Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6312–6332
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Generalized hybrid algorithms for fixed point and mixed equilibrium problems in Banach spaces

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Communicated by Y. J. Cho

Abstract

The purpose of this paper is to introduce and investigate a more generalized hybrid shrinking projection algorithm for finding a common solution for a system of generalized mixed equilibrium problems. A accelerated strong convergence theorem of common solutions is established in the framework of a non-uniformly convex Banach space. These new results improve and extend the previously known ones in the literature.

Keywords: Hybrid projection algorithm, monotone mapping, equilibrium problem, variational inequality, quasi-nonexpansive mapping, fixed point.
2010 MSC: 47H05, 47H09, 47H10.

1. Introduction

The equilibrium problem, which was introduced by Fan [11] in 1972, has been intensively investigated by many authors (see [1–10, 12–24]) and they have captured lots of applications in various disciplines such as in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization see ([2, 6, 8, 12, 14, 19, 23], and the references therein). The projection method which was first introduced by Haugazeau [13] has been investigated for the approximation of fixed points of nonlinear operators. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions imposed on mappings or spaces.

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Received 2016-10-15

Let E be a Banach space, C be a nonempty convex and closed subset of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real valued function and $S : C \to E^*$ be a nonlinear mapping. We use $Sol(F, S, \varphi)$ to denote the solution set of the following generalized mixed equilibrium problem,

$$F(x, y) + \langle Sx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

If $F = 0$, then the problem (1.1) is equivalent to find $\overline{x} \in C$ such that

$$\langle S\overline{x}, y - \overline{x}\rangle + \varphi(y) - \varphi(\overline{x}) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which is called the mixed variational inequality of Browder type and the solution set of (1.2) is denoted by $VI(F, S, \varphi)$.

If $S = 0$, then the problem (1.1) is equivalent to find $\overline{x} \in C$ such that

$$F(\overline{x}, y) + \varphi(y) - \varphi(\overline{x}) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which is called the mixed equilibrium problem and the solution set of (1.3) is denoted by $Sol(F, \varphi)$.

If $\varphi = 0$, then the problem (1.1) is equivalent to find $\overline{x} \in C$ such that

$$F(\overline{x}, y) + \langle S\overline{x}, y - \overline{x}\rangle \geq 0, \quad \forall y \in C, \tag{1.4}$$

which is called the generalized equilibrium problem and the solution set of (1.4) is denoted by $Sol(F, S)$.

If $S = 0$ and $\varphi = 0$, then the problem (1.1) is equivalent to find $\overline{x} \in C$ such that

$$F(\overline{x}, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The problem was first introduced by Fan [11] and called the equilibrium problem and the solution set of (1.5) is denoted by $Sol(F)$.

In this paper, we introduce and investigate a more generalized hybrid shrinking projection algorithm for finding a common solution for a system of generalized mixed equilibrium problems. A accelerated strong convergence theorem of common solutions is established in the framework of a non-uniformly convex Banach space.

2. Preliminaries and lemmas

Let E be a real Banach space and let E^* be the dual space of E. Let S_E be the unit sphere of E. E is said to be strictly convex, if and only if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in S_E$ and $x \neq y$. E is said to be uniformly convex, if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in S_E$, $\|x-y\| \geq \epsilon$ implies $\|x+y\| \leq 2-2\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Then the Banach space E is said to be smooth, if $\lim_{t\to 0} \frac{\|x\|-\|x+ty\|}{t}$ exists for each $x, y \in S_E$. It is also said to be uniformly smooth, if and only if the above limit is attained uniformly for $x, y \in S_E$. It is known that a uniformly smooth Banach space is reflexive and smooth, and E is uniformly smooth, if and only if E^* is uniformly convex.

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that E is said to have the Kadec-Klee property, if $\lim_{n\to\infty} \|x_n - x\| = 0$, for any sequence $\{x_n\} \subset E$, and $x \in E$ with $\lim_{n\to\infty} x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n\| \to \|x\|$. It is known that every uniformly convex Banach space has the Kadec-Klee property (see [9] and the references therein).

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \|x\|^2 = \langle x, f^*\rangle = \|f^*\|^2\},$$

where $\langle\cdot, \cdot\rangle$ denotes the generalized duality pairing. It is known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a smooth Banach space, then J is single-valued and demi-continuous, i.e., continuous from the strong topology of E to the weak star topology of E; if E is smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Let E be a smooth Banach space which means the mapping J is single-valued, consider the following functional defined on E,

$$\phi(x,y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x, y \in E.$$

In a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, for all $x, y \in H$. As we all know that, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C such that $\|x - P_C x\| \leq \|x - y\|$, for all $y \in C$, then P_C is firmly nonexpansive. In [1], Alber studied a new mapping Π_C in a Banach space which is an analogue of P_C. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x,y)$, that is, $\Pi_C = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem $\phi(\overline{x}, x) = \min_{y\in C} \phi(y,x)$. Obviously, in Hilbert space, $\Pi_C = P_C$.

