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On certain multivalent functions involving the generalized Srivastava-Attiya operator

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Abstract

In this paper, we introduce certain new classes of multivalent functions involving the generalized Srivastava-Attiya operator. Such results as inclusion relationships, integral representation and arc length problems for these classes of functions are obtained. The behavior of these classes under a certain integral operator is also discussed.

Keywords: Srivastava-Attiya operator, starlike function, subordination.
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1. Introduction and preliminaries

Let $\mathcal{A}(p)$ denote the class of all multivalent functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p},$$

which are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. It is easy to see that $\mathcal{A}(1) = \mathcal{A}$, the well-known class of normalized analytic functions.

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If f and g are analytic functions in $\mathbb{D}$, then we say that f is subordinate to g, denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{D}$ with $|w(z)| < |z|$ such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

For arbitrary fixed numbers A, B, σ and β satisfying $-1 \le B < A \le 1$, $0 < \beta \le 1$ and $0 \le \sigma < 1$, let $\mathcal{P}_\beta(A,B,\sigma)$ denote the family of functions

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n,$$

holomorphic in $\mathbb{D}$ and such that q is in the class $\mathcal{P}_\beta(A,B,\sigma)$, if and only if

$$q(z) \prec (1-\sigma)\left(\frac{1+Az}{1+Bz}\right)^{\beta} + \sigma.$$

Therefore, $q \in \mathcal{P}_\beta(A,B,\sigma)$, if and only if for some w with $|w(z)| < |z|$, we have

$$q(z) = \frac{(1-\sigma)(1+Aw(z))^{\beta} + \sigma(1+Bw(z))^{\beta}}{(1+Bw(z))^{\beta}}.$$

We note that the class $\mathcal{P}_1(A,B,\sigma) \equiv \mathcal{P}(A,B,\sigma)$ was defined by Polatoğlu et al. [18], and further by putting $\sigma = 0$ in $\mathcal{P}(A,B,\sigma)$, we get the class $\mathcal{P}(A,B)$ introduced by Janowski [8]. Also the class $\mathcal{P}_\beta(1,-1,\sigma) \equiv \mathcal{P}_\beta(\sigma)$ investigated by Dziok [5] recently, and further by setting $\sigma = 0$ and $\beta = 1$ in $\mathcal{P}_\beta(\sigma)$, we obtain the class $\mathcal{P}$ of functions with positive real part.

The Herglotz representation of the function $q \in \mathcal{P}_\beta(A,B,\sigma)$ is given by

$$q(z) = \sigma + \frac{1-\sigma}{2}\int_0^{2\pi}\left(\frac{1+Aze^{-i\theta}}{1+Bze^{-i\theta}}\right)^{\beta} d\mu(\theta),$$

where $\mu(\theta)$ is a non-decreasing function in $[0,2\pi]$ such that $\int_0^{2\pi} d\mu(\theta) = 2$.

Now, we define the subclass $\mathcal{P}_{m,\beta}(A,B,\sigma)$ of analytic functions.

Definition 1.1. A function p analytic in $\mathbb{D}$ belongs to the class $\mathcal{P}_{m,\beta}(A,B,\sigma)$, $m \ge 2$, $-1 \le B < A \le 1$, $0 < \beta \le 1$, $0 \le \sigma < 1$, if and only if

$$p(z) = \sigma + \frac{1-\sigma}{2}\int_0^{2\pi}\left(\frac{1+Aze^{-i\theta}}{1+Bze^{-i\theta}}\right)^{\beta} d\mu(\theta), \tag{1.1}$$

where $\mu(\theta)$ is a non-decreasing function in $[0,2\pi]$ with

$$\int_0^{2\pi} d\mu(\theta) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(\theta)| \le m.$$

