_{m_k}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Hence, it follows from (4.14) that

$$\begin{aligned}\psi(s\epsilon) &= \psi\left(s^3(\frac{\epsilon}{s^2})\right)\\ &\leq \psi\left(s^3 \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1})\right)\\ &\leq \alpha(x_{n_k}, x_{n_k+1})\beta(x_{m_k}, x_{m_k+1})\psi\left(s^3 \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1})\right)\\ &\leq \theta\left(\psi(\limsup_{k\to\infty} N(x_{n_k}, x_{m_k}))\right)\psi(\limsup_{k\to\infty} N(x_{n_k}, x_{m_k}))\\ &\leq \theta\left(\psi(s\epsilon)\right)\psi(s\epsilon)\\ &< \psi(s\epsilon),\end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. First, suppose that T is continuous. Then we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx^*.$$

Now, suppose that X is (α, β)-regular. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k+1}, x_{n_k}) \geq 1$ and $\beta(x_{n_k+1}, x_{n_k}) \geq 1$ for all $k \in \mathbb{N}$ and $\alpha(x^*, Tx^*) \geq 1$ and $\beta(x^*, Tx^*) \geq 1$. Now from (4.1), with $x = x_{n_k}$ and $y = x^*$, we obtain

$$\begin{aligned} \psi(d(x_{n_k+1}, Tx^*)) &= \psi(d(Tx_{n_k}, Tx^*)) \\ &\leq \psi\left(s^3 d(Tx_{n_k}, Tx^*)\right) \\ &\leq \alpha(x_{n_k}, Tx_{n_k})\beta(x^*, Tx^*)\psi\left(s^3 d(Tx_{n_k}, Tx^*)\right) \\ &\leq \theta\left(\psi(N(x_{n_k}, x^*))\,\psi(N(x_{n_k}, x^*)\right), \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} N(x_{n_k}, x^*) &= \max\left\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2s}\right\} \\ &= \max\left\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})}{2s}\right\} \\ &\leq \max\{d(x_{n_k}, x^*), s[d(x_{n_k}, x^*) + d(x_{n_k+1}, x^*)], d(x^*, Tx^*), \\ &\quad \left.\frac{s[d(x_{n_k}, x^*) + d(x^*, Tx^*)] + d(x^*, x_{n_k+1})}{2s}\right\}. \end{aligned}$$

By letting $k \to \infty$, we get

$$\begin{aligned} \lim_{k\to\infty} N(x_{n_k}, x^*) &\leq \max\left\{d(x^*, Tx^*), \frac{d(x^*, Tx^*)}{2}\right\} \\ &= d(x^*, Tx^*). \end{aligned}$$

Therefore, by taking the limit as $k \to \infty$ in (4.16), we get

$$\psi(d(x^*, Tx^*)) \leq \lim_{k\to\infty} \theta(\psi(N(x_{n_k}, x^*)))\psi(d(x^*, Tx^*)).$$

That is, $1 \leq \lim_{k\to\infty} \theta(\psi(N(x_{n_k}, x^*)))$, which implies that $\lim_{k\to\infty} \theta(\psi(N(x_{n_k}, x^*))) = 1$. Consequently, we obtain $\lim_{k\to\infty} N(x_{n_k}, x^*) = 0$. Hence $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$.

Further, suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$ and $\alpha(x^*, Tx^*) \geq 1$, $\alpha(y^*, Ty^*) \geq 1$ and $\beta(x^*, Tx^*) \geq 1$, $\beta(y^*, Ty^*) \geq 1$. Now by applying (4.1), we have

$$\begin{aligned} \psi(d(x^*, y^*)) &= \psi(d(Tx^*, Ty^*)) \\ &\leq \psi\left(s^3 d(Tx^*, Ty^*)\right) \\ &\leq \alpha(x^*, Tx^*)\beta(y^*, Ty^*)\psi\left(s^3 d(Tx^*, Ty^*)\right) \\ &\leq \theta\left(\psi(N(x^*, y^*))\right)\psi(N(x^*, y^*)), \end{aligned}$$

where

$$\begin{aligned} N(x^*, y^*) &= \max\left\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \frac{d(x^*, Ty^*) + d(y^*, Tx^*)}{2s}\right\} \\ &= d(x^*, y^*). \end{aligned}$$

