Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5756–5765
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Multivariate best proximity point theorems in metric spaces

Yinglin Luo, Yongfu Su*, Wenbiao Gao

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China.

Communicated by R. Saadati

Abstract

The purpose of this paper is to prove an existence and uniqueness theorems of the multivariate best proximity point in the complete metric spaces. The concept of multivariate best proximity point is firstly introduced in this article. These new results improve and extend the previously known ones in the literature.

Keywords: Contraction mapping principle, complete metric spaces, multivariate mapping, multivariate fixed point, multiply metric function, best proximity point theorem.
2010 MSC: 47H05, 47H09, 47H10.

1. Introduction and preliminaries

The Banach contraction mapping principle is a classical and powerful tool in nonlinear analysis, which first appeared in 1922. This principle has been generalized in many ways over the years. In 1969, Fan [2] introduced and established a classical best approximation theorem which is regarded as a natural generalization of fixed point theorems. Let (X, d) be a complete metric space and let T be a contraction mapping. Then T has a unique fixed point, i.e., the equation $Tx = x$ has a unique solution. The point $x \in X$ such that $x = Tx$ is called a fixed point of T. However, if T is a non-self-mapping from A to B, it is plausible that the equation $x = Tx$ has no solution. In this situation, we may find an element x in A such that the error $d(x, Tx)$ is minimum, where d is the distance function. A point x in A for which $d(x, Tx) = d(A, B)$ is called a best proximity point of T. Research on the best proximity point is an important topic in the nonlinear functional analysis and it has been studied by several authors (see [1–12]).

*Corresponding author
Email addresses: luoyinglink@163.com (Yinglin Luo), tjsuyongfu@163.com (Yongfu Su), 15822752271@163.com (Wenbiao Gao)

Received 2016-05-01

Let (A, B) be a pair of nonempty subsets of a metric space (X, d), consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. In fact, if $A = B$, then $d(A, B) = 0$ and hence a best proximity point of T becomes a fixed point of T. Since $d(x, Tx) \geq d(A, B)$ for any $x \in A$, the optimal solution to this problem is the one for which the value $d(A, B)$ is attained. We denote the following sets by A_0 and B_0,

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\},$$

where $d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$.

It is interesting that A_0 and B_0 are contained in the boundaries of A and B, respectively, provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ [11, 12].

Let (X, d) be a metric space, $T : X^N \to X$ be a N variable mapping, an element p is called a multivariate fixed point of T if $p = T(p, p, ..., p)$. In [7], Su and his partners proved the existence and uniqueness of the multivariate fixed point for contraction type mappings in complete metric spaces. If (A, B) is a pair of nonempty closed subsets of a complete metric space (X, d) and $T : A^N \to B$ is a N variable mapping, then T does not necessarily have a multivariate fixed point. Eventually, it is quite natural to seek an element p such that $d(p, T(p, p, ..., p))$ is minimum, which implies that p and $T(p, p, ..., p)$ are in close proximity to each other. The purpose of this paper is to prove an existence and uniqueness theorems of the multivariate best proximity point in the complete metric spaces. The concept of multivariate best proximity point is firstly introduced in this article in the complete metric spaces. These new results improve and extend the previous known ones in the literature.

Definition 1.1 ([11]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d), with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

Any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property, (see [11]).

Definition 1.2 ([11]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d), with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2).$$

In [11], P-property was weakened to weak P-property and an example satisfying P-property can be found there.

Example 1.3 ([11]). Consider (R^2, d), where d is the Euclidean distance and the subsets $A = \{(0, 0)\}$ and $B = \{y = 1 + \sqrt{1 - x^2}\}$. Obviously, $A_0 = \{(0, 0)\}$, $B_0 = \{(-1, 1), (1, 1)\}$ and $d(A, B) = \sqrt{2}$. Furthermore,

$$d((0, 0), (-1, 1)) = d((0, 0), (1, 1)) = \sqrt{2};$$

however,

$$0 = d((0, 0), (0, 0)) < d((-1, 1), (1, 1)) = 2.$$

We can see that the pair (A, B) satisfies the weak P-property but not the P-property.