Let $T : C \to C$ be a mapping on C. T is said to be closed, if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. A point p is said to be a fixed point of T, if and only if $Tp = p$. In this paper, we use $Fix(T)$ to denote the fixed point set of T. A point p is said to be an asymptotic fixed point of T, if and only if $x_n \rightharpoonup p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{Fix}(T)$.

Recall that T is said to be relatively nonexpansive [5], if and only if

$$\phi(p, Tp) \leq \phi(p, x), \quad \forall x \in C,\ \forall \widetilde{Fix}(T) = Fix(T) \neq \emptyset.$$

T is said to be quasi-ϕ-nonexpansive [17], if and only if

$$\phi(p, Tp) \leq \phi(p, x), \quad \forall x \in C,\ \forall Fix(T) \neq \emptyset.$$

The class of quasi-ϕ-nonexpansive mappings is more desirable than the class of relatively nonexpansive mappings because of strong restriction $Fix(T) = \widetilde{Fix}(T)$.

For solving the above equilibrium problems, the following restrictions on bifunction F are essential in this paper.

(A1) $F(x,x) = 0,\ \forall x \in C$;

(A2) $F(x,y) + F(y,x) \leq 0, \quad \forall x, y \in C$;

(A3) $F(x,y) \geq \limsup_{t\downarrow 0} F(tz + (1-t)x, y),\ \forall x, y, z \in C$;

(A4) for each $x \in C$, $\quad y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Remark 2.1. F is said to be monotone, if and only if $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$. $y \mapsto F(x,y)$ is convex, if and only if $F(x, ty + (1-t)z) \leq tF(x,y) + (1-t)F(x,z)$ for all $x, y, z \in C$ and $t \in (0,1)$. $y \mapsto F(x,y)$ is lower semi-continuous, if and only if $F(x, y_n) \to F(x,y)$ whenever $y_n \to y$ as $n \to \infty$.

In addition, we also need the following lemmas in this paper.

Lemma 2.2 ([20]). *Let r be a positive real number and let E be uniformly convex. Then there exists a convex, strictly increasing and continuous function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|(1-t)b + ta\|^2 + t(1-t)g(\|b - a\|) \leq t\|a\|^2 + (1-t)\|b\|^2,$$

for all $a, b \in B^r := \{a \in E : \|a\| \leq r\}$ and $t \in [0,1]$.

Lemma 2.3 ([1]). *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$, then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

and $x_0 = \Pi_C x$, if and only if

$$\langle y - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.4 ([18]). *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let T be a closed quasi-ϕ-nonexpansive mapping on C. Then $Fix(T)$ is closed and convex.*

Lemma 2.5 ([4, 17]). *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let $F : C \times C \to \mathbb{R}$ be a function with restrictions* (A1), (A2), (A3), *and* (A4). *Let $x \in E$ and $r > 0$. Then there exists $z \in C$ such that*

$$rF(z, y) + \langle z - y, Jz - Jx \rangle \leq 0, \quad \forall y \in C.$$

Define a mapping $C^{F,r}$ by

$$C^{F,r} x = \{z \in C : rF(z, y) + \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) *$C^{F,r}$ is single-valued quasi-ϕ-nonexpansive;*

(2) *$Sol(F) = Fix(C^{F,r})$ is closed and convex.*

3. Main results

We now prove the following theorems.

Theorem 3.1. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let F be a bifunction from $C \times C$ to $\mathbb{R}$ with* (A1), (A2), (A3) *and* (A4). *Let $S : C \to E^*$ be a continuous and monotone mapping and let $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let T be a quasi-ϕ-nonexpansive mapping on C. Assume that $Sol(F, S, \varphi) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{\beta_n\}$ be a real sequence such that $\liminf_{n\to\infty} \beta_n > 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{0,1}, x_{0,2}, x_{0,3}, ..., x_{0,N} \in C, \ \textit{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i=1}^N C_{1,i}, \ (i = 1, 2, 3, ..., N), \\ x_{1,i} = \Pi_{C_1} x_{0,i}, \\ x_1 = \sum_{i=1}^N \lambda_i x_{1,i}, \ \sum_{i=1}^N \lambda_i = 1, \ \lambda_i \in [0, 1], \\ F(z_{n,i}, z) + (\varphi(z) - \varphi(z_{n,i})) + \langle S z_{n,i}, z - z_{n,i} \rangle \geq \frac{1}{\beta_n} \langle z_{n,i} - z, J z_{n,i} - J x_{n,i} \rangle, \ \forall z \in C_n, \\ J y_{n,i} = \alpha_n JT x_{n,i} + (1 - \alpha_n) J z_{n,i}, \\ C_{n+1,i} = \{z \in C_n : \phi(z, x_{n,i}) \geq \phi(z, y_{n,i})\}, \\ C_{n+1} = \cap_{i=1}^N C_{n+1,i}, \\ x_{n+1,i} = \Pi_{C_{n+1}} x_{1,i}, \\ x_{n+1} = \sum_{i=1}^N \lambda_i x_{n,i}. \end{cases}$$