Hence, $\psi(d(x^*, y^*)) \leq \theta\left(\psi(N(x^*, y^*))\right)\psi(d(x^*, y^*)) < \psi(d(x^*, y^*))$, which is a contradiction unless $d(x^*, y^*) = 0$ and T has a unique fixed point. □

Corollary 4.5. *Let (X,d) be a complete b-metric space, $T : X \to X$ and $\alpha, \beta : X \times X \to [0,\infty)$. Suppose the following conditions hold:*

(a) *T is an α-admissible mapping;*

(b) *T is an α-Geraghty type contractive mapping;*

(c) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(d) *either T is continuous or X is α-regular.*

Then T has a unique fixed point.

Example 4.6. Let $X = [0,\infty)$ be endowed with the b-metric $d : X\times X \to [0,\infty)$ defined by $d(x, y) = |x-y|^2$. Then (X, d) is a complete b-metric space with $s = 2$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{1-x^2}{8} & \text{if } x \in [0,1], \\ 2x & \text{otherwise.} \end{cases}$$

Define $\alpha, \beta : X \times X \to [0,\infty)$, $\theta : [0,\infty) \to [0, 1)$ and $\psi : [0,\infty) \to [0,\infty)$ as

$$\alpha(x,y) = \begin{cases} \dfrac{3}{2} & \text{if } (x,y) \in [0,1], \\ 1 & \text{otherwise.} \end{cases} ; \quad \beta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1], \\ 0 & \text{otherwise.} \end{cases} ; \quad \theta(t) = \frac{3}{4} \quad \text{and} \quad \psi(t) = t.$$

First we show that T is an (α, β)-admissible mapping.

If $x, y \in [0, 1]$, then $\alpha(x, y) > 1$, $\beta(x, y) \geq 1$, $Tx \leq 1$ and $Ty \leq 1$. By the definition of α and β, it follows that $\alpha(Tx, Ty) > 1$ and $\beta(Tx, Ty) \geq 1$. Further, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then $x_n \subseteq [0, 1]$ and hence $x \in [0, 1]$. This implies that $\alpha(x, Tx) \geq 1$ and $\beta(x, Tx) \geq 1$.

For $x, y \in [0, 1]$, we have

$$\begin{aligned} \alpha(x,Tx)\beta(y,Ty)\psi(s^3 d(Tx,Ty)) &= 12|Tx-Ty|^2 \\ &= \frac{3}{16}|x^2-y^2|^2 = \frac{3}{16}|x-y|^2|x+y|^2 \leq \frac{3}{4}|x-y|^2 \\ &= \theta(\psi(d(x,y)))\psi(d(x,y)) \leq \theta(\psi(M(x,y)))\psi(M(x,y)). \end{aligned}$$

Hence the contractive condition of Theorem 4.4 is satisfied. If $x, y \in (1,\infty)$, then $Tx > 1$ and $\alpha(x, Tx) \geq 1$. Then we have

$$\begin{aligned} \alpha(x,Tx)\psi(s^3 d(Tx,Ty)) &= 8|2x-2y|^2 \\ &= 32|x-y|^2 > \theta(\psi(M(x,y))\psi(M(x,y)). \end{aligned}$$

Hence the contractive condition of Corollary 4.5 is not satisfied by T. However,

$$\alpha(x,Tx)\beta(y,Ty)\psi(s^3 d(Tx,Ty)) = 0 \leq \theta(\psi(M(x,y)))\psi(M(x,y)).$$

Again, if $x \in [0, 1]$ and $y > 1$, $\alpha(x, Tx)\beta(y, Ty)\psi(s^3 d(Tx, Ty)) = 0 \leq \theta(\psi(M(x, y)))\psi(M(x, y))$. Therefore, all the conditions of Theorem 4.4 are satisfied and T has a fixed point $x^* = \sqrt{17} - 4$.

