Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6396–6407
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed points of the multifunction concerning F-contractions in partial metric spaces

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Communicated by S. S. Chang

Abstract

In this paper, we prove some new fixed point theorems for multi-valued mappings under new contractions by proposing a new class of functions. The results of this paper improve several results in the literatures. And we extend the results into metric-like spaces, which expand the application range of the results.

Keywords: $\mathcal{F}$-contraction, fixed point, multi-valued mappings, partial metric space.
2010 MSC: 47H10, 54H25, 47H04.

1. Preliminaries

In 1992, Matthews [13] proposed the concept of partial metric space on the basis of metric space, and studied the Banach contraction mapping principle in partial metric space. Then, the scholars have enriched the partial metric and established the partial metric theory. In 1996, O'Neill [18] used R instead of R^+ in Matthews's result and proved the theorems. Oltra and Valero [17] combined the both sides of the Banach contractive condition with the absolute value and generalized the results of the predecessors. In 2011, Karapinar and Erhan [11] gave a new mapping called orbitally continuous operator in partial metric space and proved the related fixed point theorems. After that, Abdeljawad et al. [1] denoted a general form of the weak ϕ-contraction and proved the common fixed point theorem with such mapping. Moradi and Farajzadeh [15] proved the fixed point theorems for (ψ, φ)-weak and generalized (ψ, φ)-weak contraction mappings in partial metric spaces. In 2012, Huang et al. [9] proved some fixed point theorems for expanding mapping in partial metric spaces.

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Received 2016-09-21

Throughout the whole paper, the letters R, R^+, N and N^* will denote the set of all real numbers, the set of all nonnegative real numbers, the set of all nonnegative integer numbers and the set of all positive integer numbers, respectively.

Recently, Wardowski (with Van Dung) [21] introduced the notion of an f-weak contraction mapping which improved his work in 2012 (see [20]) and proved the existence of fixed points with such mapping. The results of Wardowski [21] extended and unified several fixed point results in the literature.

Definition 1.1 ([21])**.** Let $\mathcal{F}$ be the family of all functions $F : (0,\infty) \to R$ such that

(F1) F is strictly increasing, i.e., for all $x, y \in R^+$ such that $x < y$, $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$, if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0} \alpha^k F(\alpha) = 0$.

Definition 1.2 ([21])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a F-contraction on (X, d), if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)), \tag{1.1}$$

where $x, y \in X$.

Remark 1.3. From (F1) and (1.1), it is easy to see that F-contraction is continuous.

In 2012, Piri [19] extended the results of Wardowski by using the following condition instead of (F3):

(F3′) F is continuous on $(0,\infty)$.

Piri denoted by $\mathfrak{F}$ the set of all functions satisfying the conditions (F1), (F2) and (F3′). Piri denoted the F-Suzuki contraction as follows.

Definition 1.4 ([19])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-Suzuki contraction, if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \ne Ty$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)),$$

where $F \in \mathfrak{F}$.

Karapinar and Kutbi [12] introduced the notion of a conditionally F-contraction in the setting of complete metric-like spaces and proved the fixed point theorems.

Definition 1.5 ([12])**.** Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (A), if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \le F(M_T(x, y)),$$

where

$$M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{4}\}.$$

Definition 1.6 ([12])**.** Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (B) if there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \le F(\max\{d(x, y), d(x, Tx), d(y, Ty)\}).$$

Definition 1.7 ([12])**.** Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (C), if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

The notion of a partial metric space was introduced by Matthews [13] in 1992. The partial metric space is a generalization of the usual metric space in which $d(x, x)$ is no longer necessarily zero.

Let X be a nonempty set. A function $p : X \times X \to R^+$ is said to be a partial metric on X, if for every $x, y, z \in X$, the following conditions hold:

(P_1) $p(x, x) = p(y, y) = p(x, y)$, if and only if $x = y$;

(P_2) $p(x, x) \leq p(x, y)$;

(P_3) $p(x, y) = p(y, x)$;

(P_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Then the pair (X, p) is called a partial metric space.

