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A characteristic splitting mixed finite element method for three-dimensional saltwater intrusion problem

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Abstract

A combined method is developed for solving saltwater intrusion problem. A splitting positive definite mixed element method is used to solve the parabolic-type water head equation and a characteristic finite element method is used to solve the convection-diffusion type concentration equation. The convergence of this method is considered and the optimal L^2-norm error estimate is also derived. ©2016 All rights reserved.

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1. Introduction

In recent years, saltwater intrusion has occurred in many countries and regions all over the world, and caused great damages to industrial and agricultural productions, it is urgent to be tackled. With the increasing interest, there are more and more literatures on the problem in past decades, see [2, 3, 7, 9, 15, 19]. Yuan et al. have done a lot of work on numerical methods for this problem including characteristic finite difference methods [28], characteristic finite element methods [12, 24], upwind fractional-step finite difference methods [23, 25, 27, 29], and alternating-direction methods [26, 30].

However, solving the water head equation with the standard finite element method or finite difference method cannot directly obtain the approximate flux which appears in the concentration equation. The way

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to obtain the flux through differentiating the water head function will cause an extra error and reduce the accuracy. To obtain more accurate approximation of the flux function, Lian and Rui gave a mixed finite element method combined with a discontinuous Galerkin procedure in [11]. But the technique of the classical mixed finite element method leads to some saddle point problem whose numerical solutions have been quite difficult because of losing positive definite properties. In [13, 14, 21, 31–34], Yang et al. proposed a class of splitting positive definite mixed finite element methods, in which the mixed system is symmetric positive definite and the flux equation is separated from the original equation.

Moreover, the concentration equation is parabolic and normally convection-dominated. The standard Galerkin methods applied to the convection-dominated problems do not work well, and produce excessive numerical diffusion or nonphysical oscillation. A variety of numerical techniques have been introduced to obtain better approximations, such as higher-order Godunov scheme [4], streamline diffusion method [10], least-squares mixed finite element method [20], and the Eulerian-Lagrangian localized adjoint method (ELLAM) [5, 16–18]. Godunov schemes require that a CFL time-step constraint be imposed. Streamline diffusion method and least-squares mixed finite element method reduce the amount of diffusion but add a user-defined amount biased in the direction of the streamline. ELLAM conserves mass locally but it is difficult to evaluate the resulting integrals. The characteristic finite element methods [22, 35, 36], have much smaller numerical diffusion, nonphysical oscillations and time-truncation than those of standard methods, and can be used with a larger time step.

In this paper, a combined numerical method is constructed for solving saltwater intrusion problem: A splitting positive definite mixed finite element method is used to solve the water head equation of parabolic type and a characteristic finite element method is used to solve the concentration equation of convection-diffusion type. The application of the splitting positive definite mixed finite element method results in a symmetric positive definite coefficient matrix of the mixed element system and separating the flux equation from the water head equation so that one can obtain an approximate solution of the flux function fast and independently by using various effective algorithms. Meantime, the characteristic finite element method does well in handling convection-dominated diffusion problem. The convergence of this combined method is analyzed and an optimal L^2-norm error estimate under the classical mixed finite element spaces is also derived.

In order to illustrate our method, the following mathematical model of saltwater intrusion problem is considered: a coupled system composed of the water head equation and the concentration (of Cl^-) equation

$$\begin{aligned} &\text{(a)}\quad S_s\frac{\partial \mathcal{H}}{\partial t} - \nabla\cdot(\widetilde{\kappa}(\nabla\mathcal{H} - \eta c\mathbf{e}_3)) = -\phi\eta\frac{\partial c}{\partial t} + \frac{\rho}{\rho_0}q,\\ &\text{(b)}\quad \phi\frac{\rho_0}{\rho}\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(\phi D\nabla c) = (\widetilde{c} - c)q,\\ &\qquad\qquad x\in\Omega,\ 0<t\le T, \end{aligned} \tag{1.1}$$

with the initial-boundary conditions:

$$\begin{gathered} \mathbf{u}\cdot\nu = 0,\quad D\nabla c\cdot\nu = 0,\quad \text{on } \partial\Omega,\\ \mathcal{H}(x,0) = \mathcal{H}^0(x),\quad c(x,0) = c^0(x),\quad x\in\Omega, \end{gathered} \tag{1.2}$$

where Ω is a convex bounded domain in R^3 with the boundary $\partial\Omega$, S_s is the specific retention, $\mathcal{H} = \frac{p}{\rho_0 g} - z$ is water head function, p stands for pressure, ρ_0 represents the density of reference water (fresh water), g is gravitational acceleration, z is the height of water containing layer; ρ is the density dependent only on the concentration of salt c, Hugakorn's linearization $\rho = \rho_0(1 + \frac{c}{c_s})$ is adopted, c_s is the concentration corresponding to the maximum density, and ε is the density difference ratio $\varepsilon = \frac{\rho_s - \rho_0}{\rho_0}$. $\widetilde{\kappa} = \frac{\rho g}{\mu}\kappa$, μ is the viscosity of the fluid,

$$\kappa = \begin{bmatrix} \kappa_1 & 0 & 0\\ 0 & \kappa_2 & 0\\ 0 & 0 & \kappa_3 \end{bmatrix}$$

is the permeability; $\eta = \frac{\varepsilon}{c_s}$ is the density coupling coefficient; $\mathbf{e}_3 = (0,0,1)^T$; ϕ is the porosity; and q is the source or sink term; c stands for the concentration of Cl^-, $\widetilde{c}$ is the salt concentration near the source well,

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}$$

is the diffusion matrix, $\mathbf{u} = -\frac{\rho}{\rho_0}\kappa(\nabla\mathcal{H} - \eta c\mathbf{e}_3)$ is Darcy velocity; and ν is the unit vector outer normal to $\partial\Omega$.

2. Formulation of the method

Throughout this paper, usual definitions, notations, and norms of Sobolev spaces as in [1] are used. Let $(\,\cdot\,,\,\cdot)$ be the inner product in $L^2(\Omega)$. Introduce the spaces $H(\text{div};\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^3;\ \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$, $\mathcal{V} = \{\mathbf{v} \in H(\text{div};\Omega);\ \mathbf{v}\cdot\nu = 0 \quad \text{on } \partial\Omega\}$ and $\mathcal{W} = L^2(\Omega)$.

2.1. The splitting mixed variational formulation for water head and flux

The water head equation is a parabolic type equation, and we deal it with a splitting positive definite mixed finite element method. Define the flux $\boldsymbol{\sigma}$ as follows:

$$\boldsymbol{\sigma} = -\widetilde{\kappa}(\nabla\mathcal{H} - \eta c\mathbf{e}_3) = \frac{g\rho^2}{\rho_0\mu}\mathbf{u}.$$

So we have $\mathbf{u} = a(c)\boldsymbol{\sigma}$, $a(c) = \frac{\rho_0\mu}{g\rho^2}$. A mixed weak form of the system (1.1) (a) is given by:

$$\begin{aligned} &\text{(a)} \quad (\frac{\partial\mathcal{H}}{\partial t}, w) + (B\nabla\cdot\boldsymbol{\sigma}, w) = (B\beta q, w) - (B\phi\eta\frac{\partial c}{\partial t}, w), \quad \forall\, w \in \mathcal{W}, \\ &\text{(b)} \quad (\alpha(c)\boldsymbol{\sigma}, \mathbf{v}) - (\mathcal{H}, \nabla\cdot\mathbf{v}) = -(\eta c\mathbf{e}_3, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathcal{V}, \end{aligned} \tag{2.1}$$

where $\alpha(c) = 1/\widetilde{\kappa}$, $B = 1/S_s$ and $\beta = \rho(c)/\rho_0$.