Definition 1.4 ([8]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d), with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the (ψ, φ)-P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow \quad \psi(d(x_1, x_2)) \leq \varphi(d(y_1, y_2)),$$

where $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ are two functions.

In 2016, Su and his partners [7] proved the existence and uniqueness of the multivariate fixed point for contraction type mappings in complete metric spaces. The following concepts are useful in the discussion.

Definition 1.5 ([7])**.** A multiply metric function $\Delta(a_1, a_2, ..., a_N)$ is a continuous N variable non-negative real function with the domain

$$\{(a_1, a_2, ..., a_N) \in R_N : a_i \geq 0, i \in \{1, 2, 3, ..., N\}\},$$

which satisfies the following conditions:

(1) $\Delta(a_1, a_2, ..., a_N)$ is non-decreasing for each variable $a_i, i \in \{1, 2, 3, ..., N\}$;
(2) $\Delta(a_1 + b_1, a_2 + b_2, ..., a_N + b_N) \leq \Delta(a_1, a_2, ..., a_N) + \Delta(b_1, b_2, ..., b_N)$;
(3) $\Delta(a, a, ..., a) = a$;
(4) $\Delta(a_1, a_2, ..., a_N) \to 0 \Leftrightarrow a_i \to 0, i \in \{1, 2, 3, ..., N\}$ for all $a_i, b_i, a \in R$,

where R denotes the set of all real numbers.

Example 1.6 ([7])**.** The following are some basic examples of multiply metric functions.

(1) $\Delta_1(a_1, a_2, ..., a_N) = \frac{1}{N}\sum_{i=1}^{N} a_i$;
(2) $\Delta_2(a_1, a_2, ..., a_N) = \frac{1}{h}\sum_{i=1}^{N} q_i a_i,\ q_i \in [0, 1), 1 \leq i \leq N,\ 0 < h := \sum_{i=1}^{N} q_i < 1$;
(3) $\Delta_3(a_1, a_2, ..., a_N) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} a_i^2}$;
(4) $\Delta(a_1, a_2, ..., a_N) = \max\{a_1, a_2, ..., a_N\}$.

Definition 1.7 ([7])**.** Let (X, d) be a metric space, $T : X^N \to X$ be a N variable mapping, an element p is called a multivariate fixed point of T if $p = T(p, p, ..., p)$.

In 2016, Su et al. [7] proved the following result.

Theorem 1.8 ([7])**.** *Let (X, d) be a complete metric space, and $T : X^N \to X$ be an N variable mapping that satisfies the following condition:*

$$d(Tx, Ty) \leq h\triangle(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)),$$

where Δ is a multiply metric function,

$$x = (x_1, x_2, ..., x_N) \in X^N, \quad y = (y_1, y_2, ..., y_N) \in X^N,$$

and $h \in (0, 1)$ is a constant. Then T has a unique multivariate fixed point $p \in X$ and, for any $p_0 \in X^N$, the iterative sequence $\{p_n\} \subset X$ defined by

$$\begin{aligned} p_1 &= (Tp_0, Tp_0, ..., Tp_0), \\ p_2 &= (Tp_1, Tp_1, ..., Tp_1), \\ p_3 &= (Tp_2, Tp_2, ..., Tp_2), \\ &\vdots \\ p_{n+1} &= (Tp_n, Tp_n, ..., Tp_n), \\ &\vdots \end{aligned}$$

converges, in the multiply metric Δ, to $(p, p, ..., p) \in X^N$ and the iterative sequence $\{Tp_n\} \subset X$ converges, with respect to d, to $p \in X$.

In 2015, Su et al. [8] proved the existence and uniqueness of the fixed point for the generalized contraction type mappings in complete metric spaces.