Remark 1.8. If $p(x, y) = 0$, then (P_1) and (P_2) imply that $x = y$. But the converse is not true.

Each partial metric p on X generates a T topology τ_p on X which has a base of the family open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$. And in the work of Matthews [14], we can find that a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

If p is a partial metric on X, then the function $p^* : X \times X \to R^+$ defines a metric on X, where $p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)$. Even more, a sequence $\{x_n\}$ in (X, p^*) converges to a point $x \in X$, if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x). \tag{1.2}$$

Definition 1.9 ([14])**.** Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\} \in X$ is said to be a Cauchy sequence, if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(2) (X, p) is said to be complete, if every Cauchy sequence $\{x_n\} \in X$ converges to a point $x \in X$ with respect to τ_p such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Then, we say that the partial metric p is complete.

Lemma 1.10 ([14])**.** Let (X, p) be a partial metric space. Then:

(1) A sequence $\{x_n\} \in X$ is said to ba a Cauchy sequence in (X, p), if and only if it is a Cauchy sequence in metric space (X, p^*).

(2) A partial metric space (X, p) is complete, if and only if the metric space (X, p^*) is complete.

In 1941, Kakutani [10] proved the fixed point theorem for the set-valued mapping. Then some fixed point theorems for the multifunction on metric space are given in [4, 3, 8, 16, 22]. Aleomraninejad et al. [2] gave a new way to prove the common fixed point of a multifunction on partial metric space by denoting two classes of functions called R_1 and R_2.

We denote the family of all nonempty subsets of X by 2^X, the family of all closed and bounded subsets of X by $CB(X)$. Let $T : X \to 2^X$ be a multi-valued function. We say that $x \in X$ is a fixed point of T, if $x \in Tx$. Consistent with other literatures, the following definitions and results will be needed in the sequel.

Definition 1.11 ([6])**.** Let (X, p) be a partial metric space. For all $A, B \in CB(X)$, define

$$H_p(A, B) = \max\{\sup_{a\in A} p(a, B), \sup_{b\in B} p(b, A)\},$$

where $p(x, A) = \inf_{a\in A} p(x, a)$.

It is known that H_p is a partial Hausdorff distance on $CB(X)$ introduced by Aydi et al. [6].

Lemma 1.12 ([6])**.** Let (X, p) be a partial metric space. For $A, B \in CB(X)$, we have

(a) $H_p(A, B) = 0 \Rightarrow A = B$. The converse is not true.

(b) There exists $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that

$$p(a, b) \leq hH_p(A, B).$$

(c) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$.

Lemma 1.13 ([5])**.** Let (X, p) be a partial metric space, $A \subseteq X$, and $x \in X$. Then $x \in \overline{A}$, if and only if $p(x, A) = p(x, x)$.

In this paper, we introduce a new type of F-contraction with some weaker condition and prove the fixed point theorems of multi-valued mappings in partial metric space, and we extend the results into metric-like spaces, which expand the application range of the results.

2. Main results

In this paper, we use the following condition instead of the condition (F3) in Definition 1.1:

(F*) For every sequence $\{\beta_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \beta_n = \infty$ implies $F(\beta_n)$ exists and is finite.

We denote the set of all functions satisfying the conditions (F1), (F2) and (F*) by $\mathscr{F}$.

Example 2.1. Let $F_1(x) = -\frac{1}{x}$, $F_2(x) = 1 - \frac{1}{x}$, $F_3(x) = -\frac{1}{[x]}$. Then $F_1, F_2, F_3 \in \mathscr{F}$.

Remark 2.2. Notice that the conditions (F3) and (F*) work independently of each other. Indeed, for any $p \geq 1$ and $0 < \alpha < +\infty$, $F(x) = \alpha - \frac{1}{x^p}$ satisfies the condition (F*), but it does not satisfy the condition (F3). Therefore $\mathscr{F} \subsetneq \mathcal{F}$. If we take $F(x) = \ln x$. It is easy to see that $F(x)$ satisfies the condition (F3), not the condition (F*). Therefore, $\mathcal{F} \subsetneq \mathscr{F}$. Also, if we take $F(x) = 1 - \frac{1}{\ln x}$, then $F \in \mathcal{F}$ and $F \in \mathscr{F}$. Therefore, $\mathcal{F} \bigcap \mathscr{F} \neq \emptyset$.