5. Applications to nonlinear integral equations

In this section, we discuss an application to nonlinear quadratic integral equation.

Consider the integral equation

$$x(t) = h(t) + \lambda \int_0^1 k(t,s) f(s,x(s))ds, \qquad t \in I = [0,1], \quad \lambda \geq 0. \tag{5.1}$$

Also, consider the following conditions:

(a) $h : I \to \mathbb{R}$ is a continuous function;

(b) $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t, x) \geq 0$ and there exists a constant $0 \leq L < 1$ such that for all $x, y \in \mathbb{R}$,
$$|f(t, x) - f(t, y)| \leq L|x(t) - y(t)|;$$

(c) $k : I \times I \to \mathbb{R}$ is continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $k(t, x) \geq 0$ and $\int_0^1 k(t, s)ds \leq K$;

(d) $\lambda^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(e) the space $X = C(I)$ of continuous functions defined on $I = [0, 1]$ with the standard metric given by
$$\rho(x, y) = \sup_{t\in I} |x(t) - y(t)| \quad \text{for } x, y \in C(I).$$

Now, for $p \geq 1$, we define
$$d(x, y) = (\rho(x, y))^p = \left(\sup_{t\in I} |x(t) - y(t)|\right)^p = \sup_{t\in I} |x(t) - y(t)|^p, \quad \text{for } x, y \in C(I).$$

Then (X, d) is a complete b-metric space with $s = 2^{p-1}$ (cf. [1, 3]).

Theorem 5.1. *Under assumptions* (a)-(e) *the nonlinear quadratic integral equation* (5.1) *has a unique solution in* $C(I)$.

Proof. Define an operator $T : X \to X$ by
$$Tx(t) = h(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \qquad t \in I = [0, 1], \quad \lambda \geq 0.$$

Now, for $x, y \in X$, we have
$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| h(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds - h(t) - \lambda \int_0^1 k(t, s)f(s, y(s))ds \right| \\ &\leq \lambda \int_0^1 k(t, s)|f(s, x(s)) - f(s, y(s))|ds \\ &\leq \lambda \int_0^1 k(t, s)L|x(s) - y(s)|ds. \end{aligned}$$

Since $|x(s) - y(s)| \leq \sup_{s\in I} |x(s) - y(s)| = \rho(x, y)$, we get
$$|Tx(t) - Ty(t)| \leq \lambda KL\rho(x, y).$$

Now, we can write
$$\begin{aligned} d(Tx, Ty) &= \sup_{t\in I} |Tx(t) - Ty(t)|^p \\ &\leq (\lambda KL(p(x, y)))^p \\ &\leq \lambda^p K^p L^p d(x, y) \\ &\leq \frac{1}{2^{3p-3}} M(x, y). \end{aligned}$$

Therefore, all the assumptions of Corollary 3.7 are satisfied by the operator T and (5.1) has a unique solution in $C(I)$. □

Example 5.2. Consider the following functional integral equation:

$$x(t) = \frac{t}{1+t^2} + \frac{1}{18}\int_0^1 \frac{s}{9e^t(1+t)}\frac{|x(s)|}{1+|x(s)|}ds, \qquad t \in I = [0,1].$$

It is observed that the above equation is a special case of (5.1) with

$$\begin{aligned} h(t) &= \frac{t}{1+t^2}, \\ k(t,s) &= \frac{s}{1+t}, \\ f(t,x) &= \frac{|x|}{9e^t(1+|x|)}. \end{aligned}$$

Now, for arbitrary $x, y \in \mathbb{R}$ such that $x \geq y$ and for $t \in [0,1]$, we obtain

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left|\frac{|x|}{9e^t(1+|x|)} - \frac{|y|}{9e^t(1+|y|)}\right| \\ &= \frac{1}{9e^t}\left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| \\ &\leq \frac{1}{9}|x-y|. \end{aligned}$$

