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3-variable Jensen ρ-functional inequalities and equations

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Abstract

In this paper, we introduce and investigate Jensen ρ-functional inequalities associated with the following Jensen functional equations

$$f(x+y+z)+f(x+y-z)-2f(x)-2f(y)=0,$$
$$f(x+y+z)-f(x-y-z)-2f(y)-2f(z)=0.$$

We prove the Hyers-Ulam-Rassias stability of the Jensen ρ-functional inequalities in complex Banach spaces and prove the Hyers-Ulam-Rassias stability of the Jensen ρ-functional equations associated with the ρ-functional inequalities in complex Banach spaces.

Keywords: Jensen functional inequalities, Hyers-Ulam-Rassias stability, complex Banach spaces.
2010 MSC: 39B62, 39B52, 46B25.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. The functional equation

$$f(x+y)=f(x)+f(y),$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is called to be an additive mapping. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [25] for linear

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mappings by considering an unbounded Cauchy difference. The paper of Rassias [25] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2–9, 12–16, 18–27, 29]).

In [17], Park et al. investigated the following inequalities

$$\|f(x)+f(y)+f(z)\| \le \left\|2f\left(\frac{x+y+z}{2}\right)\right\|,$$
$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|,$$
$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|,$$

in Banach spaces. Recently, Cho et al. [5] investigated the following functional inequality

$$\|f(x)+f(y)+f(z) \le \left\|Kf\left(\frac{x+y+z}{K}\right)\right\|, \quad (0<|K|<|3|),$$

in non-Archimedean Banach spaces.

The function equations

$$f(x+y+z)+f(x+y-z)-2f(x)=0, \tag{1.1}$$
$$f(x+y+z)-f(x-y-z)-2f(y)-2f(z)=0, \tag{1.2}$$

is called 3-variable Jensen. In this paper, we investigate the 3-variable Jensen functional equations and prove the Hyers-Ulam-Rassias stability of the functional inequalities in complex Banach spaces.

Throughout this paper, assume that X is a complex normed vector space with norm $\|\cdot\|$ and that $(Y,\|\cdot\|)$ is a complex Banach space.

2. Hyers-Ulam-Rassias stability of (1.1)

In this section, we prove that the Hyers-Ulam-Rassias stability of the 3-variable functional inequality

$$\begin{aligned}\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| \le{}& \|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|,\end{aligned} \tag{2.1}$$

in the complex Banach space, where ρ_1 and ρ_2 are the fixed complex numbers with $\|\rho_1\|<\frac{1}{2}$, $\|\rho_2\|<\frac{1}{2}$.

Lemma 2.1. *Let $f: X\to Y$ be a mapping. If it satisfies (2.1) for all $x,y,z\in X$, then f is additive.*

Proof. By letting $x=y=z=0$ in (2.1) for all $x,y,z\in X$, we get

$$\|2f(0)\| \le \|2\rho_1 f(0)\|,$$

thus $f(0)=0$.

By letting $x=y=0$ in (2.1), we get

$$\|f(z)+f(-z)\| \le \|\rho_2(f(-z)+f(z))\|,$$

and so $f(-x)=-f(x)$ for all $x\in X$.

Let $z=0$ in (2.1), so we have

$$\begin{aligned}\|2f(x+y)-2f(x)-2f(y)\| &\le \|\rho_1(f(x+y)-f(x)-f(y))\| \\ &\quad+\|\rho_2(f(x+y)-f(x)-f(y))\| \\ &=(|\rho_1|+|\rho_2|)\|f(x+y)-f(x)-f(y)\|,\end{aligned}$$

and so $f(x+y)=f(x)+f(y)$ for all $x,y\in X$. Hence $f: X\to Y$ is additive. $\square$

Corollary 2.2. *Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| = &\|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|\end{aligned}$$

for all $x, y, z \in X$. Then $F : X \to Y$ is additive.

We prove the Hyers-Ulam-Rassias stability of the additive functional inequality (2.1) in complex Banach spaces.

Theorem 2.3. *Let $f : X \to Y$ be a mapping. If there is a function $\varphi : X^3 \to [0,\infty)$ with $\varphi(0,0,0) = 0$ such that*

$$\begin{aligned}\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| \le &\|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|+\varphi(x,y,z),\end{aligned} \tag{2.2}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y, 2^j z\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \widetilde{\varphi}(x,x,0) \tag{2.3}$$

for all $x \in X$.

Proof. By letting $x = y = z = 0$ in (2.2), we get

$$\|2f(0)\| \le \|2\rho_1 f(0)\|,$$

so $f(0) = 0$. Let $y = x$ and $z = 0$ in (2.2), so we get

$$\|2f(2x) - 4f(x)\| \le |\rho_1|\|f(2x)-2f(x)\| + |\rho_2|\|f(2x)-2f(x)\| + \varphi(x,x,0)$$

for all $x \in X$. Thus

$$\begin{aligned}\left\|f(x) - \frac{f(2x)}{2}\right\| &\le \frac{1}{2-|\rho_1|-|\rho_2|}\frac{1}{2}\varphi(x,x,0) \\ &\le \varphi(x,x,0)\end{aligned}$$

for all $x \in X$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\left\|\frac{1}{(2)^l} f\left((2)^l x\right) - \frac{1}{(2)^m} f\left((2)^m x\right)\right\| \le \sum_{i=l}^{m-1} \frac{1}{2^i}\varphi\left(2^i x, 2^i x, 0\right) \tag{2.4}$$

for all $x \in X$.