Remark 2.3. The conditions (F3$'$) and (F*) work independently of each other. Indeed, if we take $F(x) = x - \frac{1}{x}$. It is easy to see that $F(x)$ satisfies the condition (F3$'$), not (F*). Therefore $\mathfrak{F} \subsetneq \mathscr{F}$. Then, if we take $F(x) = 1 - \frac{1}{[x]}$, it satisfies the condition (F^*), not the condition $(F3')$. So $\mathscr{F} \subsetneq \mathfrak{F}$. But if $F(x) = 1 - \frac{1}{x}$, then $F \in \mathscr{F}$ and $F \in \mathfrak{F}$. Therefore, $\mathfrak{F} \bigcap \mathscr{F} \neq \emptyset$.

In view of Remarks 2.2 and 2.3, it is meaningful to consider the result of Wardowski [21] and the result of Piri [19] with the mapping $F \in \mathscr{F}$ instead $F \in \mathcal{F}$ and $F \in \mathfrak{F}$. Also, we define F_w-contraction as follows.

Definition 2.4. Let (X, p) be a partial metric space. A multi-valued mapping $T : X \to CB(X)$ is said to be an F_w-contraction of type (A), if there exists $\tau > 0$ such that for all $x, y \in X$ with $H_p(Tx, Ty) \neq 0$,

$$\frac{1}{2}p(x, Tx) < p(x, y) \Rightarrow \tau + F(H_p(Tx, Ty)) \leq F(M(x, y)),$$

where $F \in \mathscr{F}$ and $M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x,Ty)+p(y,Tx)}{4}\}$.

Remark 2.5. From (F1) and (1.1) it is easy to conclude that every F_w-contraction is a continuous mapping.

Now, we are ready to present our main results.

Theorem 2.6. Let (X, p) be a complete partial metric space and let $T : X \to CB(X)$ be an F_w-contraction of type (A), then T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence as follows

$$x_n \in Tx_{n-1} \quad \text{and} \quad p(x_n, x_{n+1}) < hH_p(Tx_{n-1}, Tx_n), \quad \text{where} \quad h > 1.$$

If there exists $n \in N$ such that $H_p(Tx_{n-1}, Tx_n) = 0$, then x_n is a fixed point which completes the proof. So we assume that, for every $n \in N$,

$$H_p(Tx_{n-1}, Tx_n) > 0.$$

Hence, by the definition of $p(a, B)$ which B is a nonempty set, we have for all $n \in N^*$,

$$\frac{1}{2}p(x_n, Tx_n) < p(x_n, x_{n+1}).$$

Since T is an F_w-contraction, from Definition (2.4), we have

$$\begin{aligned}
\tau + F(H_p(Tx_n, Tx_{n+1})) &\le F(\max\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \\
&\quad \frac{p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)}{4}\}) \\
&\le F(\max\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \\
&\quad \frac{H_p(Tx_{n-1}, Tx_{n+1}) + H_p(Tx_n, Tx_n)}{4}\}) \\
&\le F(\max\{p(x_n, x_{n+1}), H_p(Tx_{n-1}, Tx_n), p(x_{n+1}, Tx_{n+1}), \\
&\quad \frac{H_p(Tx_n, Tx_{n+1}) + H_p(Tx_{n-1}, Tx_n) + H_p(Tx_n, Tx_n)}{4}\}) \\
&\le F(\max\{p(x_n, x_{n+1}), H_p(Tx_{n-1}, Tx_n), H_p(Tx_n, Tx_{n+1})\}) \\
&\le F(\max\{hH_p(Tx_{n-1}, Tx_n), H_p(Tx_{n-1}, Tx_n), H_p(Tx_n, Tx_{n+1})\}) \\
&\le F(\max\{hH_p(Tx_{n-1}, Tx_n), H_p(Tx_n, Tx_{n+1})\}).
\end{aligned} \tag{2.1}$$