Then the sequence $\{x_{n,i}\}$ converges strongly to a common solution $\overline{x}_i$ and the sequence $\{x_n\}$ converges strongly to a special common solution $\overline{x} = \sum_{i=1}^N \lambda_i \overline{x}_i$, where $\overline{x}_i = \Pi_{Sol(F,S,\varphi)\cap Fix(T)} x_{1,i}$ and $\overline{x} \in Sol(F, S, \varphi) \cap Fix(T)$.

Proof. First, we define

$$G(x,y) = F(x,y) + (\varphi(y) - \varphi(x)) + \langle Sx, y-x\rangle, \quad \forall x,y \in C.$$

Next, we prove that bifunction G satisfies (A1), (A2), (A3) and (A4). Therefore, generalized mixed equilibrium problem is equivalent to the following equilibrium problem

$$\text{find} \quad (x \in C) \quad \text{such that} \quad G(x,y) \geq 0, \quad \forall y \in C.$$

Next we prove G is monotone. Since S is a continuous and monotone operator, we find from the definition of G that

$$\begin{aligned} G(y,z) + G(z,y) &= F(y,z) + (\varphi(z) - \varphi(y)) + \langle Sy, z-y\rangle + F(z,y) \\ &\quad + (\varphi(y) - \varphi(z)) + \langle Sz, y-z\rangle \\ &= F(y,z) + F(z,y) + \langle Sy, z-y\rangle + \langle Sz, y-z\rangle \\ &\leq \langle Sz - Sy, y-z\rangle \leq 0. \end{aligned}$$

It is clear that G satisfies (A2). Next, we show that for each $x \in C, y \mapsto G(x,y)$ is convex and lower semicontinuous. For each $x \in C$, for all $t \in (0,1)$ and for all $y,z \in C$, since φ is convex, we have

$$\begin{aligned} G(x, ty + (1-t)z) &= F(x, ty + (1-t)z) + \varphi(ty + (1-t)z) - \varphi(x) \\ &\quad + \langle Sx, ty + (1-t)z\rangle \\ &\leq tF(x,y) + (1-t)F(x,z) + t\varphi(y) + (1-t)\varphi(z) \\ &\quad - \varphi(x) + t\langle Sx, y\rangle + (1-t)\langle Sx, z\rangle \\ &= t(F(x,y) + \varphi(y) - \varphi(x) + \langle Sx, y\rangle) \\ &\quad + (1-t)(F(x,z) + \varphi(z) - \varphi(x) + \langle Sx, z\rangle) \\ &= tG(x,y) + (1-t)G(x,z), \end{aligned}$$

so, for each $x \in C, y \mapsto G(x,y)$ is convex. Similarly, we find that $y \mapsto G(x,y)$ is also lower semicontinuous.

Next, we show G satisfies (A4). That is,

$$\limsup_{t \downarrow 0} G(tx + (1-t)y, z) \leq G(y,z), \quad \forall x,y \in C.$$

Since S is continuous and φ is lower semicontinuous, we have

$$\begin{aligned} \limsup_{t \downarrow 0} G(tx + (1-t)y, z) &= \limsup_{t \downarrow 0} F(tx + (1-t)y, z) \\ &\quad + \limsup_{t \downarrow 0} \varphi(z) - \varphi(tx + (1-t)y) \\ &\quad + \limsup_{t \downarrow 0} \langle S(tx + (1-t)y), z - (tx + (1-t)y)\rangle \\ &\leq F(y,z) + (\varphi(z) - \varphi(y)) + \langle Sy, z-y\rangle \\ &= G(y,z). \end{aligned}$$

By using Lemma 2.5, one sees that $Sol(G) = Sol(F, S, \varphi)$ is closed and convex. By using Lemma 2.4, one sees that $Fix(T)$ is also convex and closed. Hence, $Sol(F, S, \varphi) \cap Fix(T)$ is convex and closed.

