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Integral inequalities of Simpson's type for (α, m)-convex functions

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Abstract

In this paper, we establish some integral inequalities of Simpson's type for (α, m)-convex functions.

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1. Introduction

The following definition is well-known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [10] the concept of m-convex functions below was innovated.

Definition 1.2 ([10])**.** For $f : [0, b] \to \mathbb{R}$, and $b > 0$ and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

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Definition 1.3 ([6])**.** For $f:[0,b]\to\mathbb{R}$, $b>0$, and $(\alpha,m)\in(0,1]^2$, if

$$f(\lambda x+m(1-\lambda)y)\le\lambda^\alpha f(x)+m(1-\lambda^\alpha)f(y)$$

is valid for all $x,y\in[0,b]$ and $\lambda\in[0,1]$, then we say that $f(x)$ is an (α,m)-convex function on $[0,b]$.

Theorem 1.4 ([3, Theorem 2.2])**.** *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.5 ([7, Theorem 1 and 2])**.** *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ and $q\ge1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q},$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

In [4], the following Hermite–Hadamard type inequality for m-convex functions was proved.

Theorem 1.6 ([4])**.** *Let $f:\mathbb{R}_0\to\mathbb{R}$ be m-convex and $m\in(0,1]$. If $f\in L_1([a,b])$ for $0\le a<b<\infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\le\min\left\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 1.7 ([2, Theorem 2.2])**.** *Let $f:\mathbb{R}_0\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L_1([a,b])$, then*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b\frac{f(x)+mf(x/m)}{2}\,\mathrm{d}\,x\le\frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\frac{f(a/m)+f(b/m)}{2}\right].$$

Theorem 1.8 ([5, Theorem 3.1])**.** *Let $I\supseteq\mathbb{R}_0$ be an open real interval and let $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([a,b])$ for $0\le a<b<\infty$. If $|f'|^q$ is (α,m)-convex on $[a,b]$ for some given numbers $m,\alpha\in(0,1]$ and $q\ge1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\times\min\left\{\left[v_1|f'(a)|^q+v_2m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q},\left[v_2m\left|f'\left(\frac{a}{m}\right)\right|^q+v_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned}$$

where

$$v_1=\frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha+\frac{1}{2^\alpha}\right),\quad v_2=\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2}-\frac{1}{2^\alpha}\right).$$

For more information on this topic, we refer to recent papers [1, 8, 9, 11–13] and closely related references therein.

In this paper, we establish some integral inequalities of Simpson's type for (α,m)-convex functions.

2. A lemma

To establish some new Simpson's type inequalities for (α, m)-convex functions, we need the following lemma.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \\ =\frac{b-a}{4}\int_0^1\left[\left(\frac{3}{4}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)+\left(\frac{1}{4}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}\,t.$$

Proof. By integration by parts, we have

$$\begin{aligned}
&\int_0^1\left(\frac{3}{4}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}\,t\\
&=-\frac{2}{b-a}\left[\left(\frac{3}{4}-t\right)f\left(ta+(1-t)\frac{a+b}{2}\right)\Big|_0^1+\int_0^1 f\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}\,t\right]\\
&=-\frac{2}{b-a}\left[-\frac{1}{4}f(a)-\frac{3}{4}f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_0^1 f\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}\,t\\
&=\frac{2}{b-a}\left[\frac{1}{4}f(a)+\frac{3}{4}f\left(\frac{a+b}{2}\right)\right]-\frac{4}{(b-a)^2}\int_a^{(a+b)/2} f(x)\,\mathrm{d}\,x,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left(\frac{1}{4}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}\,t\\
&=-\frac{2}{b-a}\left[\left(\frac{1}{4}-t\right)f\left(t\frac{a+b}{2}+(1-t)b\right)\Big|_0^1+\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}\,t\right]\\
&=-\frac{2}{b-a}\left[-\frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{1}{4}f(b)\right]-\frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}\,t\\
&=\frac{2}{b-a}\left[\frac{3}{4}f\left(\frac{a+b}{2}\right)+\frac{1}{4}f(b)\right]-\frac{4}{(b-a)^2}\int_{(a+b)/2}^b f(x)\,\mathrm{d}\,x.
\end{aligned}$$

The proof is completed. □

3. Some new integral inequalities of Simpson's type

In this section, the integral inequalities of Simpson's type related to (α, m)-convex function are discussed.