From (2.1) (b) we derive

$$(\frac{\partial}{\partial t}(\alpha(c)\boldsymbol{\sigma}), \mathbf{v})) - (\frac{\partial\mathcal{H}}{\partial t}, \nabla\cdot\mathbf{v}) = -(\eta\frac{\partial c}{\partial t}\mathbf{e}_3, \mathbf{v}). \tag{2.2}$$

Taking $w = \nabla\cdot\mathbf{v}$ in (2.1) (a) and substituting it into (2.2), we get the mixed system

$$\begin{aligned} &\text{(a)} \quad (\frac{\partial}{\partial t}(\alpha(c)\boldsymbol{\sigma}), \mathbf{v}) + (B\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\mathbf{v}) = (B\beta q, \nabla\cdot\mathbf{v}) - (B\phi\eta\frac{\partial c}{\partial t}, \nabla\cdot\mathbf{v}) - (\eta\frac{\partial c}{\partial t}\mathbf{e}_3, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathcal{V}, \\ &\text{(b)} \quad (\frac{\partial\mathcal{H}}{\partial t}, w) = (B[\beta q - \nabla\cdot\boldsymbol{\sigma}], w) - (B\phi\eta\frac{\partial c}{\partial t}, w), \quad \forall\, w \in \mathcal{W}. \end{aligned} \tag{2.3}$$

From the system (2.3) we know that the flux equation is separated from the water head equation and then the water head function $\mathcal{H}$, if required, can be obtained from (2.3) (b) straightly.

2.2. The characteristic weak variational formulation for the concentration

Define the differentiation along the characteristic curves of the transport $\frac{\phi}{\beta}\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ by

$$\psi(x, c, \mathbf{u})\frac{\partial}{\partial\tau} = \frac{\phi}{\beta}\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla,$$

where $\psi(x, c, \mathbf{u}) = \sqrt{\phi^2/\beta^2(c) + |\mathbf{u}|^2}$. Note that the characteristic direction τ depends on x, the concentration c and Darcy velocity $\mathbf{u}$, which vary in space and time. It follows easily that the concentration equation can be rewritten in the equivalent form

$$\psi\frac{\partial c}{\partial\tau} - \nabla\cdot(\phi D\nabla c) = (\widetilde{c} - c)q.$$

And then, a variational form can be obtained as follows:

$$(\psi\frac{\partial c}{\partial\tau}, z) + (\phi D\nabla c, \nabla z) = ((\tilde{c} - c)q, z), \quad \forall\, z \in H^1(\Omega).$$

2.3. The combined approximation procedure

In this section, we will present a characteristic splitting mixed finite element (CSMFE) method for solving saltwater intrusion problem.

Define a uniform time partition: $0 =: t_0 < t_1 < \cdots < t_n = n\triangle t < \cdots < t_{N-1} < t_N := T$, with $\triangle t =: t_n - t_{n-1}$. The characteristic derivative is approximated by

$$\psi\frac{\partial c}{\partial\tau}\Big|_{t_n} \approx \frac{\psi(x, c^{n-1}, \mathbf{u}^{n-1})(c^n - \bar{c}^{n-1})}{\sqrt{(x-\bar{x})^2 + (\triangle t)^2}} = \frac{\phi}{\beta^{n-1}}\frac{c^n - \bar{c}^{n-1}}{\triangle t},$$

where

$$\bar{c}^{n-1} = c(\bar{x}^{n-1}), \quad \bar{x}^n = x - \frac{\mathbf{u}^{n-1}\beta^{n-1}}{\phi}\triangle t, \quad \beta^{n-1} = \frac{\rho(c^{n-1})}{\rho_0}.$$

Let $\mathcal{T}_{h_\sigma}$, $\mathcal{T}_{h_H}$, and $\mathcal{T}_{h_c}$ be triple families of quasi-regular finite element partitions of the domain Ω which may be the same one or not, such that the elements in the partitions have the diameters bounded by h_σ, h_H, and h_c, respectively. Let $\mathcal{V}_h \subset \mathcal{V}$, $\mathcal{W}_h \subset \mathcal{W}$, and $\mathcal{M}_h \subset H^1(\Omega)$ be finite element spaces defined on the partitions $\mathcal{T}_{h_\sigma}$, $\mathcal{T}_{h_H}$, and $\mathcal{T}_{h_c}$, respectively. Combined the method of characteristics with the splitting positive definite mixed element procedure, a new numerical method can be established:

CSMFE Algorithm: *Give an initial approximation* $(c_h^0, \mathcal{H}_h^0, \boldsymbol{\sigma}_h^0) \in \mathcal{M}_h \times \mathcal{W}_h \times \mathcal{V}_h$ *such that*

$$\begin{aligned}
&\text{(a)} \quad (c_h^0, z_h) = (c^0, z_h), \quad \forall\ z_h \in \mathcal{M}_h,\\
&\text{(b)} \quad (\mathcal{H}_h^0, w_h) = (H^0, w_h), \quad \forall\ w_h \in \mathcal{W}_h,\\
&\text{(c)} \quad (\alpha(c_h^0)\boldsymbol{\sigma}_h^0, \boldsymbol{v}_h) = -(\nabla\mathcal{H}_h^0 - \eta c_h^0\mathbf{e}_0, \boldsymbol{v}_h), \quad \forall\ \boldsymbol{v}_h \in \mathcal{V}_h.
\end{aligned}$$

For $n = 1, 2, \ldots, N$, *seek* $(c_h^n, \boldsymbol{\sigma}_h^n, \mathcal{H}_h^n) \in \mathcal{M}_h \times \mathcal{V}_h \times \mathcal{W}_h$ *such that*

$$\begin{aligned}
&\text{(a)} \quad (\frac{\phi}{\beta_h^{n-1}}\frac{c_h^n - \widehat{c}_h^{n-1}}{\triangle t}, z_h) + (\phi D\nabla c_h^n, \nabla z_h) = ((\widetilde{c}^n - c_h^n)q^n, z_h), \quad \forall\, z_h \in \mathcal{M}_h,\\
&\text{(b)} \quad (\frac{\alpha(c_h^n)\boldsymbol{\sigma}_h^n - \alpha(c_h^{n-1})\boldsymbol{\sigma}_h^{n-1}}{\triangle t}, \mathbf{v}_h) + (B\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\mathbf{v}_h)\\
&\qquad = (B\beta_h^n q^n, \nabla\cdot\boldsymbol{v}_h) - (B\phi\eta\frac{c_h^n - c_h^{n-1}}{\triangle t}, \nabla\cdot\boldsymbol{v}_h) - (\eta\frac{c_h^n - c_h^{n-1}}{\triangle t}\mathbf{e}_3, \boldsymbol{v}_h), \quad \forall\,\mathbf{v}_h \in \mathcal{V}_h,\\
&\text{(c)} \quad (\frac{\mathcal{H}_h^n - \mathcal{H}_h^{n-1}}{\triangle t}, w_h) = (B[\beta_h^n q^n - \nabla\cdot\boldsymbol{\sigma}_h^n], w_h) - (B\phi\eta\frac{c_h^n - c_h^{n-1}}{\triangle t}, w_h), \quad \forall\, w_h \in \mathcal{W}_h,
\end{aligned}$$

where

$$\widehat{c}_h^{n-1} = c_h^{n-1}(\widehat{x}^{n-1}), \quad \widehat{x}^{n-1} = x - \frac{\mathbf{u}_h^{n-1}\beta_h^{n-1}}{\phi}\triangle t, \quad \beta_h^n = \frac{\rho(c_h^n)}{\rho_0}.$$

3. Preliminaries and some lemmas

In this and the following sections, K and δ indicate a generic constant and a small positive constant independent of mesh parameters h_σ, h_H, h_c and time increment $\triangle t$, which may be different at their occurrences. We assume that finite element spaces $\mathcal{V}_h$, $\mathcal{W}_h$, and $\mathcal{M}_h$ have the inverse property (see [6]) and approximate properties that there exist some integers r, r_1, $k > 0$ and $l \geq 0$, such that, for $1 \leq q \leq \infty$,

$$\inf_{\mathbf{v}_h\in\mathcal{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_{[^q} \leq K_1 h_\sigma^{r+1}\|\mathbf{v}\|_{W^{r+1,q}}, \quad \forall\,\mathbf{v} \in H(\mathrm{div};\Omega) \cap W^{r+1,q}(\Omega),$$

$$\inf_{\mathbf{v}_h\in\mathcal{V}_h}\|\nabla\cdot(\mathbf{v}-\mathbf{v}_h)\|_{L^q}\le K_1h_\sigma^{r_1}\|\nabla\cdot\mathbf{v}\|_{W^{r_1,q}},\quad\forall\,\mathbf{v}\in H(\mathrm{div};\Omega)\cap W^{r_1+1,q}(\Omega),$$
$$\inf_{w_h\in\mathcal{W}_h}\|w-w_h\|_{L^q}\le K_1h_H^{l+1}\|w\|_{W^{l+1,q}},\quad\forall\,w\in L^2(\Omega)\cap W^{l+1,q}(\Omega),$$

where $r_1=r$ in cases of BDDM, BDM, and BDFM elements, or $r_1=r+1$ in cases of RT and Nedelec elements.