Now, we show that C_n is closed and convex. To show C_n is convex and closed, it suffices to show that, for each $i = 1, 2, .., N, C_{n,i}$ is convex and closed. It is easy to see that $C_{n,i}$ is closed. We only show that $C_{n,i}$ is convex. It is obvious that $C_{1,i} = C$ is convex. Assume that $C_{m,i}$ is convex and closed for some $m \geq 0$. Let $\omega_1, \omega_2 \in C_{m+1,i}$. It follows that $\omega = s\omega_1 + (1-s)\omega_2 \in C_{m,i}$, where $s \in (0,1)$. Since

$$\phi(\omega_1, y_{m,i}) \leq \phi(\omega_1, x_{m,i}),$$

and

$$\phi(\omega_2, y_{m,i}) \leq \phi(\omega_2, x_{m,i}),$$

one has

$$2\langle \omega_1, Jx_{m,i} - Jy_{m,i}\rangle \leq \|x_{m,i}\|^2 - \|y_{m,i}\|^2,$$

and

$$2\langle \omega_2, Jx_{m,i} - Jy_{m,i}\rangle \leq \|x_{m,i}\|^2 - \|y_{m,i}\|^2.$$

By using the above two inequalities, we obtain that $\phi(\omega, y_{m,i}) \leq \phi(\omega, x_{m,i})$. This shows that $C_{m+1,i}$ is closed and convex. Hence, $C_{m+1} = \bigcap_{i=1}^{N} C_{m+1,i}, (i = 1, 2, 3, ..., N)$ is a convex and closed set. This implies that C_n is convex and closed.

Next, we prove $Sol(F, S, \varphi) \cap Fix(T) \subset C_n$. It is obvious $Sol(F, \varphi, S) \cap Fix(T) \subset C_{1,i} = C$. Suppose that $Sol(F, \varphi, S) \cap Fix(T) \subset C_m \subset C_{m,i}$ for some positive integer m. For any $z \in Fix(T) \cap Sol(B) \subset C_{m,i}$, we see that

$$\begin{aligned}
\phi(z, y_{m,i}) &= \|z\|^2 + \|\alpha_m JTx_{m,i} + (1-\alpha_m)Jz_{m,i}\|^2 \\
&\quad \times 2\langle z, \alpha_m JTx_{m,i} + (1-\alpha_m)Jz_{m,i}\rangle \\
&\leq \|z\|^2 + \alpha_m\|Tx_{m,i}\|^2 + (1-\alpha_m)\|Jz_{m,i}\|^2 \\
&\quad - 2(1-\alpha_m)\langle z, Tx_{m,i}\rangle - 2\alpha_m\langle z, JTx_{m,i}\rangle \\
&\leq \alpha_m\phi(z, Tx_{m,i}) + (1-\alpha_m)\phi(z, C^{G,\beta_m}x_{m,i}) \\
&\leq \phi(z, x_{m,i}),
\end{aligned}$$

where

$$C^{G,\beta_m}x = \{z \in C : \beta_m G(z, y) + \langle y - z, Jz - Jy\rangle \geq 0\}.$$

This shows that $z \in C_{m+1,i}$. This implies that $Sol(F, S, \varphi) \cap Fix(T) \subset C_{n,i}$. Hence, $Sol(F, S, \varphi) \cap Fix(T) \subset \bigcap_{i=1}^{N} C_{n,i} = C_n$, this completes the proof that $Sol(F, S, \varphi) \cap Fix(T) \subset C_n$. By using Lemma 2.3, we find

$$\langle x_{n,i} - z, Jx_{1,i} - Jx_{n,i}\rangle \geq 0, \quad \forall z \in C_n.$$

It follows that

$$\langle x_{n,i} - z, Jx_{1,i} - Jx_{n,i}\rangle \geq 0, \quad \forall z \in Sol(F, S, \varphi) \cap Fix(T) \subset C_n.$$

By using Lemma 2.3, one has

$$\begin{aligned}
\phi(x_{n,i}, x_{1,i}) &\leq \phi(\Pi_{Sol(F,S,\varphi)\cap Fix(T)}x_{1,i}, x_{1,i}) - \phi(\Pi_{Sol(F,S,\varphi)\cap Fix(T)}x_{1,i}, x_{n,i}) \\
&\leq \phi(\Pi_{Sol(F,S,\varphi)\cap Fix(T)}x_{1,i}, x_{1,i}),
\end{aligned}$$