Theorem 3.1. *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a, b \in \mathbb{R}_0$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is (α, m)-convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0, 1]^2$ and $q \geq 1$, then*

$$\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le\frac{b-a}{4}\left(\frac{5}{16}\right)^{1-1/q}\left\{\left[\frac{3^{\alpha+2}+2^{2\alpha+1}\alpha-2^{2\alpha+2}}{2^{2\alpha+3}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right.\right. \\ +m\frac{9\times2^{2\alpha+1}-2\times3^{\alpha+2}+11\times2^{2\alpha}\alpha+5\times2^{2\alpha}\alpha^2}{2^{2\alpha+4}(\alpha+1)(\alpha+2)}\Bigg|$$

$$\times f'\left(\frac{a+b}{2m}\right)\Big|^q\Bigg]^{1/q} + \left[\frac{3\times 2^{2\alpha+1}\alpha + 2^{2\alpha+2}+1}{2^{2\alpha+3}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\frac{2^{2\alpha+1}+3\times 2^{2\alpha}\alpha+5\times 2^{2\alpha}\alpha^2-2}{2^{2\alpha+4}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\Bigg\}.$$

Proof. Since $|f'|^q$ is an (α,m)-convex function on $\left[0,\frac{b}{m}\right]$, from Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{b-a}{4}\left[\int_0^1\left|\frac{3}{4}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}\,t+\int_0^1\left|\frac{1}{4}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}\,t\right] \\
&\le \frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{3}{4}-t\right|\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|\frac{3}{4}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad\left.+\left(\int_0^1\left|\frac{1}{4}-t\right|\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1\left|\frac{1}{4}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\} \\
&\le \frac{b-a}{4}\left(\frac{5}{16}\right)^{1-1/q}\left\{\left[\int_0^1\left|\frac{3}{4}-t\right|\left(t^\alpha\left|f'(a)\right|^q+m\left(1-t^\alpha\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad\left.+\left[\int_0^1\left|\frac{1}{4}-t\right|\left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left(1-t^\alpha\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right\} \\
&= \frac{b-a}{4}\left(\frac{5}{16}\right)^{1-1/q}\left\{\left[\frac{3^{\alpha+2}+2^{2\alpha+1}\alpha-2^{2\alpha+2}}{2^{2\alpha+3}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right.\right. \\
&\quad\left.+m\frac{9\times 2^{2\alpha+1}-2\times 3^{\alpha+2}+11\times 2^{2\alpha}\alpha+5\times 2^{2\alpha}\alpha^2}{2^{2\alpha+4}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q} \\
&\quad\left.+\left[\frac{3\times 2^{2\alpha+1}\alpha+2^{2\alpha+2}+1}{2^{2\alpha+3}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\frac{2^{2\alpha+1}+3\times 2^{2\alpha}\alpha+5\times 2^{2\alpha}\alpha^2-2}{2^{2\alpha+4}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 3.1 is thus completed. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1*, if* $q=1$*, then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\qquad\le \frac{b-a}{4}\left[\frac{3^{\alpha+2}+2^{2\alpha+1}\alpha-2^{2\alpha+2}}{2^{2\alpha+3}(\alpha+1)(\alpha+2)}|f'(a)|\right. \\
&\qquad\quad+m\frac{9\times 2^{2\alpha+1}-2\times 3^{\alpha+2}+11\times 2^{2\alpha}\alpha+5\times 2^{2\alpha}\alpha^2}{2^{2\alpha+4}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2m}\right)\right| \\
&\qquad\quad\left.+\frac{3\times 2^{2\alpha+1}\alpha+2^{2\alpha+2}+1}{2^{2\alpha+3}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|+m\frac{2^{2\alpha+1}+3\times 2^{2\alpha}\alpha+5\times 2^{2\alpha}\alpha^2-2}{2^{2\alpha+4}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|\right].
\end{aligned}$$

Corollary 3.3. *Under the assumptions of Theorem* 3.1*, if* $\alpha=m=1$*, then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{5(b-a)}{64}\times\left\{\left[\frac{19\left|f'(a)\right|^q+41\left|f'\left(\frac{a+b}{2}\right)\right|^q}{60}\right]^{1/q}\right.
\end{aligned}$$

$$+\left[\frac{41\left|f'\left(\frac{a+b}{2}\right)\right|^q+19\left|f'(b)\right|^q}{60}\right]^{1/q}\Bigg\}.$$

Corollary 3.4. *Under the assumptions of Theorem 3.1, if $\alpha=m=q=1$, then*

$$\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{5(b-a)}{32}\left[\frac{19\left|f'(a)\right|+82\left|f'\left(\frac{a+b}{2}\right)\right|+19\left|f'(b)\right|}{120}\right].$$