If $\max\{hH_p(Tx_{n-1}, Tx_n), H_p(Tx_n, Tx_{n+1})\} = H_p(Tx_n, Tx_{n+1})$, then (2.1) becomes

$$\tau + F(H_p(Tx_n, Tx_{n+1})) \le F(H_p(Tx_n, Tx_{n+1})),$$

which is a contradiction. Thus, we conclude that

$$\max\{hH_p(Tx_{n-1}, Tx_n), H_p(Tx_n, Tx_{n+1})\} = hH_p(Tx_{n-1}, Tx_n),$$

for all $n \in N^*$. Hence, (2.1) turns into

$$F(H_p(Tx_n, Tx_{n+1})) \le F(hH_p(Tx_{n-1}, Tx_n)) - \tau,$$

for all $n \in N^*$. By iteration, we obtain

$$\begin{aligned}
F(H_p(Tx_n, Tx_{n+1})) &\le F(hH_p(Tx_{n-1}, Tx_n)) - \tau \\
&\le F(h^2 H_p(Tx_{n-1}, Tx_n)) - 2\tau \\
&\;\;\vdots \\
&\le F(h^n H_p(Tx_0, Tx_1)) - n\tau.
\end{aligned}$$

From $\lim_{n\to\infty} h^n H_p(Tx_0, Tx_1) = \infty$, we obtain the limitation of $F(h^n H_p(Tx_0, Tx_1))$ is finite. So we have $\lim_{n\to\infty} F(H_p(Tx_n, Tx_{n+1})) = -\infty$, which together with (F2) gives

$$\lim_{n\to\infty} H_p(Tx_n, Tx_{n+1}) = 0.$$

It's obvious that for any $\varepsilon > 0$, there exists $N_1 \in N^*$ such that $H_p(Tx_n, Tx_{n+1}) < \frac{\varepsilon}{h}$ when $n > N_1$. By the construction of $\{x_n\}$, we have

$$p(x_{n+1}, x_{n+2}) < hH_p(Tx_n, Tx_{n+1}) < h \cdot \frac{\varepsilon}{h} = \varepsilon,$$

which implies that

$$\lim_{n\to\infty} p(x_{n+1}, x_{n+2}) = 0. \tag{2.2}$$

Now, we claim that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

We assume that there exists $\sigma > 0$, and two sequences $\{a(n)\}_{n=1}^{\infty}$ and $\{b(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$a(n) > b(n) > n, \quad p(x_{a(n)}, x_{b(n)}) \geq \sigma, \quad p(x_{a(n)-1}, x_{b(n)}) < \sigma, \quad H_p(Tx_{a(n)}, Tx_{b(n)}) > \sigma, \tag{2.3}$$

for all $n \in N^*$. From (P_4), we have

$$\begin{aligned} \sigma \leq p(x_{a(n)}, x_{b(n)}) &\leq p(x_{a(n)}, x_{a(n)-1}) + p(x_{a(n)-1}, x_{b(n)}) \\ &\leq p(x_{a(n)}, x_{a(n)-1}) + \sigma, \end{aligned} \tag{2.4}$$

for all $n \in N^*$. Thus from (2.2), (2.4) and the Sandwich Theorem, we get

$$\lim_{n\to\infty} p(x_{a(n)}, x_{b(n)}) = \sigma. \tag{2.5}$$

By (P_4), for all $n \in N^*$, we have

$$p(x_{a(n)}, x_{b(n)}) \leq p(x_{a(n)}, x_{a(n)+1}) + p(x_{a(n)+1}, x_{b(n)+1}) + p(x_{b(n)+1}, x_{b(n)}), \tag{2.6}$$

and

$$p(x_{a(n)+1}, x_{b(n)+1}) \leq p(x_{a(n)+1}, x_{a(n)}) + p(x_{a(n)}, x_{b(n)}) + p(x_{b(n)}, x_{b(n)+1}). \tag{2.7}$$

By letting $n \to \infty$ in (2.6) and (2.7), and by using (2.2) and (2.5), we get

$$\lim_{n\to\infty} p(x_{a(n)+1}, x_{b(n)+1}) = \sigma.$$

From (2.2) and (2.3), there exists $N_2 \in N^*$ such that $\frac{1}{2}p(x_{a(n)}, Tx_{a(n)}) < \frac{\sigma}{2} < p(x_{a(n)}, x_{b(n)})$, for all $n > N_2$.