Theorem 1.9 ([8])**.** *Let (X, d) be a complete metric space, and $T : X \to X$ be a mapping such that*

$$\psi(d(Tx, Ty)) \le \phi(d(x, y)), \quad \forall x, y \in X,$$

where $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are two functions satisfying the conditions:

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\left\{ \begin{array}{l} \psi(a_n) \le \phi(b_n) \\ a_n \to \varepsilon, b_n \to \varepsilon \end{array} \right. \quad \Rightarrow \quad \varepsilon = 0.$

Then T has a unique fixed point and, for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

Example 1.10 ([8])**.** The following functions satisfy conditions (1) and (2) of Theorem 1.9:

(1) $\left\{ \begin{array}{l} \psi_1(t) = t, \\ \phi_1(t) = \alpha t, \end{array} \right.$

where $0 < \alpha < 1$ is a constant;

(2) $\left\{ \begin{array}{l} \psi_2(t) = t^2, \\ \phi_2(t) = \ln(t^2 + 1); \end{array} \right.$

(3) $\left\{ \begin{array}{l} \psi_3(t) = t, \\ \phi_3(t) = \left\{ \begin{array}{ll} t^2, & 0 \le t \le \frac{1}{2}, \\ t - \frac{3}{8}, & \frac{1}{2} < t < +\infty; \end{array} \right. \end{array} \right.$

(4) $\left\{ \begin{array}{l} \psi_4(t) = \left\{ \begin{array}{ll} t, & 0 \le t \le 1, \\ t - \frac{1}{2}, & 1 < t < +\infty, \end{array} \right. \\ \phi_4(t) = \left\{ \begin{array}{ll} \frac{t}{2}, & 0 \le t \le 1, \\ t - \frac{4}{5}, & 1 < t < +\infty; \end{array} \right. \end{array} \right.$

(5) $\left\{ \begin{array}{l} \psi_5(t) = \left\{ \begin{array}{ll} t, & 0 \le t < 1, \\ \alpha t^2, & 1 \le t < +\infty; \end{array} \right. \\ \phi_5(t) = \left\{ \begin{array}{ll} t^2, & 0 \le t < 1, \\ \beta t, & 1 \le t < +\infty. \end{array} \right. \end{array} \right.$

In 2015, Su et al. [8] also proved the following best proximity point theorem for the generalized contraction type mappings in complete metric spaces.

Theorem 1.11 ([8])**.** *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \ne \emptyset$. Let $\psi, \varphi, \phi : [0, +\infty) \to [0, +\infty)$ be three functions satisfying the conditions:*

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\left\{ \begin{array}{l} \psi(a_n) \le \phi(b_n) \\ a_n \to \varepsilon, \quad b_n \to \varepsilon \end{array} \right. \quad \Rightarrow \quad \varepsilon = 0;$

(3) $\psi(t_n) \to 0 \quad \Rightarrow \quad t_n \to 0$;

(4) $t_n \to 0 \quad \Rightarrow \quad \varphi(t_n) \to 0$;

(5) $\varphi(a) \le \varphi(b) \quad \Rightarrow \quad a \le b$.

Let $T : A \to B$ be a mapping, such that

$$\psi(d(Tx, Ty)) \le \phi(d(x, y)), \quad \forall x, y \in A.$$

Suppose that the pair (A, B) has the (ψ, φ)-P-property and $T(A_0) \subseteq B_0$, then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

Theorem 1.12 ([8])**.** *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be two functions satisfying the conditions:*

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\left\{ \begin{array}{l} \psi(a_n) \le \phi(b_n) \\ a_n \to \varepsilon, \quad b_n \to \varepsilon \end{array} \right. \quad \Rightarrow \quad \varepsilon = 0$;

(3) $\psi(t_n) \to 0 \quad \Leftrightarrow \quad t_n \to 0$;

and $\psi(t)$ is nondecreasing. Let $T : A \to B$ be a mapping such that

$$\psi(d(Tx, Ty)) \le \phi(d(x, y)), \quad \forall x, y \in A.$$

Suppose that the pair (A, B) has the weak P-property and $T(A_0) \subseteq B_0$, then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

2. Main results

In the following, the concept of multivariate best proximity point is firstly introduced in this section.

Definition 2.1. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$, $T : A^N \to B$ be an N variable mapping, an element $p \in A$ is called a best proximity point of multivariate T if

$$d(p, T(p, p, ..., p)) = d(A, B).$$

Now, we prove an existence and uniqueness theorems of the multivariate best proximity point in the complete metric spaces which generalizes the results [8] and [7].