By using Horglotz-Stieltjes formula for the functions in the class $\mathcal{P}_{m,\beta}(A,B,\sigma)$, given by (1.1), one can easily obtain that, for $p_1, p_2 \in \mathcal{P}_\beta(A,B,\sigma)$,

$$p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z).$$

For $\beta = 1$, the class $\mathcal{P}_{m,\beta}(A,B,\sigma)$ reduces to the class $\mathcal{P}_m(A,B,\sigma)$, studied by Noor [13], and for $\sigma = 0$, $\beta = 1$, $A = 1$, $B = -1$, the $\mathcal{P}_{m,\beta}(A,B,\sigma)$ coincides with $\mathcal{P}_m$ which was introduced by Pinchuk [17]. Also by setting $\beta = 1$, $A = 1$, $B = -1$ in $\mathcal{P}_{m,\beta}(A,B,\sigma)$, we get the class $\mathcal{P}_m(\sigma)$, defined in [16].

We consider the function

$$\phi(z;s,b)=\sum_{n=0}^{\infty}\frac{z^n}{(n+b)^s},$$

where $b\in\mathbb{C}\setminus\left(\overline{\mathbb{Z}_0}:=\{0,-1,-2,...\}\right)$ and $s\in\mathbb{C}$. The function $\phi(z;s,b)$ contains many well-known familiar functions such as Riemann and Hurwitz Zeta functions (for more details, see [19, 21]).

By making use of the technique of convolution and the function $\phi(z;s,b)$, Liu [9] introduced the generalized Srivastava-Attiya operator $\mathcal{J}_{s,b}f(z):\mathcal{A}(p)\to\mathcal{A}(p)$ as follows:

$$\mathcal{J}_{s,b}f(z)=G_{s,b}(z)*f(z), \tag{1.2}$$

where $b\in\mathbb{C}\setminus\overline{\mathbb{Z}_0}$, $p\in\mathbb{N}$, $s\in\mathbb{C}$ and

$$G_{s,b}(z)=(1+b)^s\left[\phi(z;s,b)-b^{-s}\right]. \tag{1.3}$$

From (1.2) and (1.3), we have

$$\mathcal{J}_{s,b}f(z)=z^p+\sum_{n=1}^{\infty}\left(\frac{b+1}{b+n+1}\right)^s a_{n+p}z^{n+p},\quad (z\in\mathbb{D}).$$

Some special cases of the operator $\mathcal{J}_{s,b}f(z)$ are presented as follows:

1. For $s=0$, the operator $\mathcal{J}_{s,b}f(z)=f(z)$, and for $p=1$, $s=1$, $b=0$, we have $\mathcal{J}_{1,0}f(z)=\int_0^z\frac{f(t)}{t}dt$, introduced by Alexander [1].
2. If $p=1$, then $\mathcal{J}_{s,b}f(z)$ is known as Srivastava-Attiya operator [21].
3. By putting $s=1$, $b=\mu+p-1$, we get the operator $\mathcal{J}_{1,\mu+p-1}f(z)=F_{\mu,p}(f(z))$ $(\mu>-p,\ p\in\mathbb{N})$, introduced by Choi et al. [4].
4. For $s=\alpha$, $b=p$, we have $\mathcal{J}_{s,b}f(z)=\mathcal{I}_p^{\alpha}f(z)$ $(\alpha>0,\ p\in\mathbb{N})$, introduced and studied by Shams et al. [20].
5. $\mathcal{J}_{\gamma,p-1}f(z)=\mathcal{J}_p^{\gamma}f(z)$ $(\gamma\in\mathbb{N}_0)$, introduced by El-Ashwah and Aouf [6].
6. For more special cases of this operator, see also [2, 10, 11, 22–26].

To avoid repetition, it is admitted once that

$$m\geq 2,\quad -1\leq B<A\leq 1,\quad 0<\beta\leq 1,\quad 0\leq\sigma<1,\quad p\in\mathbb{N},\quad s\in\mathbb{C},\quad b\in\mathbb{C}\setminus\overline{\mathbb{Z}_0}.$$

With the help of the class $\mathcal{P}_{m,\beta}(A,B,\sigma)$, along with the generalized Srivastava-Attiya operator [9], we define the following subclasses of analytic functions.