which shows that$\{\phi(x_{n,i}, x_{1,i})\}$ is bounded. Hence, one obtains the boundedness of $\{x_{n,i}\}$. Without loss of generality, we assume $x_{n,i} \rightharpoonup \overline{x}_i$. Since every $C_{n,i}$ is convex and closed, so $\overline{x}_i \in C_{n,i}$. Since $\overline{x}_i \in C_{n,i}$, one has $\phi(x_{n,1}, x_{1,i}) \leq \phi(\overline{x}_i, x_{1,i})$. This implies that

$$\begin{aligned}
\phi(\overline{x}_i, x_{1,i}) &\leq \liminf_{n\to\infty}(\|x_{n,i}\|^2 + \|x_{1,i}\|^2 - 2\langle x_{n,i}, Jx_{1,i}\rangle) \\
&= \liminf_{n\to\infty}(\phi(x_{n,i}, x_{1,i})) \\
&\leq \limsup_{n\to\infty}(\phi(x_{n,i}, x_{1,i})) \\
&\leq \phi(\overline{x}_i, x_{1,i}).
\end{aligned}$$

Hence, one has $\lim_{n\to\infty} \phi(x_{n,i}, x_{1,i}) = \phi(\overline{x}_i, x_{1,i})$. It follows that $\lim_{n\to\infty} \|x_{n,i}\| = \|\overline{x}_i\|$. By using the KKP, one obtains that $\{x_{n,i}\}$ converges strongly to $\overline{x}_i$ as $n \to \infty$.

Since $x_{n+1,i} \subset C_{n+1,i} \subset C_{n,i}$, we find that $\phi(x_{n+1,i}, x_{1,i}) \geq \phi(x_{n,i}, x_{1,i})$, which shows that $\{\phi(x_{n,i}, x_{1,i})\}$ is nondecreasing. It follows that $\lim_{n\to\infty} \phi(x_{n,i}, x_{1,i})$ exists. Since

$$\phi(x_{n+1,i}, x_{1,i}) - \phi(x_{n,i}, x_{1,i}) \geq \phi(x_{n+1,i}, x_{n,i}),$$

one has $\lim_{n\to\infty} \phi(x_{n+1,i}, x_{n,i}) = 0$. Since $x_{n+1,i} \in C_{n+1,i}$, one obtains

$$\phi(x_{n+1,i}, y_{n,i}) \leq \phi(x_{n+1,i}, x_{n,i}).$$

It follows that $\lim_{n\to\infty} \phi(x_{n+1,i}, y_{n,i}) = 0$. Therefore, $\lim_{n\to\infty}(\|y_{n,i}\| - \|x_{n+1,i}\|) = 0$. This implies that

$$\lim_{n\to\infty} \|Jy_{n,i}\| = \lim_{n\to\infty} \|y_{n,i}\| = \|\overline{x}_i\| = \|J\overline{x}_i\|.$$

This implies that $\{Jy_{n,i}\}$ is bounded. Without loss of generality, we assume that $\{Jy_{n,i}\}$ converges weakly to $y_i^* \in E^*$. Since E is reflexive, we see that $J(E) = E^*$. This shows that there exists an element $y_i \in E$ such that $Jy_i = y_i^*$. It follows that

$$\phi(x_{n+1,i}, y_{n,i}) + 2\langle x_{n+1,i}, Jy_{n,i}\rangle = \|x_{n+1,i}\|^2 + \|Jy_{n,i}\|^2.$$

By taking $\liminf_{n\to\infty}$, one has

$$\begin{aligned} 0 &= \|\overline{x}_i\|^2 - 2\langle \overline{x}_i, y_i^*\rangle + \|y_i^*\|^2 \\ &= \|\overline{x}_i\|^2 + \|Jy_i\|^2 - 2\langle \overline{x}_i, Jy_i\rangle \\ &= \phi(\overline{x}_i, y_i) \\ &\geq 0, \end{aligned}$$

that is, $\overline{x}_i = y_i$, which implies that $J\overline{x}_i = y_i^*$. Hence $Jy_{n,i} \rightharpoonup J\overline{x}_i \in E^*$. Since E is uniformly smooth, E^* is uniformly convex and it has the KKP, we obtain $\lim_{n\to\infty} Jy_{n,i} = J\overline{x}_i$. Since $J^{-1} : E^* \to E$ is demi-continuous and E has the KKP, one gets that $y_{n,i} \to \overline{x}_i$, as $n \to \infty$.