It is well-known that, in any one of the classical mixed finite element spaces, there exists an operator Π_h from $\mathcal{V}$ onto $\mathcal{V}_h$, see [6], such that, for any $1\le q\le+\infty$,

$$\begin{aligned}&\text{(a)}\quad(\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v}),\nabla\cdot\mathbf{v}_h)=0,\quad\forall\,\mathbf{v}_h\in\mathcal{V}_h,\\&\text{(b)}\quad\|\mathbf{v}-\Pi_h\mathbf{v}\|_{L^q}\le Kh_\sigma^{r+1}\|\mathbf{v}\|_{W^{r+1,q}},\\&\text{(c)}\quad\|\nabla\cdot(\mathbf{v}-\Pi_h\ \mathbf{v})\|_{L^q}\le Kh_\sigma^{r_1}\|\nabla\cdot\mathbf{v}\|_{W^{r_1,q}}.\end{aligned}$$

And we introduce a standard elliptic projection operator P_h from $H^1(\Omega)$ onto $\mathcal{M}_h$ such that, for all $z_h\in\mathcal{M}_h$ and $c\in H^1$,

$$(\phi D\nabla(c-P_hc),z_h)+\lambda(c-P_hc,z_h)=0,\tag{3.1}$$

where λ is a positive constant such that the bilinear form on the left-hand side of (3.1) is coercive in H^1. The following optimal error bounds were given in [6]:

$$\begin{aligned}&\text{(a)}\quad\|c-P_hc\|_{L^2}+h_c\|\nabla(c-P_hc)\|_{L^2}\le Kh_c^{k+1}\|c\|_{H^{k+1}}\\&\text{(b)}\quad\|\nabla P_hc\|_{L^\infty}\le K(c)<+\infty,\\&\text{(c)}\quad\|\frac{\partial(c-P_hc)}{\partial t}\|_{L^2}\le Kh_c^{k+1}\{\|c\|_{H^{k+1}}+\|\frac{\partial c}{\partial t}\|_{H^{k+1}}\}.\end{aligned}$$

Meanwhile, we also introduce the L^2 projection operator Q_h from $L^2(\Omega)$ onto $\mathcal{W}_h$ such that

$$(\mathcal{H}-Q_h\mathcal{H},w_h)=0,\quad\forall\,w_h\in\mathcal{W}_h.$$

It is well-known that the a priori error estimate

$$\|\mathcal{H}-Q_h\mathcal{H}\|_{L^2}\le Kh_H^{l+1}\|\mathcal{H}\|_{H^{l+1}},\quad\forall\ w\in H^{l+1}(\Omega)$$

holds.

Next, we will give two lemmas which are important to prove our theoretical result in the following section.

Lemma 3.1 ([21]). *Assume that the finite element space $\mathcal{V}_h$ is any one of the classical mixed finite element spaces defined in* [6]. *The super-approximation, which for any function,*

$$\begin{aligned}&(\varphi\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v}),\nabla\cdot\mathbf{v}_h)\\&\quad\le Kh_\sigma\|\nabla\cdot\mathbf{v}_h\|_{L^2}\min(\|\varphi\|_{H^1}\|\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})\|_{L^\infty},\min(\|\varphi\|_{W^{1,\infty}},h_\sigma^{-\frac{3}{2}})\|\nabla\cdot(\mathbf{v}-\Pi_h\mathbf{v})\|_{L^2})\end{aligned}$$

holds, for each function $\varphi\in W^{1,\infty}$, $\mathbf{v}\in\mathcal{V}$ and $\mathbf{v}_h\in\mathcal{V}_h$.

Lemma 3.2 ([8]). *Let $f\in L^2(\Omega)$ and $\breve{f}=f(x-g(x)\triangle t)$, where $g=(g_1,g_2,g_3)$. Assume that g_i and $\frac{\partial g_i}{\partial x_j}$ are bounded for $i,j=1,2,3$. Then,*

$$\|f-\breve{f}\|_{H^{-1}}\le K\|f\|_{L^2}\triangle t.$$

For convenience of analysis, we usually give the following hypotheses

$$\begin{aligned}&0<\phi_*\le\phi\le\phi^*,\quad 0<D_*\le D\le D^*,\quad 0<S_*\le S_s\le S^*,\\&0<\alpha_*\le\alpha\le\alpha^*,\quad 0<\beta_*\le\beta\le\beta^*,\quad 0<a_*\le a\le\ a^*,\\&\left|\frac{\partial\alpha(c)}{\partial c}\right|+\left|\frac{\partial\beta(c)}{\partial c}\right|+\left|\frac{\partial a(c)}{\partial c}\right|+\left|\frac{\partial^2a(c)}{\partial c^2}\right|\le K^*.\end{aligned}\tag{3.2}$$

We also assume the regularities of the solution of (1.1)-(1.2) as follows:

$$\begin{aligned} c \in L^\infty(H^{k+1}) \cap L^2(W^1_\infty), \quad \frac{\partial c}{\partial t} \in L^2(H^{k+1}) \cap L^\infty(L^\infty), \\ \frac{\partial^2 c}{\partial t^2} \in L^2(L^2), \quad \mathcal{H} \in L^\infty(H^{l+1}) \cap H^2(L^2), \\ \boldsymbol{\sigma} \in L^\infty(H^{r+1} \cap W^1_\infty), \quad \frac{\partial \boldsymbol{\sigma}}{\partial t} \in L^2(H^{r+1}) \cap L^\infty(L^\infty), \quad \frac{\partial^2 \boldsymbol{\sigma}}{\partial t^2} \in L^2(L^2), \end{aligned} \tag{3.3}$$

where $L^\infty(H^{k+1})$ denotes $L^\infty(0, T; H^{k+1}(\Omega))$: L^∞ is subject to time variable and H^{k+1} is subject to space variable, and the definitions of the other spaces are similar.

4. Convergence analysis and error estimate

For CSMFE Algorithm, we have the following main result:

Theorem 4.1. *Assume that the hypotheses* (3.2) *hold and the solution of system* (1.1)-(1.2) *has the regular properties* (3.3). *If the mesh parameters h_c, h_σ, and $\triangle t$ satisfy the relations*

$$\triangle t = o(h_c^{\frac{3}{2}}) = o(h_\sigma^{\frac{3}{2}}), \tag{4.1}$$

then there hold the priori error estimates

$$\text{(a)} \quad \max_n \|c^n - c_h^n\|_{L^2} + \max_n \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n\|_{L^2} \le K \left\{ h_c^{k+1} + h_\sigma^{r+1} + h_\sigma^{r_1+1} + \triangle t \right\},$$

$$\text{(b)} \quad \max_n \|\mathcal{H}^n - \mathcal{H}_h^n\|_{L^2} \le K \left\{ h_c^{k+1} + h_\sigma^{r+1} + h_\sigma^{r_1} + h_H^{l+1} + \triangle t \right\}.$$