Definition 1.2. A function $f\in\mathcal{A}(p)$ is in the class $\mathcal{R}_{m,\beta}^{s,b}[p,A,B,\sigma]$, if and only if

$$\frac{z\left(\mathcal{J}_{s,b}f(z)\right)'}{p\mathcal{J}_{s,b}f(z)}\in\mathcal{P}_{m,\beta}(A,B,\sigma),\quad (z\in\mathbb{D}).$$

Definition 1.3. A function $f\in\mathcal{A}(p)$ is in the class $\mathcal{V}_{m,\beta}^{s,b}[p,A,B,\sigma]$, if and only if

$$\frac{1}{p}+\frac{z\left(\mathcal{J}_{s,b}f(z)\right)''}{p\left(\mathcal{J}_{s,b}f(z)\right)'}\in\mathcal{P}_{m,\beta}(A,B,\sigma),\quad (z\in\mathbb{D}).$$

We note that

$$f\in\mathcal{V}_{m,\beta}^{s,b}[p,A,B,\sigma]\Longleftrightarrow\frac{zf'}{p}\in\mathcal{R}_{m,\beta}^{s,b}[p,A,B,\sigma]. \tag{1.4}$$

Definition 1.4. Let $f \in \mathcal{A}(p)$. Then the function $f \in \mathcal{M}_{m,\beta}^{s,b}[p, A, B, \sigma, \alpha]$ with $0 \leq \alpha \leq 1$, if and only if

$$(1-\alpha)\frac{z(\mathcal{J}_{s,b}f(z))'}{p\mathcal{J}_{s,b}f(z)} + \alpha\frac{\left(z(\mathcal{J}_{s,b}f(z))'\right)'}{p(\mathcal{J}_{s,b}f(z))'} \in \mathcal{P}_{m,\beta}(A, B, \sigma), \quad (z \in \mathbb{D}).$$

By giving specific values to $\alpha, \sigma, \beta, A, B, s, b, m$ and p in $\mathcal{M}_{m,\beta}^{s,b}[p, A, B, \sigma, \alpha]$, we obtain many important subclasses studied by various authors in earlier papers (see for details [3, 7, 13–17]).

To prove our main results, we need the following lemma due to Miller and Mocanu [12].

Lemma 1.5. *Let q be convex in $\mathbb{D}$ and $\Re(\mu_1 q(z) + \mu_2) > 0$, where $\mu_1, \mu_2 \in \mathbb{C} \setminus \{0\}$. If h is analytic in $\mathbb{D}$ with $q(0) = h(0)$ and*

$$h(z) + \frac{zh'(z)}{\mu_1 h(z) + \mu_2} \prec q(z), \quad (z \in \mathbb{D}),$$

then $h(z) \prec q(z)$.

The main purpose of this paper is to derive some inclusion relationships, integral representation and arc length problems for the function classes $\mathcal{R}_{m,\beta}^{s,b}[p, A, B, \sigma]$, $\mathcal{V}_{m,\beta}^{s,b}[p, A, B, \sigma]$ and $\mathcal{M}_{m,\beta}^{s,b}[p, A, B, \sigma, \alpha]$. The behavior of these classes under a certain integral operator is also discussed.

2. Main results

We begin by deriving the following inclusion relationship.

Theorem 2.1. *Let $f \in \mathcal{A}(p)$ with $\mathcal{J}_{s,b}f(z) \neq 0$. Then*

$$\mathcal{M}_{2,\beta}^{s,b}[p, A, B, \sigma, \alpha] \subset \mathcal{R}_{2,\beta}^{s,b}[p, A, B, \sigma].$$