Thus, f satisfies condition (b) of the integral equation (5.1) with $L = \frac{1}{9}$. It can be easily seen that h is a continuous function and k satisfies condition (c) with

$$\int_0^1 k(t,s)ds = \int_0^1 \frac{s}{1+t}ds = \frac{1}{2(1+t)} \leq \frac{1}{2} = K.$$

By substituting $L = \frac{1}{9}$, $K = \frac{1}{2}$ and $\lambda = \frac{1}{18}$ in condition (d), we obtain

$$\frac{1}{9^p} \times \frac{1}{18^p} \times \frac{1}{2^p} \leq \frac{1}{2^{3p-3}}.$$

The above inequality is true for each $p \geq 1$. Consequently, all the conditions of Theorem 5.1 are satisfied and hence the integral equation (5.1) has a unique solution in $C(I)$.

References

[1] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, **64** (2014), 941–960. 5

[2] S. Alizadeh, F. Moradlou, P. Salimi, *Some fixed point results for (α,β)-(ψ,ϕ)-contractive mappings*, Filomat, **28** (2014), 635–647. 1

[3] R. Allahyari, R. Arab, A. Shole Haghighi, *Fixed points of admissible almost contractive type mappings on b-metric spaces with an application to quadratic integral equations*, J. Inequal. Appl., **2015** (2015), 18 pages. 1, 5

[4] T. V. An, L. Q. Tuyen, N. V. Dung, *Stone-type theorem on b-metric spaces and applications*, Topology Appl., **185/186** (2015), 50–64. 2

[5] M. Boriceanu, M. Bota, A. Petruşel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math., **8** (2010), 367–377. 2.5

[6] M. Boriceanu, A. Petruşel, I. A. Rus, *Fixed point theorems for some multivalued generalized contractions in b-metric spaces*, Int. J. Math. Stat., **6** (2010), 65–76. 2

[7] S. Chandok, *Some fixed point theorems for (α,β)-admissible Geraghty type contractive mappings and related results*, Math. Sci. (Springer), **9** (2015), 127–135. 1, 4.1, 4.2

[8] S.-H. Cho, J.-S. Bae, *Fixed points of weak α-contraction type maps*, Fixed Point Theory Appl., **2014** (2014), 12 pages. 1

[9] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273. 1, 1.2, 1, 2.1

[10] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 2, 2.3

[11] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti. Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276.

[12] S. Czerwik, K. Dlutek, S. L. Singh, *Round-off stability of iteration procedures for operators in b-metric spaces*, J. Natur. Phys. Sci., **11** (1997), 87–94. 2, 2.3

[13] M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608. 1, 1.1, 1

[14] E. Karapınar, P. Kumam, P. Salimi, *On α-ψ-Meir-Keeler contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 2.10, 3.2

[15] E. Karapınar, B. Samet, *Generalized α-ψ contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., **2012** (2012), 17 pages. 1

[16] P. Kumam, N. V. Dung, K. Sitthithakerngkiet, *A generalization of Ćirić fixed point theorems*, Filomat, **29** (2015), 1549–1556. 1, 1.4, 1, 2.1, 2.2

[17] A. Latif, V. Parvaneh, P. Salimi, A. E. Al-Mazrooei, *Various Suzuki type theorems in b-metric spaces*, J. Nonlinear Sci. Appl., **8** (2015), 363–377. 2, 2.8

[18] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226** (1977), 257–290. 1

[19] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha\psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 2.9

[20] S. L. Singh, C. Bhatnagar, S. N. Mishra, *Stability of iterative procedures for multivalued maps in metric spaces*, Demonstratio Math., **37** (2005), 905–916. 2

[21] W. Sintunavarat, S. Plubtieng, P. Katchang, *Fixed point result and applications on a b-metric space endowed with an arbitrary binary relation*, Fixed Point Theory Appl., **2013** (2013), 13 pages. 1, 3.6