Theorem 3.5. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is (α,m)-convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]^2$ and $q>1$, then*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le\frac{b-a}{4}\left[\frac{(q-1)\left(3^{(2q-1)/(q-1)}+1\right)}{2^{2(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}\\&\times\left\{\left[\frac{1}{\alpha+1}\left|f'(a)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\&\left.+\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Proof. Since $|f'|^q$ is an (α,m)-convex function on $\left[0,\frac{b}{m}\right]$, by Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le\frac{b-a}{4}\left[\int_0^1\left|\frac{3}{4}-t\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}\,t\right.\\&\left.+\int_0^1\left|\frac{1}{4}-t\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}\,t\right]\\&\le\frac{b-a}{4}\left\{\left(\int_0^1\left|\frac{3}{4}-t\right|^{q/(q-1)}\mathrm{d}\,t\right)^{1-1/q}\right.\\&\times\left[\int_0^1\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\\&+\left(\int_0^1\left|\frac{1}{4}-t\right|^{q/(q-1)}\mathrm{d}\,t\right)^{1-1/q}\\&\left.\times\left[\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}\\&\le\frac{b-a}{4}\left[\frac{(q-1)\left(3^{(2q-1)/(q-1)}+1\right)}{2^{2(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}\\&\times\left\{\left[\int_0^1\left(t^\alpha\left|f'(a)\right|^q+m\left(1-t^\alpha\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right.\\&\left.+\left[\int_0^1\left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left(1-t^\alpha\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}\\&=\frac{b-a}{4}\left[\frac{(q-1)\left(3^{(2q-1)/(q-1)}+1\right)}{2^{2(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}\\&\times\left\{\left[\frac{1}{\alpha+1}\left|f'(a)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\end{aligned}$$

$$+\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\Bigg\}.$$

Theorem 3.5 is proved. □

Corollary 3.6. *Under the assumptions of Theorem 3.5, if $\alpha=m=1$, then*

$$\left|\frac{1}{8}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left[\frac{(q-1)\big(3^{(2q-1)/(q-1)}+1\big)}{2^{2(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}$$
$$\times\left\{\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right]^{1/q}\right\}.$$

4. Applications to means

In this final section, we apply some inequalities of the Hermite–Hadamard type for (α,m)-convex functions to construct some inequalities for means.

For two positive numbers $b>a>0$, define

$$A(a,b)=\frac{a+b}{2},\quad H(a,b)=\frac{2ab}{a+b},\quad I(a,b)=\frac{b-a}{\ln b-\ln a},\quad\text{and}\quad L_s(a,b)=\left[\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{1/s},$$

for $s\ne0,-1$. These means are respectively called the arithmetic, harmonic, logarithmic and generalized logarithmic means of two positive number a and b.

Let $f(x)=x^s$ for $x>0$, $s>1$, $q\ge1$, and $(s-1)q\ge1$. Then the function $|f'(x)|^q=s^qx^{(s-1)q}$ is convex on $(0,\infty)$. Applying Corollary 3.3 to $|s|^qx^{(s-1)q}$ yields:

Theorem 4.1. *Let $b>a>0$, $s>1$, $q\ge1$, and $(s-1)q\ge1$. Then*

$$\left|\frac{A(a^s,b^s)+3A^s(a,b)}{4}-L_s^s(a,b)\right|$$
$$\le\frac{5s(b-a)}{64}\left\{\left[\frac{19a^{(s-1)q}+41[A(a,b)]^{(s-1)q}}{60}\right]^{1/q}+\left[\frac{41[A(a,b)]^{(s-1)q}+19b^{(s-1)q}}{60}\right]^{1/q}\right\}.$$

Furthermore, if $s\ge2$, then

$$\left|\frac{A(a^s,b^s)+3A^s(a,b)}{4}-L_s^s(a,b)\right|\le\frac{5s(b-a)}{32}\left[\frac{19a^{s-1}+82[A(a,b)]^{s-1}+19b^{s-1}}{120}\right].$$

Taking $f(x)=x^s$ for $x>0$, $s>1$, $q>1$, and $(s-1)q\ge1$ in Corollary 3.6 derives the following inequalities for means.

Theorem 4.2. *Let $b>a>0$, $s>1$, $q\ge1$, and $(s-1)q\ge1$. Then*

$$\left|\frac{A(a^s,b^s)+3A^s(a,b)}{4}-L_s^s(a,b)\right|\le\frac{s(b-a)}{4}\left[\frac{(q-1)\big(3^{(2q-1)/(q-1)}+1\big)}{2^{2(2q-1)/(q-1)}(2q-1)}\right]^{1-1/q}$$
$$\times\left\{\left[\frac{a^{(s-1)q}+[A(a,b)]^{(s-1)q}}{2}\right]^{1/q}+\left[\frac{[A(a,b)]^{(s-1)q}+b^{(s-1)q}}{2}\right]^{1/q}\right\}.$$

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