Proof. Let $f \in \mathcal{M}_{2,\beta}^{s,b}[p, A, B, \sigma, \alpha]$ and set

$$\phi(z) = \frac{\mathcal{J}_{s,b}f(z)}{z^p}.$$

Then the function ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. By taking logarithmic differentiation, we have

$$\frac{z(\mathcal{J}_{s,b}f(z))'}{p\mathcal{J}_{s,b}(f(z))} = \varphi(z) + 1, \tag{2.1}$$

where

$$\varphi(z) = \frac{z\phi'(z)}{p\phi(z)}.$$

By logarithmic differentiation of (2.1) with some simplification, we obtain

$$\varphi(z) + 1 + \frac{\alpha}{p}\frac{z\varphi'(z)}{\varphi(z)+1} = (1-\alpha)\frac{z(\mathcal{J}_{s,b}f(z))'}{p\mathcal{J}_{s,b}f(z)} + \alpha\frac{\left(z(\mathcal{J}_{s,b}f(z))'\right)'}{p(\mathcal{J}_{s,b}f(z))'}.$$

Let $\varphi(z) + 1 = H(z)$. Then H is analytic in $\mathbb{D}$ with $H(0) = 1$. Now, by using hypothesis of Theorem 2.1, we have

$$H(z) + \frac{zH'(z)}{\frac{p}{\alpha}H(z)} \prec (1-\sigma)\left(\frac{1+Az}{1+Bz}\right)^\beta + \sigma.$$

By Lemma 1.5, we get

$$H(z) \prec (1-\sigma)\left(\frac{1+Az}{1+Bz}\right)^\beta + \sigma,$$

which implies $f \in \mathcal{R}_{2,\beta}^{s,b}[p, A, B, \sigma]$. Thus, the assertion of Theorem 2.1 holds true. $\square$

Theorem 2.2. *If $f \in \mathcal{A}(p)$ with $\mathcal{J}_{s,b}f(z) \neq 0$, $z \in \mathbb{D}$, then*

$$\mathcal{R}^{s,b}_{2,\beta}[p, A, B, \sigma] \subset \mathcal{R}^{s+1,b}_{2,\beta}[p, A, B, \sigma].$$

Proof. Let $f \in \mathcal{R}^{s,b}_{2,\beta}[p, A, B, \sigma]$ and put

$$\frac{z(\mathcal{J}_{s+1,b}f(z))'}{p\mathcal{J}_{s+1,b}f(z)} = h(z),$$

where h is analytic in $\mathbb{D}$ and $h(0) = 1$. By using the identity

$$z(\mathcal{J}_{s+1,b}f(z))' = [p - (1 + b)]\mathcal{J}_{s+1,b}f(z) + (1 + b)\mathcal{J}_{s,b}f(z),$$

we have

$$\frac{(1 + b)\mathcal{J}_{s,b}f(z)}{\mathcal{J}_{s+1,b}f(z)} = h(z) + \frac{b + 1}{p} - 1.$$

By differentiating the above equation logarithmically, we obtain

$$h(z) + \frac{zh'(z)}{ph(z) + b + 1 - p} = \frac{z(\mathcal{J}_{s,b}f(z))'}{p\mathcal{J}_{s,b}f(z)}.$$

By using hypothesis of Theorem 2.2 along with Lemma 1.5, we get

$$h(z) \prec (1 - \sigma)\left(\frac{1 + Az}{1 + Bz}\right)^{\beta} + \sigma.$$

This implies that $f \in \mathcal{R}^{s+1,b}_{2,\beta}[p, A, B, \sigma]$. □

Theorem 2.3. *If $f \in \mathcal{A}(p)$ with $\mathcal{J}_{s,b}f(z) \neq 0$, $z \in \mathbb{D}$, then*

$$\mathcal{V}^{s,b}_{2,\beta}[p, A, B, \sigma] \subset \mathcal{V}^{s+1,b}_{2,\beta}[p, A, B, \sigma].$$