Theorem 2.2. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) with $A_0 \neq \emptyset$, $T : A^N \to B$ be an N variable mapping, that satisfies the following condition:*

$$d(Tx, Ty) \le h\Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)),$$

where Δ is a multiply metric function,

$$x = (x_1, x_2, ..., x_N) \in A^N, \quad y = (y_1, y_2, ..., y_N) \in A^N,$$

and $h \in (0, 1)$ is a constant. Suppose that the pair (A, B) has the weak P-property and $T(A_0^N) \subseteq B_0$, then there exists a unique p in A such that

$$d(p, T(p, p, ..., p)) = d(A, B)$$

and, for any $p_0 \in A^N$, the iterative sequence $\{p_n\} \subset A^N$ defined by:

$$\begin{aligned} p_1 &= (PTp_0, PTp_0, \cdots, PTp_0), \\ p_2 &= (PTp_1, PTp_1, \cdots, PTp_1), \\ p_3 &= (PTp_2, PTp_2, \cdots, PTp_2), \\ &\vdots \\ p_{n+1} &= (PTp_n, PTp_n, \cdots, PTp_n), \\ &\vdots \end{aligned}$$

converges, in the multiply metric $\triangle$, to $(p, p, \cdots, p) \in A^N$ and the iterative sequence $\{PTp_n\} \subset A$ converges, with respect to d, to $p \in A$, where

$$d(PTp_i, Tp_i) = d(A, B),\ \ i = 0, 1, 2, \cdots, N.$$

Proof. We first prove that B_0 is closed. Let $y_n \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the weak P-property that

$$d(y_n, y_m) \to 0 \Rightarrow d(x_n, x_m) \to 0$$

as $n, m \to \infty$, where $x_n, x_m \in A_0$ and $d(x_n, y_n) = d(A, B)$, $d(x_m, y_m) = d(A, B)$. This implies that $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of metric d we have $d(p, q) = d(A, B)$, that is, $q \in B_0$, and hence B_0 is closed.

Let $\overline{A}_0$ be the closure of A_0. We claim that, $T(\overline{A}_0^N) \subseteq B_0$. In fact, if $x \in \overline{A}_0^N \setminus A_0^N$, then there exists a sequence $\{x_n\} \subseteq A_0^N$ such that $x_n \to x$. This together with the fact that B_0 is closed implies that,

$$\lim_{n\to\infty} Tx_n = Tx \in B_0.$$

That is, $T(\overline{A}_0^N) \subseteq B_0$.

Define an operator $P_{A_0} : T(\overline{A}_0^N) \to A_0$,

$$P_{A_0}y = \{x \in A_0 : d(x, y) = d(A, B)\}.$$

This shows that $P_{A_0}T : \overline{A}_0^N \to \overline{A}_0$ is an N variable mapping from $\overline{A}_0^N$ into a complete metric subspace $\overline{A}_0$. Since the pair (A, B) has the weak P-property, then we have

$$d(P_{A_0}Tx, P_{A_0}Ty) \le d(Tx, Ty) \le h\Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)),$$

where Δ is a multiply metric function, $h \in (0, 1)$ is a constant, and

$$x = (x_1, x_2, ..., x_N) \in A^N, \quad y = (y_1, y_2, ..., y_N) \in A^N.$$

By using Theorem 1.8, we can get that $P_{A_0}T$ has a unique fixed point $p \in \overline{A}_0$, and, for any $p_0 \in \overline{A}_0^N$, the iterative sequence $\{p_n\} \subset \overline{A}_0^N$ defined by

$$\begin{aligned}
p_1 &= (P_{A_0}Tp_0, P_{A_0}Tp_0, ..., P_{A_0}Tp_0),\\
p_2 &= (P_{A_0}Tp_1, P_{A_0}Tp_1, ..., P_{A_0}Tp_1),\\
p_3 &= (P_{A_0}Tp_2, P_{A_0}Tp_2, ..., P_{A_0}Tp_2),\\
&\ \ \vdots\\
p_{n+1} &= (P_{A_0}Tp_n, P_{A_0}Tp_n, ..., P_{A_0}Tp_n)
\end{aligned}$$