On the other hand, we find from Lemma 2.2 that

$$\begin{aligned} \phi(z, y_{n,i}) &\leq \|z\|^2 + \alpha_n\|JTx_{n,i}\| + (1-\alpha_n)\|Jz_{n,i}\|^2 \\ &\quad - 2(1-\alpha_n)\langle z, Jz_{n,i}\rangle - 2\alpha_n\langle z, JTx_{n,i}\rangle \\ &\quad - \alpha_n(1-\alpha_n)g(\|JTx_{n,i} - Jz_{n,i}\|) \\ &\leq \alpha_n\phi(z, Tx_{n,i}) + (1-\alpha_n)\phi(z, C^{G,\mu_i}x_{n,i}) \\ &\quad - \alpha_n(1-\alpha_n)g(\|JTx_{n,i} - Jz_{n,i}\|) \\ &\leq \phi(z, x_{n,i}) - \alpha_n(1-\alpha_n)g(\|JTx_{n,i} - Jz_{n,i}\|). \end{aligned}$$

Since

$$\phi(z, x_{n,i}) - \phi(z, y_{n,i}) \leq (\|x_{n,i}\| + \|y_{n.i}\|)\|y_{n,i} - x_{n,i}\| + 2\langle z, Jy_{n,i} - Jx_{n,i}\rangle,$$

we find

$$\lim_{n\to\infty}(\phi(z, x_{n,i}) - \phi(z, y_{n,i})) = 0, \quad \forall z \in Fix(T) \cap Sol(F).$$

This implies $\lim_{n\to\infty} \|Jz_{n,i} - JTx_{n,i}\| = 0$. Hence, one has $JTx_{n,i} \to J\overline{x}_i$ as $n \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous, one has $Tx_{n,i} \rightharpoonup \overline{x}_i$. By using the fact

$$|\|Tx_{n,i}\| - \|\overline{x}_i\|| = |\|JTx_{n,i}\| - \|J\overline{x}_i\|| \leq \|JTx_{n,i} - J\overline{x}_i\|,$$

one has $\|Tx_{n,i}\| \to \|\overline{x}_i\|$ as $n \to \infty$. Since E has the KKP, one has $\lim_{n\to\infty} \|\overline{x}_i - Tx_{n,i}\| = 0$. By using the closedness of T, we find $T\overline{x}_i = \overline{x}_i$. This proves $\overline{x}_i \in Fix(T)$. Similarly, $\{z_{n,i}\}$ converges strongly to $\overline{x}_i$. Since G is a monotone bifunction, one has $\beta_n G(z, z_{n,i}) \leq \|z - z_{n,i}\|\|Jz_{n,i} - Jx_{n,i}\|$. Since $\liminf_{n\to\infty} \beta_n > 0$, we may assume there exists $\gamma > 0$ such that $\beta_n \geq \gamma$. It follows that

$$G(z, z_{n,i}) \leq \|z - z_{n,i}\|\frac{\|Jz_{n,i} - Jx_{n,i}\|}{\gamma}.$$

Hence, one has $G(z, \overline{x}_i) \leq 0$. For $0 < p < 1$, define $z^p = (1-p)\overline{x}_i + pz$. This implies that $G(z^p, \overline{x}_i) \leq 0$. Hence, we have

$$0 = G(z^p, z^p) \leq pG(z^p, z).$$

It follows that $G(\overline{x}_i, z) \geq 0$, for all $z \in C$. This implies that $\overline{x}_i \in Sol(G) = Sol(F, S, \varphi)$. By using Lemma 2.3, we find

$$\langle x_{n,i} - z, Jx_{1,i} - Jx_{n,i} \rangle \geq 0, \quad \forall z \in Fix(T) \cap Sol(F, S, \varphi).$$

Letting $n \to \infty$, one has $\langle \overline{x}_i - z, Jx_{1,i} - J\overline{x}_i \rangle \geq 0$. It follows that $\overline{x}_i = \Pi_{Fix(T) \cap Sol(F,S,\varphi)} x_{1,i}$. By the convexity of $Fix(T) \cap Sol(F, S, \varphi)$, one has

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sum_{i=1}^{N} \lambda_i x_{n,i} = \sum_{i=1}^{N} \lambda_i \overline{x}_i = \overline{x} \in Fix(T) \cap Sol(F, S, \varphi).$$

This completes the proof. □

In the framework of Hilbert space, we have the following result.