From (2.3), for $n \geq N_3 \geq N_2$, we have $H_p(Tx_{a(n)}, Tx_{b(n)}) > \sigma$. Since T is an F_w-contraction of type (A), we have

$$\begin{aligned} \tau + F(p(x_{a(n)+1}, x_{b(n)+1})) &< \tau + F(H_p(Tx_{a(n)}, Tx_{b(n)})) \\ &\leq F(\max\{p(x_{a(n)}, x_{b(n)}), p(x_{a(n)}, Tx_{a(n)}), p(x_{b(n)}, Tx_{b(n)}), \\ &\quad \frac{p(x_{a(n)}, Tx_{b(n)}) + p(x_{b(n)}, Tx_{a(n)})}{4}\}) \\ &\leq F(\max\{p(x_{a(n)}, x_{b(n)}), p(x_{a(n)}, Tx_{a(n)}), p(x_{b(n)}, Tx_{b(n)}), \\ &\quad \frac{p(x_{a(n)}, Tx_{b(n)}) + p(x_{b(n)}, Tx_{a(n)}) + 2p(x_{a(n)}, x_{b(n)})}{4}\}) \\ &\leq F(\max\{p(x_{a(n)}, x_{b(n)}), p(x_{a(n)}, Tx_{a(n)}), p(x_{b(n)}, Tx_{b(n)})\}) \\ &\leq F(\max\{p(x_{a(n)}, x_{b(n)}), p(x_{a(n)}, x_{a(n)+1}), p(x_{b(n)}, x_{b(n)+1})\}). \end{aligned} \tag{2.8}$$

By letting $n \to \infty$ in (2.8), and by using (2.2) and (2.5) we find that

$$\tau + F(\sigma) \le F(\sigma),$$

which is a contradiction since $\tau > 0$. Hence $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

By the definition of p^*, we get $p^*(x_n, x_m) \le 2p(x_n, x_m)$. Thus $\lim_{n,m\to\infty} p^*(x_n, x_m) = 0$. This implies that $\{x_n\}$ is a Cauchy sequence in (X, p^*). Since (X, p) is complete, (X, p^*) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric p^*, that is $\lim_{n\to\infty} p^*(x_n, x^*) = 0$. From (1.2), we have

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_n) = 0.$$

Notice that

$$\begin{aligned} p(x^*, Tx^*) &\le p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) \\ &\le p(x^*, x_{n+1}) + p(x_{n+1}, x^*) + p(x^*, Tx^*) \\ &= 2p(x^*, x_{n+1}) + p(x^*, Tx^*). \end{aligned}$$

By the Sandwich Theorem, we get $\lim_{n\to\infty}[p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*)] = p(x^*, Tx^*)$, which implies that

$$\lim_{n\to\infty} p(x_{n+1}, Tx^*) = p(x^*, Tx^*). \tag{2.9}$$

Now we prove that, for every $n \in N^*$,

$$\frac{1}{2}p(x_n, Tx_n) < p(x_n, x^*), \quad \text{or} \quad \frac{1}{2}H_p(Tx_n, T^2x_n) < H_p(Tx_n, x^*). \tag{2.10}$$

By the contradiction, we assume that there exists $m \in N^*$ such that

$$\frac{1}{2}p(x_m, Tx_m) \ge p(x_m, x^*), \quad \text{and} \quad \frac{1}{2}H_p(Tx_m, T^2x_m) \ge H_p(Tx_m, x^*).$$