converges in the multiply metric Δ to $(p, p, ..., p) \in \overline{A}_0^N$, and the iterative sequence $\{P_{A_0}Tp_n\} \subset \overline{A}_0$ converges, with respect to d, to $p \in \overline{A}_0$. That is, $p = P_{A_0}T(p, p, ..., p)$. It implies that

$$d(p, T(p, p, ..., p)) = d(A, B).$$

It is easy to see that p is also the unique one in $\overline{A}_0$ such that $d(p, T(p, p, ..., p)) = d(A, B)$. This completes the proof. □

By Theorem 1.11, we can get the following results.

Theorem 2.3. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq 0$. Let $\psi, \varphi, \phi : [0, +\infty) \to [0, +\infty)$ be three functions satisfying the conditions:*

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\begin{cases} \psi(a_n) \le \phi(b_n) \\ a_n \to \varepsilon, b_n \to \varepsilon \end{cases} \quad \Rightarrow \quad \varepsilon = 0$;

(3) $\psi(t_n) \to 0 \quad \Rightarrow \quad t_n \to 0$;

(4) $t_n \to 0 \quad \Rightarrow \varphi(t_n) \quad \to \quad 0$;

(5) $\varphi(a) \le \varphi(b) \quad \Rightarrow \quad a \le b$.

Let $T : A^N \to B$ be an N variable mapping, such that

$$\psi(d(Tx, Ty)) \le \phi(\Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)),$$

where Δ is a multiply metric function, and

$$x = (x_1, x_2, ..., x_N) \in A^N, \quad y = (y_1, y_2, ..., y_N) \in A^N.$$

Suppose that the pair (A, B) has the (ψ, φ)-P-property and $T(A_0^N) \subseteq B_0$, then there exists a unique p in A such that

$$d(p, T(p, p, ..., p)) = d(A, B)$$

and, for any $p_0 \in \overline{A}_0^N$, the operator $P_{A_0} : T(\overline{A}_0^N) \to A_0$ defined by

$$P_{A_0} y = \{x \in A_0 : d(x, y)) = d(A, B)\},$$

the iterative sequence $\{p_n\} \subset \overline{A}_0^N$ defined by

$$\begin{aligned} p_1 &= (P_{A_0}Tp_0, P_{A_0}Tp_0, ..., P_{A_0}Tp_0), \\ p_2 &= (P_{A_0}Tp_1, P_{A_0}Tp_1, ..., P_{A_0}Tp_1), \\ p_3 &= (P_{A_0}Tp_2, P_{A_0}Tp_2, ..., P_{A_0}Tp_2), \\ &\vdots \\ p_{n+1} &= (P_{A_0}Tp_n, P_{A_0}Tp_n, ..., P_{A_0}Tp_n), \\ &\vdots \end{aligned}$$

converges, in the multiply metric Δ, to $(p, p, ..., p) \in \overline{A}_0^N$, and the iterative sequence $\{P_{A_0}Tp_n\} \subset \overline{A}_0$ converges, with respect to d, to $p \in \overline{A}_0$.

By Theorem 1.12, we can get the following results.

Theorem 2.4. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq 0$. Let $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be two functions satisfying the conditions:*

(1) $\psi(a) \le \phi(b) \quad \Rightarrow \quad a \le b$;

(2) $\begin{cases} \psi(a_n) \le \phi(b_n) \\ a_n \to \varepsilon, b_n \to \varepsilon \end{cases} \quad \Rightarrow \quad \varepsilon = 0$;

(3) $\psi(t_n) \to 0 \quad \Leftrightarrow \quad t_n \to 0$.