Corollary 3.2. *Let E be Hilbert space. Let C be a convex and closed subset of E and let F be a bifunction from $C \times C$ to $\mathbb{R}$ with $(A1), (A2), (A3)$ and $(A4)$. Let $S : C \to E$ be a continuous and monotone mapping and let $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let T be a quasi-nonexpansive mapping on C. Assume that $Sol(F, S, \varphi) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{\beta_n\}$ be a real sequence such that $\liminf_{n\to\infty} \beta_n > 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{0,1}, x_{0,2}, x_{0,3}, ..., x_{0,N} \in C, \ \textit{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap\limits_{i=1}^{N} C_{1,i}, \ (i = 1, 2, 3, ..., N), \\ x_{1,i} = P_{C_1} x_{0,i}, \\ x_1 = \sum_{i=1}^{N} \lambda_i x_{1,i}, \ \sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i \in [0, 1], \\ F(z_{n,i}, z) + (\varphi(z) - \varphi(z_{n,i})) + \langle Sz_{n,i}, z - z_{n,i} \rangle \geq \frac{1}{\beta_n} \langle z_{n,i} - z, Jz_{n,i} - Jx_{n,i} \rangle, \ \forall z \in C_n, \\ Jy_{n,i} = \alpha_n JTx_{n,i} + (1-\alpha_n) Jz_{n,i}, \\ C_{n+1,i} = \{z \in C_n : \|z, x_{n,i}\| \geq \|z, y_{n,i}\|\}, \\ C_{n+1} = \bigcap\limits_{i=1}^{N} C_{n+1,i}, \\ x_{n+1,i} = P_{C_{n+1}} x_{1,i}, \\ x_{n+1} = \sum_{i=1}^{N} \lambda_i x_{n,i}. \end{cases}$$

Then the sequence $\{x_{n,i}\}$ converges strongly to a common solution $\overline{x}_i$ and the sequence $\{x_n\}$ converges strongly to a special common solution $\overline{x} = \sum_{i=1}^{N} \lambda_i \overline{x}_i$, where $\overline{x}_i = P_{Sol(F,S,\varphi) \cap Fix(T)} x_{1,i}$ and $\overline{x} \in Sol(F, S, \varphi) \cap Fix(T)$.

Proof. In Hilbert space, the generalized projection is reduced to the metric projection and the class of quasi-ϕ-nonexpansive mappings is reduced to the class of quasi-nonexpansive mappings. By using Theorem 3.1, we find the results of Corollary 3.2. □

From Theorem 3.1, we also have the following result on the generalized equilibrium problem (1.4).

Corollary 3.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let F be a bifunction from $C \times C$ to $\mathbb{R}$ with*

$(A1),(A2),(A3)$ and $(A4)$. Let $S : C \to E^*$ be a continuous and monotone mapping and let T be a quasi-ϕ-nonexpansive mapping on C. Assume that $Sol(F,S) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{\beta_n\}$ be a real sequence such that $\liminf\limits_{n\to\infty} \beta_n > 0$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_{0,1}, x_{0,2}, x_{0,3}, ..., x_{0,N} \in C, \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap\limits_{i=1}^{N} C_{1,i}, \ (i = 1,2,3,...,N), \\ x_{1,i} = \Pi_{C_1} x_{0,i}, \\ x_1 = \sum_{i=1}^{N} \lambda_i x_{1,i}, \ \sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i \in [0,1], \\ F(z_{n,i}, z) + \langle S z_{n,i}, z - z_{n,i} \rangle \geq \frac{1}{\beta_n} \langle z_{n,i} - z, J z_{n,i} - J x_{n,i} \rangle, \ \forall z \in C_n, \\ J y_{n,i} = \alpha_n J T x_{n,i} + (1-\alpha_n) J z_{n,i}, \\ C_{n+1,i} = \{ z \in C_n : \phi(z, x_{n,i}) \geq \phi(z, y_{n,i}) \}, \\ C_{n+1} = \bigcap\limits_{i=1}^{N} C_{n+1,i}, \\ x_{n+1,i} = P_{C_{n+1}} x_{1,i}, \\ x_{n+1} = \sum_{i=1}^{N} \lambda_i x_{n,i}. \end{cases}$$

Then the sequence $\{x_{n,i}\}$ converges strongly to a common solution $\overline{x}_i$ and the sequence $\{x_n\}$ converges strongly to a special common solution $\overline{x} = \sum_{i=1}^{N} \lambda_i \overline{x}_i$, where $\overline{x}_i = \Pi_{Sol(F,S) \cap Fix(T)} x_{1,i}$ and $\overline{x} \in Sol(F,S) \cap Fix(T)$.

From Theorem 3.1, we also have the following result on the mixed equilibrium problem (1.3).