Set $\xi_c^n = c_h^n - P_h c^n$, $\zeta_c^n = c^n - P_h c^n$, $\xi_\sigma^n = \boldsymbol{\sigma}_h^n - \Pi_h \boldsymbol{\sigma}^n$, $\zeta_\sigma^n = \boldsymbol{\sigma}^n - \Pi_h \boldsymbol{\sigma}^n$, $\xi_H^n = \mathcal{H}_h^n - Q_h \mathcal{H}^n$, and $\zeta_H^n = \mathcal{H}^n - Q_h \mathcal{H}^n$. We have to estimate bounds of ξ_c, ξ_σ, and ξ_H, which satisfy the error residual equations:

$$\begin{aligned} &(\frac{\phi}{\beta_h^{n-1}} \frac{\xi_c^n - \xi_c^{n-1}}{\triangle t}, z_h) + (\phi D \nabla \xi_c^n, \nabla z_h) + (q^n \xi_c^n, z_h) \\ &\quad = (\frac{\phi}{\beta_h^{n-1}} \frac{\partial c}{\partial t} + \mathbf{u}_h^{n-1} \cdot \nabla c^n - \frac{\phi}{\beta_h^{n-1}} \frac{c^n - \widehat{c}^{n-1}}{\triangle t}, z_h) + (\frac{\phi(\beta_h^{n-1} - \beta^n)}{\beta^n \beta_h^{n-1}} \frac{\partial c}{\partial t}, z_h) \\ &\qquad + (\frac{\phi}{\beta_h^{n-1}} \frac{\widehat{\xi}_c^{n-1} - \xi_c^{n-1}}{\triangle t}, z_h) + (\frac{\phi}{\beta_h^{n-1}} \frac{\zeta_c^n - \widehat{\zeta}_c^{n-1}}{\triangle t}, z_h) \\ &\qquad + ((\mathbf{u}^n - \mathbf{u}_h^{n-1}) \cdot \nabla c^n, z_h) - \lambda(\zeta_c^n, z_h) + (q^n \zeta_c^n, z_h), \quad \forall\, z_h \in \mathcal{M}_h, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &(\frac{\alpha(c_h^n)\xi_\sigma^n - \alpha(c_h^{n-1})\xi_\sigma^{n-1}}{\triangle t}, \mathbf{v}_h) + (B \nabla \cdot \xi_\sigma^n, \nabla \cdot \mathbf{v}_h) \\ &\quad = (\frac{\partial}{\partial t}(\alpha(c)\boldsymbol{\sigma}) - \frac{\alpha(c^n)\boldsymbol{\sigma}^n - \alpha(c^{n-1})\boldsymbol{\sigma}^{n-1}}{\triangle t}, \mathbf{v}_h) + (B\phi\eta(\frac{\partial c}{\partial t} - \frac{c^n - c^{n-1}}{\triangle t}), \nabla \cdot \mathbf{v}_h) \\ &\quad + (\eta(\frac{\partial c}{\partial t} - \frac{c^n - c^{n-1}}{\triangle t})\mathbf{e}_3, \mathbf{v}_h) + (\frac{\alpha(c_h^n)\zeta_\sigma^n - \alpha(c_h^{n-1})\zeta_\sigma^{n-1}}{\triangle t}, \mathbf{v}_h) \\ &\quad + (\frac{[\alpha(c^n) - \alpha(c_h^n)]\boldsymbol{\sigma}^n - [\alpha(c^{n-1}) - \alpha(c_h^{n-1})]\boldsymbol{\sigma}^{n-1}}{\triangle t}, \mathbf{v}_h) \\ &\quad + (B \nabla \cdot \zeta_\sigma^n, \nabla \cdot \mathbf{v}_h) + (B q^n (\beta_h^n - \beta^n), \nabla \cdot \mathbf{v}_h) \\ &\quad + (B\phi\eta \frac{\zeta_c^n - \zeta_c^{n-1}}{\triangle t}, \nabla \cdot \mathbf{v}_h) - (B\phi\eta \frac{\xi_c^n - \xi_c^{n-1}}{\triangle t}, \nabla \cdot \mathbf{v}_h) \\ &\quad + (\eta \frac{\zeta_c^n - \zeta_c^{n-1}}{\triangle t}\mathbf{e}_3, \mathbf{v}_h) - (\eta \frac{\xi_c^n - \xi_c^{n-1}}{\triangle t}\mathbf{e}_3, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathcal{V}_h, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}(\frac{\xi_H^n-\xi_H^{n-1}}{\triangle t},w_h)=&(Bq^n(\beta_h^n-\beta^n),w_h)-(B\nabla\cdot(\boldsymbol{\sigma}_h^n-\boldsymbol{\sigma}^n),w_h)\\&+(B\phi\eta(\frac{\partial c}{\partial t}-\frac{c^n-c^{n-1}}{\triangle t}),w_h)+(B\phi\eta\frac{\zeta_c^n-\zeta_c^{n-1}}{\triangle t},\nabla\cdot\mathbf{v}_h)\\&-(B\phi\eta\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},w_h),\quad\forall\, w_h\in\mathcal{W}_h.\end{aligned}\tag{4.4}$$

Lemma 4.2. *Assume that β, β', a, a', ϕ, and ϕ' are bounded and set $J_n=[t_{n-1},t_n]$, then there exists an estimate*

$$\begin{aligned}&(\frac{\phi}{\beta_h^{n-1}}\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},z_h)+(\phi D\nabla\xi_c^n,\nabla z_h)+(q^n\xi_c^n,z_h)\\&\quad\le K\Big\{\|\xi_c^{n-1}\|_{L^2}^2+\|\xi_\sigma^{n-1}\|_{L^2}^2+\|\zeta_c^{n-1}\|_{L^2}^2+\|\zeta_c^n\|_{L^2}^2+\|\zeta_\sigma^{n-1}\|_{L^2}^2\\&\qquad+\|z_h\|_{L^2}^2+\triangle t\|\frac{\partial^2c}{\partial\tau^2}\|_{L^2(J^n;L^2(\Omega))}^2+\triangle t\|\frac{\partial c}{\partial t}\|_{L^2(J^n;L^2(\Omega))}^2+\triangle t\|\frac{\partial\mathbf{u}}{\partial t}\|_{L^2(J^n;L^2)}^2\Big\}+\delta\|\nabla z_h\|_{L^2}^2.\end{aligned}\tag{4.5}$$

Proof. Set

$$\begin{aligned}&T_1=(\frac{\phi}{\beta_h^{n-1}}\frac{\partial c}{\partial t}+\mathbf{u}_h^{n-1}\cdot\nabla c^n-\frac{\phi}{\beta_h^{n-1}}\frac{c^n-\widehat{c}^{n-1}}{\triangle t},z_h),\\&T_2=(\frac{\phi(\beta_h^{n-1}-\beta^n)}{\beta^n\beta_h^{n-1}}\frac{\partial c}{\partial t},z_h),\\&T_3=(\frac{\phi}{\beta_h^{n-1}}\frac{\widehat{\xi}_c^{n-1}-\xi_c^{n-1}}{\triangle t},z_h),\qquad T_4=(\frac{\phi}{\beta_h^{n-1}}\frac{\zeta_c^n-\widehat{\zeta}_c^{n-1}}{\triangle t},z_h),\\&T_5=((\mathbf{u}^n-\mathbf{u}_h^{n-1})\cdot\nabla c^n,z_h),\qquad T_6=-\lambda(\zeta_c^n,z_h)+(q^n\zeta_c^n,z_h).\end{aligned}$$

To handle T_1, we require an induction hypothesis. Assume that

$$\|\mathbf{u}_h^{n-1}\|_{L^\infty}\le Kh_\sigma^{-\frac12}\left[\frac{h_\sigma^{\frac32}}{\triangle t}\right]^{\frac12}.\tag{4.6}$$

Then we have

$$\begin{aligned}|T_1|&\le K\left\{\|\frac{\phi}{\beta_h^{n-1}}\frac{\partial c}{\partial t}+\mathbf{u}_h^{n-1}\cdot\nabla c^n-\frac{\phi}{\beta_h^{n-1}}\frac{c^n-\widehat{c}^{n-1}}{\triangle t}\|^2+\|z_h\|_{L^2}^2\right\}\\&\le K\left\{\int_\Omega(\frac{\phi}{\beta_h^{n-1}\triangle t})^2(\frac{\psi(x,c_h^{n-1},\mathbf{u}_h^{n-1})\beta_h^{n-1}\triangle t}{\phi})^2|\int_{(\hat{x},t_{n-1})}^{(x,t_n)}\frac{\partial^2c}{\partial\tau^2}d\tau|^2dx+\|z_h\|_{L^2}^2\right\}\\&\le K\left\{\triangle t\|\frac{\psi^3(x,c_h^{n-1},\mathbf{u}_h^{n-1})\beta_h^{n-1}}{\phi}\|_\infty\int_\Omega\int_{(\hat{x},t_{n-1})}^{(x,t_n)}|\frac{\partial^2c}{\partial\tau^2}|^2d\tau dx+\|z_h\|^2\right\}\\&\le K\left\{\triangle t\int_\Omega\int_{t_{n-1}}^{t_n}|\frac{\partial^2c}{\partial\tau^2}|^2dtdx+\|z_h\|_{L^2}^2\right\}\\&\le K\left\{\triangle t\|\frac{\partial^2c}{\partial\tau^2}\|_{L^2(J^n;L^2(\Omega))}^2+\|z_h\|_{L^2}^2\right\},\end{aligned}$$

where we have used the fact that β, ϕ are bounded.