Now from (2.1) and (F1), we obtain $H_p(Tx_m, T^2x_m) < H_p(x_m, Tx_m)$. Thus

$$\begin{aligned} H_p(x_m, Tx_m) &\le H_p(x_m, x^*) + H_p(x^*, Tx_m) \\ &\le p(x_m, x^*) + \frac{1}{2}H_p(Tx_m, T^2x_m) \\ &< \frac{1}{2}p(x_m, Tx_m) + \frac{1}{2}H_p(x_m, Tx_m) \\ &\le \frac{1}{2}H_p(x_m, Tx_m) + \frac{1}{2}H_p(x_m, Tx_m) = H_p(x_m, Tx_m), \end{aligned}$$

which is a contradiction. Hence (2.10) holds.

Suppose $p(x^*, Tx^*) > 0$ and part (1) of (2.10) is satisfied. Then from our assumption, we have

$$\begin{aligned} \tau + F(H_p(Tx_n, Tx^*)) &\le F(\max\{p(x_n, x^*), p(x_n, Tx_n), p(x^*, Tx^*), \frac{p(x_n, Tx^*) + p(x^*, Tx_n)}{4}\}) \\ &\le F(\max\{p(x_n, x^*), p(x_n, Tx_{n+1}), p(x^*, Tx^*)\}) \\ &\le F(p(x^*, Tx^*)). \end{aligned}$$

Now $x_{n+1} \in Tx_n$ gives that

$$p(x_{n+1}, Tx^*) \le H_p(Tx_n, Tx^*). \tag{2.11}$$

From (2.11), we obtain $\tau + F(p(x_{n+1}, Tx^*)) \leq \tau + F(H_p(Tx_n, Tx^*)) \leq F(p(x^*, Tx^*))$. With (F1), we have $p(x_{n+1}, Tx^*) < p(x^*, Tx^*)$ which is a contradiction when $n \to \infty$. Suppose $p(x^*, Tx^*) > 0$ and part (2) of (2.10) is satisfied. Then from our assumption, we have

$$\begin{aligned}\tau + F(H_p(x_{n+2}, Tx^*)) &\leq \tau + F(H_p(Tx_{n+1}, Tx^*)) \\ &\leq F(\max\{p(x_{n+1}, x^*), p(x_{n+1}, Tx_{n+1}), p(x^*, Tx^*), \\ &\quad \frac{p(x_{n+1}, Tx^*) + p(x^*, Tx_{n+1})}{4}\}).\end{aligned}$$

From (2.2) and (2.9), there exists $N_4 \in N^*$ such that for all $n > N_4$,

$$\max\{p(x_{n+1}, x^*), p(x_{n+1}, Tx_{n+1}), p(x^*, Tx^*), \frac{p(x_{n+1}, Tx^*) + p(x^*, Tx_{n+1})}{4}\} = p(x^*, Tx^*).$$

Then, we get

$$\lim_{n\to\infty} [\tau + F(H_p(x_{n+2}, Tx^*))] = \tau + F(H_p(x^*, Tx^*)) \leq F(p(x^*, Tx^*)),$$

which is a contradiction. Thus, $p(x^*, Tx^*) = 0 = p(x^*, x^*)$. From Lemma 1.13, we obtain $x^* \in \overline{Tx^*} = Tx^*$. This completes the proof. □

Definition 2.7. Let (X, p) be a complete partial metric space. A mapping $T : X \to X$ is said to be an F_w-contraction of type (B), if there exist $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $H_p(Tx, Ty) \neq 0$,

$$\frac{1}{2} p(x, Tx) < p(x, y) \Rightarrow \tau + F(H_p(Tx, Ty)) \leq F(\max\{p(x, y), p(x, Tx), p(y, Ty)\}).$$

Definition 2.8. Let (X, p) be a complete partial metric space. A mapping $T : X \to X$ is said to be an F_w-contraction of type (C), if there exist $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $H_p(Tx, Ty) \neq 0$,

$$\frac{1}{2} p(x, Tx) < p(x, y) \Rightarrow \tau + F(H_p(Tx, Ty)) \leq F(p(x, y)).$$

Theorem 2.9. Let (X, p) be a complete partial metric space and let $T : X \to CB(X)$ be an F_w-contraction of type (B), then T has a fixed point $x^* \in X$.