Proof. By Theorem 2.2 and (1.4), we see that

$$\begin{aligned}
f \in \mathcal{V}^{s,b}_{2,\beta}[p, A, B, \sigma] &\iff \mathcal{J}_{s,b}f \in \mathcal{V}_{2,\beta}[p, A, B, \sigma] \\
&\iff \frac{z(\mathcal{J}_{s,b}f)'}{p} \in \mathcal{R}_{2,\beta}[p, A, B, \sigma] \\
&\iff \mathcal{J}_{s,b}\left(\frac{zf'(z)}{p}\right) \in \mathcal{R}_{2,\beta}[p, A, B, \sigma] \\
&\iff \frac{zf'}{p} \in \mathcal{R}^{s,b}_{2,\beta}[p, A, B, \sigma] \\
&\implies \frac{zf'}{p} \in \mathcal{R}^{s+1,b}_{2,\beta}[p, A, B, \sigma] \\
&\iff \mathcal{J}_{s+1,b}\left(\frac{zf'}{p}\right) \in \mathcal{R}_{2,\beta}[p, A, B, \sigma] \\
&\iff \frac{z}{p}(\mathcal{J}_{s+1,b}f)' \in \mathcal{R}_{2,\beta}[p, A, B, \sigma] \\
&\iff \mathcal{J}_{s+1,b}f \in \mathcal{V}_{2,\beta}[p, A, B, \sigma] \\
&\iff f \in \mathcal{V}^{s+1,b}_{2,\beta}[p, A, B, \sigma].
\end{aligned}$$

The proof of Theorem 2.3 is thus completed. □

Theorem 2.4. *If* $0 < \alpha_1 \leq \alpha_2 < 1$, *then*

$$\mathcal{M}^{s,b}_{2,\beta}\left[p, A, B, \sigma, \alpha_2\right] \subset \mathcal{M}^{s,b}_{2,\beta}\left[p, A, B, \sigma, \alpha_1\right].$$

Proof. Let $f \in \mathcal{M}^{s,b}_{2,\beta}\left[p, A, B, \sigma, \alpha_2\right]$. Then

$$(1-\alpha_1)\frac{z\left(\mathcal{J}_{s,b}f(z)\right)'}{p\mathcal{J}_{s,b}f(z)} + \alpha_1\frac{\left(z\left(\mathcal{J}_{s,b}f(z)\right)'\right)'}{p\left(\mathcal{J}_{s,b}f(z)\right)'} = \left(1-\frac{\alpha_1}{\alpha_2}\right)h_1(z) + \frac{\alpha_1}{\alpha_2}h_2(z),$$

with

$$h_1(z) = \frac{z\left(\mathcal{J}_{s,b}f(z)\right)'}{pJ_{s,b}f(z)},$$

and

$$h_2(z) = (1-\alpha_2)\frac{z\left(\mathcal{J}_{s,b}f(z)\right)'}{p\mathcal{J}_{s,b}f(z)} + \alpha_2\frac{\left(z\left(\mathcal{J}_{s,b}f(z)\right)'\right)'}{p\left(\mathcal{J}_{s,b}f(z)\right)'}.$$

From hypothesis and Theorem 2.1, we easily obtain

$$h_1, h_2 \in \mathcal{P}_{2,\beta}\left(A, B, \sigma\right).$$

Since the class $\mathcal{P}_{2,\beta}\left(A, B, \sigma\right)$ is a convex set, it follows that

$$\left(1-\frac{\alpha_1}{\alpha_2}\right)h_1(z) + \frac{\alpha_1}{\alpha_2}h_2(z) \in \mathcal{P}_{2,\beta}\left(A, B, \sigma\right).$$

This implies that $f \in \mathcal{M}^{s,b}_{2,\beta}\left[p, A, B, \sigma, \alpha_1\right]$. □

Theorem 2.5. *Let* $f \in \mathcal{R}^{s,b}_{m,\beta}\left[p, A, B, \sigma\right]$. *If* $s_1, s_2 \in \mathcal{R}^{0,b}_{2,\beta}\left[p, A, B, \sigma\right]$, *then*

$$\mathcal{J}_{s,b}f(z) = \frac{\left(s_1(z)\right)^{\frac{m}{4}+\frac{1}{2}}}{\left(s_2(z)\right)^{\frac{m}{4}-\frac{1}{2}}}. \tag{2.2}$$