Let $T : A^N \to B$ be an N variable mapping such that

$$\psi(d(Tx, Ty)) \le \phi(\Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)),$$

where Δ is a multiply metric function, and

$$x = (x_1, x_2, ..., x_N) \in A^N, \quad y = (y_1, y_2, ..., y_N) \in A^N.$$

Suppose that the pair (A, B) has the weak P-property and $T(A_0^N) \subseteq B_0$, then there exists a unique p in A

such that

$$d(p, T(p, p, ..., p)) = d(A, B)$$

and, for any $p_0 \in \overline{A}_0^N$, *the operator* $P_{A_0} : T(\overline{A}_0^N) \to A_0$ *defined by*

$$P_{A_0} y = \{x \in A_0 : d(x, y)) = d(A, B)\},$$

the iterative sequence $\{p_n\} \subset \overline{A}_0^N$ *defined by*

$$\begin{aligned} p_1 &= (P_{A_0} T p_0, P_{A_0} T p_0, ..., P_{A_0} T p_0), \\ p_2 &= (P_{A_0} T p_1, P_{A_0} T p_1, ..., P_{A_0} T p_1), \\ p_3 &= (P_{A_0} T p_2, P_{A_0} T p_2, ..., P_{A_0} T p_2), \\ &\vdots \\ p_{n+1} &= (P_{A_0} T p_n, P_{A_0} T p_n, ..., P_{A_0} T p_n), \\ &\vdots \end{aligned}$$

converges, in the multiply metric Δ, *to* $(p, p, ..., p) \in \overline{A}_0^N$, *and the iterative sequence* $\{P_{A_0} T p_n\} \subset \overline{A}_0$ *converges, with respect to* d, *to* $p \in \overline{A}_0$.

If we choose ψ_3, ϕ_3 in Example 1.10, by Theorem 2.2, we can get the following result.

Theorem 2.5. *Let* (A, B) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* $A_0 \neq 0$. *Let* $T : A^N \to B$ *be an* N *variable mapping such that*

(1) $0 \leq \Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)) \leq \frac{1}{2} \Rightarrow d(Tx, Ty) \leq (\Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)))^2$;

(2) $\frac{1}{2} < \Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)) \Rightarrow d(Tx, Ty) \leq \Delta(d(x_1, y_1), d(x_2, y_2), \cdots, d(x_N, y_N)) - \frac{3}{8}$,

where Δ *is a multiply metric function, and*

$$x = (x_1, x_2, ..., x_N) \in A^N, \quad y = (y_1, y_2, ..., y_N) \in A^N.$$

Suppose that the pair (A, B) *has the weak* P*-property and* $T(A_0^N) \subseteq B_0$, *then there exists a unique* p *in* A *such that*

$$d(p, T(p, p, ..., p)) = d(A, B)$$

and, for any $p_0 \in \overline{A}_0^N$, *the operator* $P_{A_0} : T(\overline{A}_0^N) \to A_0$ *defined by*

$$P_{A_0} y = \{x \in A_0 : d(x, y)) = d(A, B)\},$$

the iterative sequence $\{p_n\} \subset \overline{A}_0^N$ *defined by*

$$\begin{aligned} p_1 &= (P_{A_0} T p_0, P_{A_0} T p_0, ..., P_{A_0} T p_0), \\ p_2 &= (P_{A_0} T p_1, P_{A_0} T p_1, ..., P_{A_0} T p_1), \\ p_3 &= (P_{A_0} T p_2, P_{A_0} T p_2, ..., P_{A_0} T p_2), \\ &\vdots \\ p_{n+1} &= (P_{A_0} T p_n, P_{A_0} T p_n, ..., P_{A_0} T p_n), \\ &\vdots \end{aligned}$$

converges, in the multiply metric Δ, *to* $(p, p, ..., p) \in \overline{A}_0^N$, *and the iterative sequence* $\{P_{A_0} T p_n\} \subset \overline{A}_0$ *converges, with respect to* d, *to* $p \in \overline{A}_0$.

If we choose ψ_4, ϕ_4 in Example 1.10, by Theorem 2.2, we can get the following result.