Proof. By following the proof in Theorem 2.6, we can conclude the result. □

Theorem 2.10. Let (X, p) be a complete partial metric space and let $T : X \to CB(X)$ be an F_w-contraction of type (C), then T has a fixed point $x^* \in X$.

Proof. It is easy to conclude the result by following the proof of Theorem 2.6. □

3. Application

Now we consider an example to illustrate our main result. We consider a mapping T which is not continuous, so not an F-contraction but it is an F_w-contraction of type (A).

Example 3.1. Consider $X = \{0, 1, 2\}$. Let $p : X \times X \to [0, \infty)$ be a mapping defined by

$$p(0,0) = p(1,1) = 0, \quad p(2,2) = \frac{5}{2}, \quad p(0,2) = p(2,0) = 2,$$

$$p(1,2) = p(2,1) = 3, \quad p(0,1) = p(1,0) = \frac{3}{2}.$$

It is clear that p is a partial metric. Since $p(2,2) \neq 0$, so p is not a metric. Thus, (X,p) is a complete partial metric space. Let $T : X \to CB(X)$ be given by

$$T0 = 0 = T1, \quad \text{and} \quad T2 = \{0,1\}.$$

Suppose that $F(x) = 1 - \frac{1}{[x]} \in \mathscr{F}$ and $\tau \in (0, \frac{1}{2})$. Since T is not continuous, T is not an F-contraction by Remark 1.3.

We will consider the inequality

$$\frac{1}{2}p(x,Tx) < p(x,y), \tag{3.1}$$

where $x,y \in X$ with $H_p(Tx,Ty) \neq 0$ and the inequality

$$\tau + F(H_p(Tx,Ty)) \leq F(\max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{4}\}), \tag{3.2}$$

for all $x,y \in X$ with $H_p(Tx,Ty) \neq 0$ which satisfy (2.1).

Case 1: Let $x = 0$.

Now $H_p(T0,T0) = H_p(T0,T1) = p(0,0) = 0$, so we need only consider the case $y = 2$ in (3.1) and (3.2). Now (3.1) is true since

$$\frac{1}{2}p(0,T0) = 0 < p(0,2) = 3.$$

We also find

$$H_p(T0,T2) = \frac{3}{2} < \max\{p(0,2), p(0,T0), p(2,T2), \frac{p(0,T2)+p(2,T0)}{4}\} = \max\{2,0,2,\frac{1}{2}\} = 2.$$

Now (3.2) is satisfied since

$$\begin{aligned}\tau + F(H_p(T0,T2)) &= \tau + 1 - \frac{1}{[\frac{3}{2}]} = \tau \\ &\leq F(\max\{p(0,2), p(0,T0), p(2,T2), \frac{p(0,T2)+p(2,T0)}{4}\}) \\ &\leq F(2) = 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Case 2: Let $x = 1$.

We need only consider the case $y = 2$, since $H_p(T0,T1) = H_p(T1,T1) = p(0,0) = 0$. Now (3.1) is true since

$$\frac{1}{2}p(1,T1) = \frac{3}{2} < 3 = p(1,2).$$

We also know

$$H_p(T1,T2) = \frac{3}{2} < \max\{p(1,2), p(1,T1), p(2,T2), \frac{p(1,T2)+p(2,T1)}{4}\} = \max\{3,\frac{3}{2},2,\frac{1}{2}\} = 3.$$

And the inequality (3.2) is true since

$$\begin{aligned}\tau + F(H_p(T1,T2)) &= \tau + 1 - \frac{1}{[\frac{3}{2}]} = \tau \\ &\leq F(\max\{p(1,2), p(1,T1), p(2,T2), \frac{p(1,T2)+p(2,T1)}{4}\}) \\ &\leq F(3) = 1 - \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

Case 3: Let $x = 2$.