It follows from (2.4) that the sequence $\left\{\frac{f(2^k x)}{2^k}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is a Banach space, the sequence $\left\{\frac{f(2^k x)}{2^k}\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty}\left\{\frac{f(2^k x)}{2^k}\right\}, \quad \forall x \in X.$$

By taking $m = 0$ and letting $l \to \infty$ in (2.4), we get (2.3).

It follows from (2.2) that

$$\begin{aligned}
&\|A(x+y+z)+A(x+y-z)-2A(x)-2A(y)\| \\
&\qquad = \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x+y-z}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right\| \\
&\qquad \le \lim_{n\to\infty} 2^n \left\| \rho_1\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right) \right\| \\
&\qquad\quad + \lim_{n\to\infty} 2^n \left\| \rho_2\left(f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right) \right\| \\
&\qquad\quad + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\qquad = \|\rho_1(A(x+y+z)-A(x)-A(y)-A(z))\| \\
&\qquad\quad + \|\rho_2(A(x+y-z)-A(x)-A(y)+A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. One can see that A satisfies the inequality (2.1) and so it is additive by Lemma 2.1.

Now, we show the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.2). Then one has

$$\begin{aligned}
\|A(x)-T(x)\| &= \left\| \frac{1}{2^k} A\left(2^k x\right) - \frac{1}{2^k} T\left(2^k x\right) \right\| \\
&\le \frac{1}{2^k} \left(\left\| A\left(2^k x\right) - f\left(2^k x\right) \right\| \right. \\
&\quad \left. + \left\| T\left(2^k x\right) - f\left(2^k x\right) \right\| \right) \\
&\le 2 \frac{1}{2^k} \widetilde{\varphi}(2^k x, 2^k x, 0),
\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. □

Corollary 2.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ ba a mapping such that*

$$\begin{aligned}
&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| \\
&\qquad \le \|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\
&\qquad\quad + \|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{2\theta}{2-2^r}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.2) such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \widetilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$.

Proof. The proof is similar to Theorem 2.3, we can get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2},\frac{y}{2},0\right)$$

for all $x \in X$.

Next, we can prove that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in X$, and define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$ that is similar to the corresponding part of the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ ba a mapping such that*

$$\begin{aligned}\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| \leq &\|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\|\\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|\\ &+\theta(\|x\|^r+\|y\|^r+\|z\|^r)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \frac{2^{1+r}\theta}{2^r-1}\|x\|^r$$

for all $x \in X$.

3. Hyers-Ulam-Rassias stability of (1.2)

In this section, we prove that the Hyers-Ulam-Rassias stability of the 3-variable functional inequality

$$\begin{aligned}\|f(x+y+z)-f(x-y-z)-2f(y)-2f(z)\| \leq &\|\rho_1(f(x+y+z)-f(x+y)-f(z))\|\\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|,\end{aligned} \tag{3.1}$$

in the complex Banach space, where ρ_1 and ρ_2 are the fixed complex numbers with $\|\rho_1\| < \frac{1}{2}$, $\|\rho_2\| < \frac{1}{2}$.

Lemma 3.1. *Let $f : X \to Y$ be a mapping. If it satisfies (3.1) for all $x, y, z \in X$, then f is additive.*

Proof. By letting $x = y = z = 0$ in (3.1) for all $x, y, z \in X$, we get

$$\|4f(0)\| \leq \|\rho_1 f(0)\|,$$

thus $f(0) = 0$ and by letting $x = y = 0$ in (3.1), we get

$$(1-|\rho_2|)\|f(z)+f(-z)\| \leq 0,$$

and so $f(-z) = -f(z)$ for all $z \in X$.

Let $x = 0$ in (3.1), so we have

$$\begin{aligned}\|f(y+z)-f(-y-z)-2f(y)-2f(z)\| \leq &\|\rho_1(f(y+z)-f(y)-f(z))\|\\ &+\|\rho_2(f(y-z)-f(y)+f(z))\|\end{aligned}$$

for all $y, z \in X$.

Thus

$$(2-|\rho_1|)\|f(y+z)-f(y)-f(z)\| \leq |\rho_2|\|f(y-z)-f(y)+f(z)\| \tag{3.2}$$

for all $y, z \in X$.

By replacing z by $-z$ in (3.2), we have

$$(2-|\rho_1|)\|f(y-z)-f(y)+f(z)\| \leq |\rho_2|\|f(y+z)-f(y)-f(z)\| \tag{3.3}$$

for all $y, z \in X$.