Corollary 3.4. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let F be a bifunction from $C \times C$ to $\mathbb{R}$ with $(A1),(A2),(A3)$ and $(A4)$. Let $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function and let T be a quasi-ϕ-nonexpansive mapping on C. Assume that $Sol(F,\varphi) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{\beta_n\}$ be a real sequence such that $\liminf_{n\to\infty} \beta_n > 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{0,1}, x_{0,2}, x_{0,3}, ..., x_{0,N} \in C, \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap\limits_{i=1}^{N} C_{1,i}, \ (i = 1,2,3,...,N), \\ x_{1,i} = \Pi_{C_1} x_{0,i}, \\ x_1 = \sum_{i=1}^{N} \lambda_i x_{1,i}, \ \sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i \in [0,1], \\ F(z_{n,i}, z) + \varphi(z) - \varphi(z_{n,i}) \geq \frac{1}{\beta_n} \langle z_{n,i} - z, J z_{n,i} - J x_{n,i} \rangle, \ \forall z \in C_n, \\ J y_{n,i} = \alpha_n J T x_{n,i} + (1-\alpha_n) J z_{n,i}, \\ C_{n+1,i} = \{ z \in C_n : \phi(z, x_{n,i}) \geq \phi(z, y_{n,i}) \}, \\ C_{n+1} = \bigcap\limits_{i=1}^{N} C_{n+1,i}, \\ x_{n+1,i} = \Pi_{C_{n+1}} x_{1,i}, \\ x_{n+1} = \sum_{i=1}^{N} \lambda_i x_{n,i}. \end{cases}$$

Then the sequence $\{x_{n,i}\}$ converges strongly to a common solution $\overline{x}_i$ and the sequence $\{x_n\}$ converges strongly to a special common solution $\overline{x} = \sum_{i=1}^{N} \lambda_i \overline{x}_i$, where $\overline{x}_i = \Pi_{Sol(F,\varphi) \cap Fix(T)} x_{1,i}$ and $\overline{x} \in Sol(F,\varphi) \cap Fix(T)$.

Finally, we give a result on equilibrium problem (1.5).

Corollary 3.5. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let F be a bifunction from $C \times C$ to $\mathbb{R}$ with $(A1), (A2), (A3)$ and $(A4)$. Let T be a quasi-ϕ-nonexpansive mappings on C. Assume that $Sol(F) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{\beta_n\}$ be real sequence such that $\liminf_{n\to\infty} \beta_n > 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{0,1}, x_{0,2}, x_{0,3}, ..., x_{0,N} \in C, \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap_{i=1}^{N} C_{1,i}, \ (i = 1, 2, 3, ..., N), \\ x_{1,i} = \Pi_{C_1} x_{0,i}, \\ x_1 = \sum_{i=1}^{N} \lambda_i x_{1,i}, \ \sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i \in [0,1], \\ F(z_{n,i}, z) \geq \frac{1}{\beta_n} \langle z_{n,i} - z, Jz_{n,i} - Jx_{n,i} \rangle, \ \forall z \in C_n, \\ Jy_{n,i} = \alpha_n JTx_{n,i} + (1-\alpha_n) Jz_{n,i}, \\ C_{n+1,i} = \{z \in C_n : \phi(z, x_{n,i}) \geq \phi(z, y_{n,i})\}, \\ C_{n+1} = \bigcap_{i=1}^{N} C_{n+1,i}, \\ x_{n+1,i} = \Pi_{C_{n+1}} x_{1,i}, \\ x_{n+1} = \sum_{i=1}^{N} \lambda_i x_{n,i}. \end{cases}$$

Then the sequence $\{x_{n,i}\}$ converges strongly to a common solution $\overline{x}_i$ and the sequence $\{x_n\}$ converges strongly to a special common solution $\overline{x} = \sum_{i=1}^{N} \lambda_i \overline{x}_i$, where $\overline{x}_i = \Pi_{Sol(F)\cap Fix(T)} x_{1,i}$ and $\overline{x} \in Sol(F) \cap Fix(T)$.

Remark 3.6. Now, we give an example to show that, the multidirectional iteration algorithm can accelerate the convergence speed of iterative sequence. Let $X = R^2$, $C_n = \{(x,y) \in R^2 : x^2 + y^2 \leq 1\}$, $x_{1,1} = (1,1)$, $x_{1,2} = (-1,1)$, $F = \{0\}$.
Case 1, take only one initial $x_{1,1}$, $x_n = P_{C_n} x_{1,1} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, then $d(x_n, F) = 1$.
Case 2, take two initials $x_{1,1}, x_{1,2}$,

$$x_n = \frac{1}{2} P_{C_n} x_{1,1} + \frac{1}{2} P_{C_n} x_{1,2} = (0, \frac{\sqrt{2}}{2}),$$

then $d(x_n, F) = \frac{\sqrt{2}}{2}$. It is obvious, the inequality "$\frac{\sqrt{2}}{2} < 1$" show that, the multidirectional iteration algorithm can accelerate the convergence speed of iterative sequence $\{x_n\}$.

Acknowledgment

This work is dedicated to Professor Yeol Je Cho on the occasion of his 65th birthday.

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