Proof. If $f \in \mathcal{R}^{s,b}_{m,\beta}\left[p, A, B, \sigma\right]$, then there exist two functions $h_1, h_2 \in \mathcal{P}_{2,\beta}\left(A, B, \sigma\right)$ such that

$$\frac{z\left(\mathcal{J}_{s,b}f(z)\right)'}{p\mathcal{J}_{s,b}f(z)} = \left(\frac{m}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)h_2(z),$$

which is equivalent to

$$\frac{z\left(\mathcal{J}_{s,b}f(z)\right)'}{p\mathcal{J}_{s,b}f(z)} = \left(\frac{m}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{ps_1(z)} - \left(\frac{m}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{ps_2(z)}, \tag{2.3}$$

where $s_1, s_2 \in \mathcal{R}^{0,b}_{2,\beta}\left[p, A, B, \sigma\right]$. By integrating both sides of (2.3), we have

$$\log \mathcal{J}_{s,b}f(z) = \left(\frac{m}{4}+\frac{1}{2}\right)\log s_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)\log s_2(z). \tag{2.4}$$

From (2.4), we readily get (2.2). □

Theorem 2.6. *Let* $f \in \mathcal{M}^{s,b}_{m,\beta}\left[p, A, B, \sigma, \alpha\right]$. *Then* $g \in \mathcal{R}^{s,b}_{m,\beta}\left[p, A, B, \sigma\right]$, *where*

$$\left(\frac{\mathcal{J}_{s,b}g(z)}{z}\right)^{\frac{1}{p}} = \left(\frac{\mathcal{J}_{s,b}f(z)}{z}\right)^{\frac{1-\alpha}{p}}\left[\left(\mathcal{J}_{s,b}f(z)\right)'\right]^{\frac{\alpha}{p}}. \tag{2.5}$$

Proof. By differentiating both sides of (2.5) logarithmically, with some simplification, we have

$$\frac{z\left(\mathcal{J}_{s,b}g\left(z\right)\right)'}{p\mathcal{J}_{s,b}g\left(z\right)}=(1-\alpha)\frac{z\left(\mathcal{J}_{s,b}f\left(z\right)\right)'}{p\mathcal{J}_{s,b}f\left(z\right)}+\alpha\frac{\left(z\left(\mathcal{J}_{s,b}f\left(z\right)\right)'\right)'}{p\left(\mathcal{J}_{s,b}f\left(z\right)\right)'}\in\mathcal{P}_{m,\beta}\left(A,B,\sigma\right).$$

Hence $g\in\mathcal{R}_{m,\beta}^{s,b}\left[p,A,B,\sigma\right]$. This completes the proof of Theorem 2.6. □

Theorem 2.7. *A function $f\in\mathcal{M}_{m,\beta}^{s,b}\left[p,A,B,\sigma,\alpha\right]$, if and only if there exists a function $g\in\mathcal{R}_{m,\beta}^{s,b}\left[p,A,B,\sigma\right]$ such that*

$$\mathcal{J}_{s,b}f(z)=\left[\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}\left(\frac{\mathcal{J}_{s,b}g(z)}{z}\right)^{\frac{1}{\alpha}}dt\right]^{\alpha}. \tag{2.6}$$

Proof. Suppose that $f\in\mathcal{M}_{m,\beta}^{s,b}\left[p,A,B,\sigma,\alpha\right]$ and $g\in\mathcal{R}_{m,\beta}^{s,b}\left[p,A,B,\sigma\right]$. From (2.5), we have

$$\left(\mathcal{J}_{s,b}f(z)\right)^{\frac{1-\alpha}{\alpha}}\left(\mathcal{J}_{s,b}f(z)\right)'=\left(\frac{\mathcal{J}_{s,b}g(z)}{z}\right)^{\frac{1}{\alpha}}z^{\frac{1-\alpha}{\alpha}}. \tag{2.7}$$