We need to consider the case $y \in \{0, 1\}$, since $H_p(T2, T2) = 0$. Note that

$$\frac{1}{2}p(2, T2) = 1 < 2 = p(2, 0),$$

and

$$\frac{1}{2}p(2, T2) = 1 < 3 = p(2, 1).$$

It is easy to check that

$$H_p(T0, T2) = \frac{3}{2} < \max\{p(0, 2), p(0, T0), p(2, T2), \frac{p(0, T2) + p(2, T0)}{4}\} = \max\{2, 0, 2, \frac{1}{2}\} = 2,$$

and

$$H_p(T1, T2) = \frac{3}{2} < \max\{p(1, 2), p(1, T1), p(2, T2), \frac{p(1, T2) + p(2, T1)}{4}\} = \max\{3, \frac{3}{2}, 2, \frac{1}{2}\} = 3.$$

The inequality (3.2) is true since

$$\tau + F(H_p(T0, T2)) \le F(\max\{p(0, 2), p(0, T0), p(2, T2), \frac{p(0, T2) + p(2, T0)}{4}\}),$$

in Case 1 and

$$\tau + F(H_p(T1, T2)) \le F(\max\{p(1, 2), p(1, T1), p(2, T2), \frac{p(1, T2) + p(2, T1)}{4}\}),$$

in Case 2. Hence T is an F_w-contraction. Obviously, the fixed point of T is 0.

Notice that if we replace the condition (P_4) with

$$p(x, y) \le p(x, z) + p(z, y),$$

then (X, p) turns to be a metric-like space. So based on the definition, it is easily to know that every partial metric is metric-like.

Definition 3.2. Let X be a non-empty set. A mapping $d : X \times X \longrightarrow R^+$ is said to be a metric-like on X, if for all $x, y, z \in X$ the following conditions are satisfied:

(D1) if $d(x, y) = 0$ then $x = y$;

(D2) $d(x, y) = d(y, x)$;

(D3) $d(x, y) \le d(x, z) + d(z, y)$.

Then we call (X, d) a metric-like space.

Let (X, d) be a metric-like space. A sequence $\{x_n\}_{n=1}^{\infty} \in X$ converges to $x \in X$, if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$. If $\lim_{n,m\to\infty} d(x_n, x_m)$ exists and is finite, we call it a Cauchy sequence in (X, d), and a metric-like space (X, d) is said to be complete, if and only if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ so that $\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n\to\infty} d(x_n, x) = d(x, x)$.

Now we derive the result of Theorem 2.6 in the context of metric-like spaces.

Theorem 3.3. Let (X, d) be a complete metric-like space and let $T : X \longrightarrow CB(X)$ be an F_w-contraction of type (A), then T has a fixed point $x^* \in X$.

Proof. Notice that we only need the partial metric p satisfies the condition

$$p(x,y) \leq p(x,z) + p(z,y) - p(z,z) \leq p(x,z) + p(z,y),$$

which is clearly satisfied in metric-like spaces. Hence, following the proof in Theorem 2.6 yields the existence of a fixed point of T. □

The following two theorems can be obtained easily by repeating the steps in the proof of Theorem 2.6.

Theorem 3.4. Let (X,d) be a complete metric-like space and let $T : X \longrightarrow CB(X)$ be an F_w-contraction of type (B), then T has a fixed point $x^* \in X$.

Theorem 3.5. Let (X,d) be a complete metric-like space and let $T : X \longrightarrow CB(X)$ be an F_w-contraction of type (C), then T has a fixed point $x^* \in X$.

4. Conclusions

The author uses the condition (F^*) instead of the condition (F3) and defines a new type of F-contraction called F_w-contraction. Consequently, the related fixed point theorems are proved and an example is given in the end. The paper provides a new method to prove the fixed point theorem for multi-valued mappings in partial metric spaces. And we extend the results into metric-like spaces, which expand the application range of the results.

Acknowledgment

The authors would like to thank the referees for careful reading of the manuscript and many helpful suggestions. The first author is supported by the National Natural Science Foundation of China and the Provincial Natural Science Foundation of Jiangxi.

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