For T_2 we have

$$|T_2| \leq K\left\{\|\xi_c^{n-1}\|_{L^2}^2 + \|\zeta_c^{n-1}\|_{L^2}^2 + \triangle t\|\frac{\partial c}{\partial t}\|_{L^2(J^n;L^2(\Omega))}^2 + \|z_h\|_{L^2}^2\right\}.$$

To estimate the term T_3, we first show that the $g(x) = \mathbf{u}_h^{n-1}\beta_h^{n-1}/\phi$ and $\partial g(x)/\partial x_j$, $(j = 1,2,3)$ are bounded.

By the induction hypothesis (4.6), we can easily show the boundedness of $g(x)$. For $\partial g(x)/\partial x_j$, we know that

$$\begin{aligned}
\frac{\partial g(x)}{\partial x_j} =& \frac{1}{\phi^2}\left\{\mathbf{u}_h^{n-1}\phi\frac{\beta(c_h^{n-1})}{\partial c}\frac{\partial c_h^{n-1}}{\partial x_j} + \frac{\partial \mathbf{u}_h^{n-1}}{\partial x_j}\beta_h^{n-1}\phi - \mathbf{u}_h^{n-1}\beta_h^{n-1}\frac{\partial \phi}{\partial x_j}\right\} \\
=& \frac{1}{\phi^2}\left\{\mathbf{u}_h^{n-1}\phi\frac{\beta(c_h^{n-1})}{\partial c}\frac{\partial c_h^{n-1}}{\partial x_j} + \frac{\partial[a(c_h^{n-1})\boldsymbol{\sigma}_h^{n-1}]}{\partial x_j}\beta_h^{n-1}\phi - \mathbf{u}_h^{n-1}\beta_h^{n-1}\frac{\partial \phi}{\partial x_j}\right\} \\
=& \frac{1}{\phi^2}\{\mathbf{u}_h^{n-1}\phi\frac{\beta(c_h^{n-1})}{\partial c} + \frac{\partial a(c_h^{n-1})}{\partial c}\frac{\partial c_h^{n-1}}{\partial x_j}\boldsymbol{\sigma}_h^{n-1}\beta_h^{n-1}\phi \\
&+ a(c_h^{n-1})\frac{\partial \boldsymbol{\sigma}_h^{n-1}}{\partial x_j}\beta_h^{n-1}\phi - \mathbf{u}_h^{n-1}\beta_h^{n-1}\frac{\partial \phi}{\partial x_j}\}.
\end{aligned}$$

By inverse inequality and the induction hypothesis (4.6), we know that

$$\|\boldsymbol{\sigma}_h^{n-1}\|_{W^{1,\infty}}\triangle t = \|\frac{1}{a(c_h^{n-1})}\mathbf{u}_h^{n-1}\|_{W^{1,\infty}} \leq Kh_\sigma^{-1}\|\boldsymbol{\sigma}_h^{n-1}\|_{L^\infty}\triangle t \leq K\left[\frac{\triangle t}{h_\sigma^{\frac{3}{2}}}\right]^{\frac{1}{2}} = o(1).$$

Make another induction hypothesis

$$\|c_h^{n-1}\|_{L^\infty} \leq Kh_c^{-\frac{1}{2}}\left[\frac{h_c^{\frac{3}{2}}}{\triangle t}\right]^{\frac{1}{2}}. \tag{4.7}$$

Hence we can obtain

$$\|c_h^{n-1}\|_{W^{1,\infty}}\triangle t \leq Kh_c^{-1}\|c_h^{n-1}\|_{L^\infty}\triangle t \leq K\left[\frac{\triangle t}{h_c^{\frac{3}{2}}}\right]^{\frac{1}{2}} = o(1),$$

where we have used the condition (4.1).

Under the induction hypotheses (4.6) and (4.7), using the fact that β, $\partial\beta/\partial c$, $a(c)$, $\partial a(c)/\partial c$, ϕ, and $\partial\phi/\partial x_j$ are bounded, we have by Lemma 3.2 that

$$|T_3| \leq \|\frac{\phi}{\beta_h^{n-1}}\frac{\widehat{\xi}_c^{n-1} - \xi_c^{n-1}}{\triangle t}\|_{H^{-1}}\|z_h\|_{H^1} \leq K\|\xi_c^{n-1}\|_{L^2}^2 + \delta\|z_h\|_{H^1}^2.$$

Using the similar technique, we can get

$$|T_4| \leq \|\frac{\phi}{\beta_h^{n-1}}\frac{\zeta_c^{n-1} - \widehat{\zeta}_c^{n-1}}{\triangle t}\|_{H^{-1}}\|z_h\|_{H^1} \leq K\|\zeta_c^{n-1}\|_{L^2}^2 + \delta\|z_h\|_{H^1}^2.$$

For T_5 and T_6, we have

$$|T_5| \leq |((\mathbf{u}^n - \mathbf{u}^{n-1})\cdot\nabla c^n, z_h)| + |((\mathbf{u}^{n-1} - \mathbf{u}_h^{n-1})\cdot\nabla c^n, z_h)|$$

$$
\begin{aligned}
&\le |(\int_{t_{n-1}}^{t_n} \frac{\partial \mathbf{u}}{\partial t} dt \cdot \nabla c^n, z_h)| + |((a(c^{n-1}) - a(c_h^{n-1}))\boldsymbol{\sigma}^{n-1} \cdot \nabla c^n, z_h)| \\
&\quad + |(a(c_h^{n-1})\xi_\sigma^{n-1} \cdot \nabla c^n, z_h)| + |(a(c_h^{n-1})\zeta_\sigma^{n-1} \cdot \nabla c^n, z_h)| \\
&\le K\Big\{ \|\xi_c^{n-1}\|_{L^2}^2 + \|\zeta_c^{n-1}\|_{L^2}^2 + \|\xi_\sigma^{n-1}\|_{L^2}^2 + \|\zeta_\sigma^{n-1}\|_{L^2}^2 \\
&\quad + \|z_h\|_{L^2}^2 + \triangle t \|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(J^n;L^2)}^2 \Big\}
\end{aligned}
$$

and

$$
|T_6| \le K \left\{ \|\zeta_c^n\|_{L^2}^2 + \|z_h\|_{L^2}^2 \right\}.
$$

Substituting these estimates into (4.2), we get the inequality (4.5). This ends the proof of Lemma 4.2. □

To complete the proof of convergence theorem, we make another induction hypothesis as follows:

$$
\max_{0\le n\le N-1} \|\xi_\sigma^n\|_{L^2} + \max_{0\le n\le N-1} \|\xi_c^n\|_{L^2} = o\left(\max(h_\sigma^{\frac{3}{2}}, h_c^{\frac{3}{2}}) \right). \tag{4.8}
$$

Lemma 4.3. *Under the conditions of Lemma 4.2, the priori estimate*

$$
\|\xi_c^n\|_{L^2}^2 + \sum_{i=1}^{n} \|\nabla \xi_c^i\|_{L^2}^2 \triangle t \le K \left\{ \sum_{i=0}^{n-1} \left[\|\xi_c^i\|_{L^2}^2 + \|\xi_\sigma^i\|_{L^2}^2 \right] \triangle t + h_c^{2k+2} + h_\sigma^{2r+2} + (\triangle t)^2 \right\} \tag{4.9}
$$

holds for $0 < n \le N$.