By (3.2) and (3.3), we get

$$(2-|\rho_1|)^2\|f(y+z)-f(y)-f(z)\| \leq |\rho_2|^2\|f(y+z)-f(y)-f(z)\|$$

for all $y, z \in X$.

Hence $f : X \to Y$ is additive. □

Corollary 3.2. *Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}\|f(x+y+z)-f(x-y-z)-2f(y)-2f(z)\| = &\|\rho_1(f(x+y+z)-f(x+y)-f(z))\| \\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|\end{aligned}$$

for all $x, y, z \in X$. Then $f : X \to Y$ is additive.

We prove the Hyers-Ulam-Rassias stability of the additive functional inequality (3.1) in complex Banach spaces.

Theorem 3.3. *Let $f : X \to Y$ be a mapping. If there is a function $\varphi : X^3 \to [0,\infty)$ with $\varphi(0,0,0) = 0$ such that*

$$\begin{aligned}\|f(x+y+z)-f(x-y-z)-2f(y)-2f(z)\| \leq &\|\rho_1(f(x+y+z)-f(x+y)-f(z))\| \\ &+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|+\varphi(x,y,z),\end{aligned} \tag{3.4}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^jx, 2^jy, 2^jz\right) < \infty$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \widetilde{\varphi}(x,x,0) \tag{3.5}$$

for all $x \in X$.

Proof. By letting $x = y = z = 0$ in (3.4), we get

$$\|4f(0)\| \leq \|\rho_1 f(0)\|.$$

So $f(0) = 0$.

Let $y = x$ and $z = 0$ in (3.4), so we get

$$\|f(2x)-2f(x)\| \leq |\rho_2|\|f(2x)-2f(x)\|+\varphi(x,x,0)$$

for all $x \in X$. Thus

$$\left\|f(x)-\frac{f(2x)}{2}\right\| \leq \frac{1}{1-|\rho_2|}\frac{1}{2}\varphi\left(x,x,0\right) \leq \varphi\left(x,x,0\right)$$

for all $x \in X$, since $|\rho_2| < \frac{1}{2}$, $\frac{1}{1-|\rho_2|} < 2$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\left\| \frac{1}{(2)^l} f\left((2)^l x\right) - \frac{1}{(2)^m} f\left((2)^m x\right) \right\| \leq \sum_{i=l}^{m-1} \frac{1}{2^i} \varphi\left(2^i x, 2^i x, 0\right) \tag{3.6}$$

for all $x \in X$.

It follows from (3.6) that the sequence $\left\{\frac{f(2^k x)}{2^k}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is a Banach space, the sequence $\left\{\frac{f(2^k x)}{2^k}\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} \left\{\frac{f(2^k x)}{2^k}\right\}, \quad \forall x \in X.$$

By taking $m = 0$ and letting $l \to \infty$ in (3.6), we get (3.5).

It follows from (3.4) that

$$\begin{aligned}
\|A(x+y+z) &- A(x-y-z) - 2A(y) - 2A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x-y-z}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right) \right\| \\
&\leq \lim_{n\to\infty} 2^n \left\| \rho_1\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right) \right\| \\
&\quad + \lim_{n\to\infty} 2^n \left\| \rho_2\left(f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right) \right\| \\
&\quad + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\rho_1(A(x+y+z) - A(x+y) - A(z))\| \\
&\quad + \|\rho_2(A(x+y-z) - A(x) - A(y) + A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. One can see that A satisfies the inequality (3.1) and so it is additive by Lemma 3.1.

Now, we show the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (3.4). Then one has

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\| \frac{1}{2^k} A\left(2^k x\right) - \frac{1}{2^k} T\left(2^k x\right) \right\| \\
&\leq \frac{1}{2^k} \left(\left\| A\left(2^k x\right) - f\left(2^k x\right) \right\| \right. \\
&\quad \left. + \left\| T\left(2^k x\right) - f\left(2^k x\right) \right\| \right) \\
&\leq 2 \frac{1}{2^k} \widetilde{\varphi}(2^k x, 2^k x, 0),
\end{aligned}$$

which tends to zero as $k \to \infty$, for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. □

Corollary 3.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
\|f(x+y+z) - f(x-y-z) - 2f(y) - 2f(z)\| \\
\leq \|\rho_1(f(x+y+z) - f(x+y) - f(z))\| \\
+ \|\rho_2(f(x+y-z) - f(x) - f(y) + f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r$$

for all $x \in X$.

Theorem 3.5. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (3.4) such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \widetilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$.

Proof. The proof is similar to Theorem 3.3, we can get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}, 0\right)$$

for all $x \in X$.

Next, we can prove that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in X$, and define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$, that is similar to the corresponding part of the proof of Theorem 3.3. □

Corollary 3.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z)\| \\
&\qquad \leq \|\rho_1(f(x + y + z) - f(x + y) - f(z))\| \\
&\qquad\quad + \|\rho_2(f(x + y - z) - f(x) - f(y) + f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{1+r}\theta}{2^r - 1}\|x\|^r$$

for all $x \in X$.

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