By integrating both sides of (2.7), we easily get (2.6). Conversely, assume that (2.6) holds with $g\in\mathcal{R}_{m,\beta}^{s,b}\left[p,A,B,\sigma\right]$, we only need to show that $f\in\mathcal{M}_{m,\beta}^{s,b}\left[p,A,B,\sigma,\alpha\right]$. From (2.6), we obtain

$$(1-\alpha)\frac{z\left(\mathcal{J}_{s,b}f\right)'}{p\mathcal{J}_{s,b}f}+\alpha\frac{\left(z\left(\mathcal{J}_{s,b}f\right)'\right)'}{p\left(\mathcal{J}_{s,b}f\right)'}=\frac{z\left(\mathcal{J}_{s,b}g\right)'}{p\mathcal{J}_{s,b}g}\in\mathcal{P}_{m,\beta}\left(A,B,\sigma\right),$$

which implies that $f\in\mathcal{M}_{m,\beta}^{s,b}\left[p,A,B,\sigma,\alpha\right]$. □

Theorem 2.8. *Suppose that $f\in\mathcal{M}_{m,0}^{s,b}\left[p,A,B,\sigma,\alpha\right]$, $L_r(f)$ denotes the length of the curve C, $C=f(re^{i\theta})$, $0<\theta\leq 2\pi$, and $M(r)=\max\limits_{0<\theta\leq 2\pi}\left|f(re^{i\theta})\right|$. Then, for $0<r<1$,*

$$L_r(f)\leq\frac{(2-\alpha)\pi pM(r)}{\alpha}\left[\frac{2+(k-2)A_1-kB}{1-B}\right],$$

where $A_1=(1-\alpha)A+\alpha B$.

Proof. Assume that $F(z)=\mathcal{J}_{s,b}f(z)$. By taking integration by parts, with $z=re^{i\theta}$, we get

$$\begin{aligned}
L_r(f)&=\int_0^{2\pi}\left|zF'(z)\right|d\theta\\
&=\int_0^{2\pi}zF'(z)e^{-i\arg(zF'(z))}d\theta\\
&=\int_0^{2\pi}F(z)e^{-i\arg(zF'(z))}\Re\left(\frac{(zF'(z))'}{F'(z)}\right)d\theta\\
&\leq\frac{pM(r)}{\alpha}\int_0^{2\pi}\left|(1-\alpha)\frac{zF'(z)}{pF(z)}+\alpha\frac{(zF'(z))'}{pF'(z)}+(\alpha-1)\frac{zF'(z)}{pF(z)}\right|d\theta\\
&\leq\frac{pM(r)}{\alpha}\left[\int_0^{2\pi}\left|(1-\alpha)\frac{zF'(z)}{pF(z)}+\alpha\frac{(zF'(z))'}{pF'(z)}\right|d\theta+(1-\alpha)\int_0^{2\pi}\left|\frac{zF'(z)}{pF(z)}\right|d\theta\right]\\
&\leq\frac{pM(r)}{\alpha}\left[\left(\frac{2+(k-2)A_1-kB}{1-B}\right)\pi+(1-\alpha)\left(\frac{2+(k-2)A_1-kB}{1-B}\right)\pi\right]\\
&=\frac{(2-\alpha)\pi pM(r)}{\alpha}\left[\frac{2+(k-2)A_1-kB}{1-B}\right].
\end{aligned}$$

We thus complete the proof of Theorem 2.8. □

Theorem 2.9. *Let $f \in \mathcal{M}_{m,0}^{s,b}[p, A, B, \sigma, \alpha]$. Then*

$$n|a_n| = O(1)M\left(1 - \frac{1}{n}\right), \quad (n \geq 2),$$

where $O(1)$ is a constant depending on A_1, B, p, α and k only.

Proof. Since $z = re^{i\theta}$, the Cauchy theorem gives

$$n|a_n| = \frac{1}{2\pi r^n} L_r(f).$$

By virtue of Theorem 2.8, we have

$$n|a_n| = \frac{1}{2r^n} \frac{(2-\alpha)pM(r)}{\alpha}\left[\frac{2 + (k-2)A_1 - kB}{1 - B}\right],$$

where $A_1 = (1-\alpha)A + \alpha B$.