Proof. Taking $z_h = \xi_c^n$ in (4.5), and noting that

$$
\begin{aligned}
(\frac{\phi}{\beta_h^{n-1}} \frac{\xi_c^n - \xi_c^{n-1}}{\triangle t}, \xi_c^n) \ge& \frac{1}{2\triangle t} \left[(\frac{\phi}{\beta_h^{n-1}} \xi_c^n, \xi_c^n) - (\frac{\phi}{\beta_h^{n-1}} \xi_c^{n-1}, \xi_c^{n-1}) \right] \\
=& \frac{1}{2\triangle t} \left[(\frac{\phi}{\beta_h^{n-1}} \xi_c^n, \xi_c^n) - (\frac{\phi}{\beta_h^{n-2}} \xi_c^{n-1}, \xi_c^{n-1}) \right] \\
&+ \frac{1}{2\triangle t} \left(\frac{\phi(\beta_h^{n-1} - \beta_h^{n-2})}{\beta_h^{n-2}} \xi_c^{n-1}, \xi_c^{n-1} \right) \\
\ge& \frac{1}{2\triangle t} \left[(\frac{\phi}{\beta_h^{n-1}} \xi_c^n, \xi_c^n) - (\frac{\phi}{\beta_h^{n-2}} \xi_c^{n-1}, \xi_c^{n-1}) \right] \\
&- \frac{1}{2\triangle t} \left| \left(\frac{\phi(\beta_h^{n-1} - \beta_h^{n-2})}{\beta_h^{n-1}\beta_h^{n-2}} \xi_c^{n-1}, \xi_c^{n-1} \right) \right|,
\end{aligned}
$$

we can get

$$
\begin{aligned}
&\frac{1}{2\triangle t} \left[(\frac{\phi}{\beta_h^{n-1}} \xi_c^n, \xi_c^n) - (\frac{\phi}{\beta_h^{n-2}} \xi_c^{n-1}, \xi_c^{n-1}) \right] + (\phi D \nabla \xi_c^n, \nabla \xi_c^n) + (q^n \xi_c^n, \xi_c^n) \\
&\quad \le K\Big\{ \|\xi_c^n\|_{L^2}^2 + \|\xi_c^{n-1}\|_{L^2}^2 + \|\xi_\sigma^{n-1}\|_{L^2}^2 + \|\zeta_c^{n-1}\|_{L^2}^2 + \|\zeta_\sigma^{n-1}\|_{L^2}^2 \\
&\quad + \|\zeta_c^n\|_{L^2}^2 + \|\frac{\partial \zeta_c}{\partial t}\|_{L^2}^2 + \triangle t \|\frac{\partial^2 c}{\partial \tau^2}\|_{L^2(J^n;L^2)}^2 + \triangle t \|\frac{\partial c}{\partial t}\|_{L^2(J^n;L^2(\Omega))}^2 \\
&\quad + \triangle t \|\frac{\partial \mathbf{u}}{\partial t}\|_{L^2(J^n;L^2)}^2 \Big\} + \delta \|\nabla \xi_c^n\|_{L^2}^2,
\end{aligned} \tag{4.10}
$$

where we have used the induction hypothesis (4.8). Multiplying (4.10) by $\triangle t$ and summing over n, for sufficiently small δ and $\triangle t$, we get the estimate (4.9). □

Lemma 4.4. *Assume that α, α', and α'' are bounded, then the priori estimate*

$$\begin{aligned}
&\Big(\frac{[\alpha(c^n)-\alpha(c_h^n)]\boldsymbol{\sigma}^n-[\alpha(c^{n-1})-\alpha(c_h^{n-1})]\boldsymbol{\sigma}^{n-1}}{\triangle t},\mathbf{v}_h\Big)\\
&\quad\le K\big\{\|\xi_c^n\|_{L^2}^2+\|\xi_c^{n-1}\|_{L^2}^2+\|\xi_c^{n-2}\|_{L^2}^2+\|\xi_\sigma^{n-1}\|_{L^2}^2\\
&\qquad+\|\xi_\sigma^{n-2}\|_{L^2}^2+\|\mathbf{v}_h\|_{L^2}^2+h_c^{2k+2}+h_\sigma^{2r+2}+(\triangle t)^2\big\}+\delta\|\nabla\cdot\mathbf{v}_h\|_{L^2}^2
\end{aligned}\tag{4.11}$$

holds for any $\mathbf{v}_h\in\mathcal{V}_h$.

Proof. Note that

$$\begin{aligned}
\boldsymbol{\sigma}^{n-1}&=\boldsymbol{\sigma}^n-\int_{t_{n-1}}^{t_n}\frac{\partial\boldsymbol{\sigma}}{\partial t}dt,\\
\alpha(c^n)-\alpha(c^{n-1})&=\int_0^1\alpha'(c^{n-1}+s(c^n-c^{n-1}))ds(c^n-c^{n-1}),\\
\alpha(c_h^n)-\alpha(c_h^{n-1})&=\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds(c_h^n-c_h^{n-1}).
\end{aligned}$$

So we have

$$\begin{aligned}
[\alpha(c^n)-\alpha(c^{n-1})]-[\alpha(c_h^n)-\alpha(c_h^{n-1})]&=\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds(\eta_c^n-\eta_c^{n-1})\\
&\quad-\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds(\xi_c^n-\xi_c^{n-1})\\
&\quad+\int_0^1[\zeta_c^{n-1}+s(\zeta_c^n-\zeta_c^{n-1})]\alpha''ds\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt\\
&\quad-\int_0^1[\xi_c^{n-1}+s(\xi_c^n-\xi_c^{n-1})]\alpha''ds\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt.
\end{aligned}$$

Utilizing this equation, we can easily get

$$\begin{aligned}
&\Big(\frac{[\alpha(c^n)-\alpha(c_h^n)]\boldsymbol{\sigma}^n-[\alpha(c^{n-1})-\alpha(c_h^{n-1})]\boldsymbol{\sigma}^{n-1}}{\triangle t},\mathbf{v}_h\Big)\\
&=\Big(\frac{[\alpha(c^n)-\alpha(c_h^n)-(\alpha(c^{n-1})-\alpha(c_h^{n-1}))]\boldsymbol{\sigma}^n}{\triangle t},\mathbf{v}_h\Big)\\
&\quad+\Big([\alpha(c^{n-1})-\alpha(c_h^{n-1})]\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\frac{\partial\boldsymbol{\sigma}}{\partial t}dt,\mathbf{v}_h\Big)\\
&=\Big(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\boldsymbol{\sigma}^n\frac{\zeta_c^n-\zeta_c^{n-1}}{\triangle t},\mathbf{v}_h\Big)\\
&\quad-\Big(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\boldsymbol{\sigma}^n\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},\mathbf{v}_h\Big)\\
&\quad+\Big(\int_0^1[\zeta_c^{n-1}+s(\zeta_c^n-\zeta_c^{n-1})]\alpha''ds\frac{\boldsymbol{\sigma}^n}{\triangle t}\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt,\mathbf{v}_h\Big)\\
&\quad-\Big(\int_0^1[\xi_c^{n-1}+s(\xi_c^n-\xi_c^{n-1})]\alpha''ds\frac{\boldsymbol{\sigma}^n}{\triangle t}\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt,\mathbf{v}_h\Big)\\
&\quad+\Big([\alpha(c^{n-1})-\alpha(c_h^{n-1})]\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\frac{\partial\boldsymbol{\sigma}}{\partial t}dt,\mathbf{v}_h\Big)\\
&=F_1+F_2+F_3+F_4+F_5.
\end{aligned}\tag{4.12}$$

Using Lemma 4.2, we can derive that

$$\begin{aligned}F_1+F_3+F_4+F_5 \leq& K\{\|\xi_c^n\|^2_{L^2}+\|\xi_c^{n-1}\|^2_{L^2}+\|\xi_c^{n-2}\|^2_{L^2}+\|\xi_\sigma^{n-1}\|^2_{L^2}\\&+\|\xi_\sigma^{n-2}\|^2_{L^2}+\|\mathbf{v}_h\|^2_{L^2}+h_c^{2k+2}+h_\sigma^{2r+2}+(\triangle t)^2\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.\end{aligned}\tag{4.13}$$