Theorem 2.6. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq 0$. Let $T : A^N \to B$ be an N variable mapping such that*

(1) $0 \leq \Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)) \leq 1 \Rightarrow d(Tx, Ty) \leq \frac{1}{2}(\Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)))$;

(2) $1 < \Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)) \quad \Rightarrow \quad d(Tx, Ty) \leq \Delta(d(x_1, y_1), d(x_2, y_2), ..., d(x_N, y_N)) - \frac{3}{10}$,

where Δ is a multiply metric function, and

$$x = (x_1, x_2, ..., x_N) \in A^N, \quad y = (y_1, y_2, ..., y_N) \in A^N.$$

Suppose that the pair (A, B) has the weak P-property and $T(A_0^N) \subseteq B_0$, then there exists a unique p in A such that

$$d(p, T(p, p, ..., p)) = d(A, B)$$

and, for any $p_0 \in \overline{A}_0^N$, the operator $P_{A_0} : T(\overline{A}_0^N) \to A_0$ defined by

$$P_{A_0}y = \{x \in A_0 : d(x, y)) = d(A, B)\},$$

the iterative sequence $\{p_n\} \subset \overline{A}_0^N$ defined by

$$\begin{aligned} p_1 &= (P_{A_0}Tp_0, P_{A_0}Tp_0, ..., P_{A_0}Tp_0),\\ p_2 &= (P_{A_0}Tp_1, P_{A_0}Tp_1, ..., P_{A_0}Tp_1),\\ p_3 &= (P_{A_0}Tp_2, P_{A_0}Tp_2, ..., P_{A_0}Tp_2),\\ &\vdots\\ p_{n+1} &= (P_{A_0}Tp_n, P_{A_0}Tp_n, ..., P_{A_0}Tp_n),\\ &\vdots \end{aligned}$$

converges, in the multiply metric Δ, to $(p, p, ..., p) \in \overline{A}_0^N$, and the iterative sequence $\{P_{A_0}Tp_n\} \subset \overline{A}_0$ converges, with respect to d, to $p \in \overline{A}_0$.

If we choose ψ_5, ϕ_5 in Example 1.10, by Theorem 2.2, we can get the relatively result, which we omit it here.

References

[1] A. Amini-Harandi, N. Hussain, F. Akbar, *Best proximity point results for generalized contractions in metric spaces*, Fixed Point Theory Appl., **2013** (2013), 13 pages. 1

[2] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math Z., **112** (1969), 234–240. 1

[3] E. Karapınar, *On best proximity point of ψ-Geraghty contractions*, Fixed Point Theory Appl., **2013** (2013), 9 pages.

[4] C. Mongkolkeha, Y. J. Cho, P. Kumam, *Best proximity points for Geraghty's proximal contraction mappings*, Fixed Point Theory Appl., **2013** (2013), 17 pages.

[5] H. K. Nashine, P. Kumam, C. Vetro, *Best proximity point theorems for rational proximal contractions*, Fixed Point Theory Appl., **2013** (2013), 11 pages.

[6] S. Sadiq Basha, *Best proximity point theorems generalizing the contraction principle*, Nonlinear Anal., **74** (2011), 5844–5850.

[7] Y. F. Su, A. Petruşel, J.-C. Yao, *Multivariate fixed point theorems for contractions and nonexpansive mappings with applications*, Fixed Point Theory Appl., **2016** (2016), 19 pages. 1, 1, 1.5, 1.6, 1.7, 1, 1.8, 2

[8] Y. F. Su, J.-C. Yao, *Further generalized contraction mapping principle and best proximity theorem in metric spaces*, Fixed Point Theory Appl., **2015** (2015), 13 pages. 1.4, 1, 1.9, 1.10, 1, 1.11, 1.12, 2

[9] F. F. Yan, Y. F. Su, Q. S. Feng, *A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory Appl., **2012** (2012), 13 pages.

[10] J. L. Zhang, Y. F. Su, *Best proximity point theorems for weakly contractive mapping and weakly Kannan mapping in partial metric spaces*, Fixed Point Theory Appl, **2014** (2014), 8 pages.

[11] J. L. Zhang, Y. F. Su, Q. Q. Cheng, *A note on 'A best proximity point theorem for Geraghty-contractions'*, Fixed Point Theory Appl., **2013** (2013), 4 pages. 1, 1.1, 1.2, 1.3

[12] J. L. Zhang, Y. F. Su, Q. Q. Cheng, *Best proximity point theorems for generalized contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2013** (2013), 7 pages. 1