By taking $r = 1 - \frac{1}{n}$, we get

$$n|a_n| = \frac{(2-\alpha)p}{2\alpha\left(1 - \frac{1}{n}\right)^n}\left[\frac{2 + (k-2)A_1 - kB}{1 - B}\right] M\left(1 - \frac{1}{n}\right),$$

which gives the desired result. □

Theorem 2.10. *Let c be a real number with $c > -p$, and $\mathcal{J}_{s,b}F_{c,p}(z) \neq 0$, for all $z \in \mathbb{D}$. If $f \in \mathcal{R}_{2,\beta}^{s,b}[p, A, B, \sigma]$, then*

$$F_{c,p}(z) \in \mathcal{R}_{2,\beta}^{s,b}[p, A, B, \sigma],$$

where $F_{c,p} : \mathcal{A}(p) \to \mathcal{A}(p)$ is defined by

$$F_{c,p}(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt = \left(z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n} z^{n+p}\right) * f(z). \tag{2.8}$$

Proof. Let $f \in \mathcal{R}_{2,\beta}^{s,b}[p, A, B, \sigma]$ and set

$$\phi(z) = \frac{\mathcal{J}_{s,b}F_{c,p}(z)}{z^p}. \tag{2.9}$$

Then ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. By differentiating both sides of (2.8), we have

$$\frac{z\left(F_{c,p}(z)\right)'}{pF_{c,p}(z)} = \frac{c+p}{p}\frac{f(z)}{pF_{c,p}(z)} - \frac{c}{p}. \tag{2.10}$$

By applying the operator $\mathcal{J}_{s,b}$ to (2.10), we get

$$\frac{z\left(\mathcal{J}_{s,b}F_{c,p}(z)\right)'}{p\mathcal{J}_{s,b}F_{c,p}(z)} = \frac{c+p}{p}\frac{\mathcal{J}_{c,b}f(z)}{p\mathcal{J}_{c,b}F_{c,p}(z)} - \frac{c}{p}. \tag{2.11}$$

Now, by taking logarithmic differentiation of (2.9), we obtain

$$\frac{z\left(\mathcal{J}_{s,b}F_{c,p}(z)\right)'}{p\mathcal{J}_{s,b}F_{c,p}(z)} - 1 = \frac{z\phi'(z)}{\phi(z)} = \varphi(z). \tag{2.12}$$

From (2.11) and (2.12), we know that

$$\frac{c+p}{p}\frac{\mathcal{J}_{c,b}f(z)}{p\mathcal{J}_{c,b}F_{c,p}(z)} = \varphi(z) + 1 + \frac{c}{p}. \tag{2.13}$$

Logarithmic differentiation of (2.13), together with (2.12) yields

$$H(z)+\frac{z\phi'(z)}{pH(z)+c}=\frac{z\left(\mathcal{J}_{s,b}f\left(z\right)\right)'}{p\mathcal{J}_{s,b}f\left(z\right)}\prec(1-\sigma)\left(\frac{1+Az}{1+Bz}\right)^{\beta}+\sigma,$$

where $H(z)=\varphi(z)+1$. By Lemma 1.5, we see that

$$H(z)\prec(1-\sigma)\left(\frac{1+Az}{1+Bz}\right)^{\beta}+\sigma.$$

This implies that $F_{c,p}\left(z\right)\in\mathcal{R}_{2,\beta}^{s,b}\left[p,A,B,\sigma\right].$ □

Theorem 2.11. *Let c be a real number with $c>-p$, and $\mathcal{J}_{s,b}F_{c,p}(z)\neq 0$ for all $z\in\mathbb{D}$. If $f\in\mathcal{V}_{2,\beta}^{s,b}\left[p,A,B,\sigma\right]$, then*

$$F_{c,p}(z)\in\mathcal{V}_{2,\beta}^{s,b}\left[p,A,B,\sigma\right],$$

where $F_{c,p}(z)$ is given by (2.8).

Proof. The proof follows directly from (1.4) and Theorem 2.10. □

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