For F_2 we have

$$\begin{aligned}F_2 =&(\frac{\phi}{\beta_h^{n-1}}\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\boldsymbol{\sigma}^n\cdot\mathbf{v}_h\beta_h^{n-1}/\phi)\\ =&(\frac{\phi}{\beta_h^{n-1}}\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},R_M[\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\boldsymbol{\sigma}^n\cdot\mathbf{v}_h\beta_h^{n-1}/\phi])\\ \leq& K\{\|\xi_c^n\|^2_{L^2}+\|\xi_c^{n-1}\|^2_{L^2}+\|\xi_\sigma^{n-1}\|^2_{L^2}+\|\mathbf{v}_h\|^2_{L^2}+h_c^{2k+2}+h_\sigma^{2r+2}+(\triangle t)^2\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2},\end{aligned}\tag{4.14}$$

where R_M is a weighted L^2-projection operator from $L^2(\Omega)$ onto $\mathcal{M}_h$ such that

$$(\frac{\phi}{\beta_h^{n-1}}(z-R_Mz),z_h)=0,\quad \forall\ z\in L^2(\Omega),\ z_h\in\mathcal{M}_h.$$

Substituting (4.13) and (4.14) into (4.12), we get the estimate (4.11). □

Lemma 4.5. *Under the conditions of Lemmas* 4.2 *and* 4.4*, we have the following estimate*

$$\begin{aligned}&(\frac{\alpha(c_h^n)\xi_\sigma^n-\alpha(c_h^{n-1})\xi_\sigma^{n-1}}{\triangle t},\mathbf{v}_h)+(B\nabla\cdot\xi_\sigma^n,\nabla\cdot\mathbf{v}_h)\\ &\leq K\{\|\xi_c^n\|^2_{L^2}+\|\xi_c^{n-1}\|^2_{L^2}+\|\xi_c^{n-2}\|^2_{L^2}+\|\xi_\sigma^{n-1}\|^2_{L^2}+\|\xi_\sigma^{n-2}\|^2_{L^2}+\|\mathbf{v}_h\|^2_{L^2}\\ &\quad+h_\sigma^{2r+2}+h_\sigma^{2r_1+2}+h_c^{2k+2}+(\triangle t)^2\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.\end{aligned}\tag{4.15}$$

Proof. It is easily seen that

$$\begin{aligned}&(\frac{\partial}{\partial t}(\alpha(c)\boldsymbol{\sigma}^n)-\frac{\alpha(c^n)\boldsymbol{\sigma}^n-\alpha(c^{n-1})\boldsymbol{\sigma}^{n-1}}{\triangle t},\mathbf{v}_h)\\ &\quad+(B\phi\eta(\frac{\partial c}{\partial t}-\frac{c^n-c^{n-1}}{\triangle t}),\nabla\cdot\mathbf{v}_h)+(\eta(\frac{\partial c}{\partial t}-\frac{c^n-c^{n-1}}{\triangle t})\mathbf{e}_3,\mathbf{v}_h)\\ &\leq K\{\triangle t[\|\frac{\partial^2\alpha(c)\boldsymbol{\sigma}}{\partial t^2}\|^2_{L^2(J^n;L^2)}+\|\frac{\partial^2 c}{\partial t^2}\|^2_{L^2(J^n;L^2)}]+\|\mathbf{v}_h\|^2_{L^2}\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2},\end{aligned}$$

$$(Bq^n(\beta_h^n-\beta^n),\nabla\cdot\mathbf{v}_h)\leq K\{\|\xi_c^n\|^2_{L^2}+\|\eta_c^n\|^2_{L^2}\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2},$$

and

$$(B\phi\eta\frac{\zeta_c^n-\zeta_c^{n-1}}{\triangle t},\nabla\cdot\mathbf{v}_h)+(\eta\frac{\zeta_c^n-\zeta_c^{n-1}}{\triangle t}\mathbf{e}_3,\mathbf{v}_h)\leq K\{\|\mathbf{v}_h\|^2_{L^2}+h_c^{2k+2}\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.$$

As noted above, we know that

$$\begin{aligned}\alpha(c_h^{n-1}) =&\alpha(c_h^n)-\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds(c_h^n-c_h^{n-1})\\ =&\alpha(c_h^n)-\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds(\xi_c^n-\xi_c^{n-1})+\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds(\zeta_c^n-\zeta_c^{n-1})\\ &-\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt.\end{aligned}$$

So we have

$$\begin{aligned}
(\frac{\alpha(c_h^n)\zeta_\sigma^n-\alpha(c_h^{n-1})\zeta_\sigma^{n-1}}{\triangle t},\mathbf{v}_h)=&(\alpha(c_h^n)\frac{\zeta_\sigma^n-\zeta_\sigma^{n-1}}{\triangle t},\mathbf{v}_h)+(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\zeta_\sigma^{n-1}\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},\mathbf{v}_h)\\
&-(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\zeta_\sigma^{n-1}\frac{\zeta_c^n-\zeta_c^{n-1}}{\triangle t},\mathbf{v}_h)\\
&+(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\zeta_\sigma^{n-1}\frac{1}{\triangle t_n}\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt,\mathbf{v}_h)\\
\le&(\frac{\phi}{\beta_h^{n-1}}\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t_n},R_M[\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\zeta_\sigma^{n-1}\cdot\mathbf{v}_h\beta_h^{n-1}/\phi])\\
&+K\big\{\frac{1}{\triangle t}\|\frac{\partial\zeta_\sigma}{\partial t}\|^2_{L^2(J^n;L^2)}+\|\zeta_\sigma^{n-1}\|^2_{L^2}+\|\mathbf{v}_h\|^2_{L^2}\big\}\\
\le&K\big\{\|\xi_c^{n-1}\|^2_{L^2}+\|\xi_\sigma^{n-1}\|^2_{L^2}+\|\mathbf{v}_h\|^2_{L^2}+h_\sigma^{2r+2}+h_c^{2k+2}+(\triangle t)^2\big\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.
\end{aligned}$$

By Lemma 3.1 and the inverse property of the finite element space $\mathcal{V}_h$, we have the estimate

$$(B\nabla\cdot\zeta_\sigma^n,\nabla\cdot\mathbf{v}_h)\le Kh_\sigma\|\nabla\cdot\zeta_\sigma^n\|_{L^2}\|\nabla\cdot\mathbf{v}_h\|_{L^2}\le Kh_\sigma^{2r_1+2}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.$$

Using the similar technique as in (4.14), we can get the following inequality

$$\begin{aligned}
&|(B\phi\eta\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t},\nabla\cdot\mathbf{v}_h)|+|(\eta\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t}\mathbf{e}_3,\mathbf{v}_h)|\\
&\quad\le K\big\{\|\xi_c^{n-1}\|^2_{L^2}+\|\xi_\sigma^{n-1}\|^2_{L^2}+\|\mathbf{v}_h\|^2_{L^2}+h_\sigma^{2r+2}+h_c^{2k+2}+(\triangle t)^2\big\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.
\end{aligned}$$

Substituting these estimates into (4.3), we can complete the proof of Lemma 4.5. □

Lemma 4.6. *Under the conditions of Lemma 4.5, we have the estimate*

$$\begin{aligned}
&\|\xi_\sigma^n\|^2_{L^2}+\triangle t\sum_{i=1}^n\|\nabla\cdot\xi_\sigma^i\|^2_{L^2}\\
&\quad\le K\Big\{\triangle t\sum_{i=1}^n\big[\|\xi_\sigma^{i-1}\|^2_{L^2}+\|\xi_c^i\|^2_{L^2}+\|\xi_c^{i-1}\|^2_{L^2}\big]+h_\sigma^{2r+2}+h_\sigma^{2r_1+2}+h_c^{2k+2}+(\triangle t)^2\Big\}.
\end{aligned}\tag{4.16}$$

Proof. Take $\mathbf{v}_h=\xi_\sigma^n$ in (4.15), and note that

$$\begin{aligned}
(\frac{\alpha(c_h^n)\xi_\sigma^n-\alpha(c_h^{n-1})\xi_\sigma^{n-1}}{\triangle t},\xi_\sigma^n)\ge&\frac{1}{2\triangle t}\big[(\alpha(c_h^n)\xi_\sigma^n,\xi_\sigma^n)-(\alpha(c_h^{n-1})\xi_\sigma^{n-1},\xi_\sigma^{n-1})\big]+\frac{1}{2\triangle t}([\alpha(c_h^n)-\alpha(c_h^{n-1})]\xi_\sigma^n,\xi_\sigma^n)\\
=&\frac{1}{2\triangle t}\big[(\alpha(c_h^n)\xi_\sigma^n,\xi_\sigma^n)-(\alpha(c_h^{n-1})\xi_\sigma^{n-1},\xi_\sigma^{n-1})\big]\\
&+\frac{1}{2}(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\frac{\xi_c^n-\xi_c^{n-1}}{\triangle t}\xi_\sigma^n,\xi_\sigma^n)\\
&-\frac{1}{2}(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\frac{\eta_c^n-\eta_c^{n-1}}{\triangle t}\xi_\sigma^n,\xi_\sigma^n)\\
&+\frac{1}{2\triangle t}(\int_0^1\alpha'(c_h^{n-1}+s(c_h^n-c_h^{n-1}))ds\int_{t_{n-1}}^{t_n}\frac{\partial c}{\partial t}dt\xi_\sigma^n,\xi_\sigma^n).
\end{aligned}$$

Under the induction hypotheses (4.8), using the similar technique as in (4.14) and Lemma 4.2, we get

$$\begin{aligned}
&\frac{1}{2\triangle t}\big[(\alpha(c_h^n)\xi_\sigma^n,\xi_\sigma^n)-(\alpha(c_h^{n-1})\xi_\sigma^{n-1},\xi_\sigma^{n-1})\big]+(B\nabla\cdot\xi_\sigma^n,\nabla\cdot\xi_\sigma^n)\\
&\quad\le K\big\{\|\xi_c^n\|^2_{L^2}+\|\xi_\sigma^n\|^2_{L^2}+\|\xi_c^{n-1}\|^2_{L^2}+\|\xi_c^{n-2}\|^2_{L^2}+\|\xi_\sigma^{n-1}\|^2_{L^2}+\|\xi_\sigma^{n-2}\|^2_{L^2}\\
&\qquad+\|\mathbf{v}_h\|^2_{L^2}+h_\sigma^{2r+2}+h_\sigma^{2r_1+2}+h_c^{2k+2}+(\triangle t)^2\big\}+\delta\|\nabla\cdot\mathbf{v}_h\|^2_{L^2}.
\end{aligned}\tag{4.17}$$

Multiplying (4.17) by $2\triangle t$ and summing it over n, for sufficiently small δ, we get the estimate (4.16). □

Now, we can complete the proof of Theorem 4.1.

Proof. Under the induction hypothesis (4.6), (4.7), and (4.8), using Lemmas 4.3 and 4.6, we can get

$$\begin{aligned}&\|\xi_\sigma^n\|_{L^2}^2+\|\xi_c^n\|_{L^2}^2+\triangle t\sum_{i=1}^{n}\left[\|\nabla\cdot\xi_\sigma^i\|_{L^2}^2+\|\nabla\xi_c^i\|_{L^2}^2\right]\\&\quad\le K\left\{\triangle t\sum_{i=0}^{n-1}[\|\xi_u^i\|_{L^2}^2+\|\xi_c^i\|_{L^2}^2]+h_\sigma^{2r+2}+h_\sigma^{2r_1+2}+h_c^{2k+2}+(\triangle t)^2\right\}.\end{aligned}$$

Using the discrete Gronwall's inequality, we have

$$\begin{aligned}&\max_n\|\xi_\sigma^n\|_{L^2}^2+\max_n\|\xi_c^n\|_{L^2}^2+\triangle t\sum_{i=1}^{n}\left[\|\nabla\cdot\xi_\sigma^i\|_{L^2}^2+\|\nabla\xi_c^i\|_{L^2}^2\right]\\&\quad\le K\left\{h_\sigma^{2r+2}+h_\sigma^{2r_1+2}+h_c^{2k+2}+(\triangle t)^2\right\}.\end{aligned}\tag{4.18}$$

It is clear that the optimal error estimate (4.18) is derived under the induction hypotheses (4.6), (4.7), and (4.8). Now we have to check it. When $n=0$, for integers r, $k>0$ we have

$$\begin{aligned}\|\mathbf{u}_h^0\|_{L^\infty}&=\|a(c_h^0)\boldsymbol{\sigma}_h^0\|_{L^\infty}\le K\left\{\|\Pi_h\boldsymbol{\sigma}^0\|_{L^\infty}+\|\xi_\sigma^0\|_{L^\infty}\right\}\\&\le K\left\{\|\Pi_h\boldsymbol{\sigma}^0\|_{L^\infty}+h_\sigma^{-\frac32}\|\xi_\sigma^0\|_{L^2}\right\}\le Kh_\sigma^{-\frac12}\left[\frac{h_\sigma^{\frac32}}{\triangle t}\right]^{\frac12},\end{aligned}$$

$$\|c_h^0\|_{L^\infty}\le K\left\{\|P_hc^0\|_{L^\infty}+\|\xi_c^0\|_{L^\infty}\right\}\le K\left\{\|P_hc^0\|_{L^\infty}+h_c^{-\frac32}\|\xi_c^0\|_{L^2}\right\}\le Kh_c^{-\frac12}\left[\frac{h_c^{\frac32}}{\triangle t}\right]^{\frac12},$$

$$\|\xi_c^0\|_{L^\infty}+\|\xi_\sigma^0\|_{L^\infty}\le K\left\{h_c^{-\frac32}\|\xi_c^0\|_{L^2}+h_\sigma^{-\frac32}\|\xi_\sigma^0\|_{L^2}\right\}\le K\left\{h_c^{k-\frac12}+h_\sigma^{r-\frac12}\right\}.$$

For sufficiently small h_σ and h_c, the induction hypotheses (4.6), (4.7), and (4.8) are true at $n=0$. By (4.18), for $n<N$, we know that

$$\begin{aligned}\|\mathbf{u}_h^n\|_{L^\infty}&=\|a(c_h^n)\boldsymbol{\sigma}_h^n\|_{L^\infty}\le K\left\{\|\Pi_h\boldsymbol{\sigma}^n\|_{L^\infty}+\|\xi_\sigma^n\|_{L^\infty}\right\}\le K\left\{\|\Pi_h\boldsymbol{\sigma}^n\|_{L^\infty}+h_\sigma^{-\frac32}\|\xi_\sigma^n\|_{L^2}\right\}\\&\le K\left\{\|\Pi_h\boldsymbol{\sigma}^n\|_{L^\infty}+h_\sigma^{-\frac32}(h_\sigma^{r+1}+h_\sigma^{r_1+1}+h_c^{k+1}+\triangle t)\right\}\le Kh_\sigma^{-\frac12}\left[\frac{h_\sigma^{\frac32}}{\triangle t}\right]^{\frac12},\\\|c_h^n\|_{L^\infty}&\le K\left\{\|P_hc^n\|_{L^\infty}+\|\xi_c^n\|_{L^\infty}\right\}\le K\left\{\|P_hc^n\|_{L^\infty}+h_c^{-\frac32}\|\xi_c^n\|_{L^2}\right\}\\&\le K\left\{\|P_hc^n\|_{L^\infty}+h_c^{-\frac32}(h_\sigma^{r+1}+h_\sigma^{r_1+1}+h_c^{k+1}+\triangle t)\right\}\le Kh_c^{-\frac12}\left[\frac{h_c^{\frac32}}{\triangle t}\right]^{\frac12},\end{aligned}$$

and

$$\max_{n\le N}\|\xi_c^n\|_{L^2}+\max_{n\le N}\|\xi_\sigma^n\|_{L^2}\le K\left\{h_\sigma^{r+1}+h_\sigma^{r_1+1}+h_c^{k+1}+\triangle t\right\}.$$

Under the condition (4.1), we know that the induction hypotheses (4.6), (4.7), and (4.8) hold.

Finally, we consider the boundedness of ξ_H. Taking $w_h = \xi_H^n$ in (4.4) and using the estimate (4.18), we can easily get

$$\max_n \|\xi_H^n\|_{L^2}^2 \leq K\{h_c^{2k+2} + h_\sigma^{2r+2} + h_\sigma^{2r_1} + h_H^{2l+2} + (\triangle t)^2\}.$$

This ends the proof of Theorem 4.1. □

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