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Fixed point theorems of nondecreasing order-Ćirić-Lipschitz mappings in normed vector spaces without normalities of cones

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Abstract

We introduce the concept of order-Ćirić-Lipschitz mappings, and prove some fixed point theorems for such kind of mappings in normed vector spaces without assuming the normalities of cones by using upper and lower solutions method, which improve many existing results of order-Lipschitz mappings in Banach spaces or Banach algebras. It is worth mentioning that even in the setting of normal cones, the main results in this paper are still new since the sum of spectral radius or the sum of restricted constants may be greater than or equal to 1.

Keywords: Fixed point, order-Ćirić-Lipschitz mapping, Picard-complete, *w*-complete.
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1. Introduction and preliminaries

Ćirić's contraction in metric spaces was primitively introduced and studied by Ćirić [3], which generalizes and includes Banach contraction [1], Kannan contraction [7], and Chatterjea contraction [2]. Let (X, d) be a metric space. Recall that a mapping $T : X \to X$ is called a Ćirić's contraction if there exist nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that

$$d(Tx, Ty) \leqslant qd(x, y) + rd(x, Tx) + sd(y, Ty) + t[d(x, Ty) + d(y, Tx)], \quad \forall\, x, y \in X.$$

Note that a Ćirić's contraction must be a Lipschitz mapping. Thus motivated by [3], we introduce the concept of Ćirić order-Lipschitz mapping in topological vector spaces as follows.

Definition 1.1. Let P be a cone of a topological vector space E, $D \subset E$ and $\preceq$ the partial order induced by P. A mapping $T : D \to E$ is called an order-Ćirić-Lipschitz mapping if there exist $A_i, B_i : P \to P$ $(i = 1, 2, 3, 4, 5)$ (or nonnegative real numbers A_i, B_i (i=1,2,3,4,5)) such that for each $x, y \in D$ with $y \preceq x$,

$$\begin{aligned} &-B_1(x-y) - B_2(x-Tx)_+ - B_3(y-Ty)_+ - B_4(x-Ty)_+ - B_5(y-Tx)_+ \preceq Tx - Ty \\ &\preceq A_1(x-y) + A_2(x-Tx)_+ + A_3(y-Ty)_+ + A_4(x-Ty)_+ + A_5(y-Tx)_+, \end{aligned} \tag{1.1}$$

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where

$$u_+ = \begin{cases} u, & u \in P, \\ -u, & u \in -P, \\ \theta, & \text{o.w.} \end{cases}$$

for each $u \in E$.

In particular when $A_i u \equiv \theta$ $(i = 2,3,4,5)$ and $B_i u \equiv \theta$ $(i = 2,3,4,5)$, the generalized order-Lipschitz mapping is reduced to an order-Lipschitz mapping considered in [8, 10, 11], where the cone is necessarily assumed to be normal. Recently, without assuming the normality of P, Jiang and Li [6] proved the following fixed point theorem of order-Lipschitz mappings by introducing the concept of Picard-completeness and applying the sandwich theorem in the sense of w-convergence.

Theorem 1.2 ([6]). *Let P be a solid cone of a Banach algebra $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$, and $T : D = [u_0, v_0] \to E$ a nondecreasing order-Lipschitz mapping restricted with vectors (i.e., there exist $A_1 \in P$ and $A_i = \theta$ $(i = 2,3,4,5)$ such that such that (1.1) is satisfied). Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$ (i.e., u_0 and v_0 are a pair of lower and upper solutions of T), $r(A_1) < 1$ and T is Picard-complete at u_0 and v_0. Then T has a unique fixed point $x^* \in [u_0, v_0]$, and $x_n \xrightarrow{w} x^*$ for each $x_0 \in [u_0, v_0]$, where $\{x_n\} = O(T, x_0)$ and $O(T, x_0)$ denotes the Picard iteration sequence of T at x_0, i.e., $x_n = T^n x_0$ for each n.*

In this paper, we shall prove some new fixed point theorems of order-Ćirić-Lipschitz mappings in normed vector spaces without assuming the normalities of cones by using upper and lower solutions method, which improve many existing results of order-Lipschitz mappings in Banach spaces or Banach algebras. It is worth mentioning that even in the setting of normal cones, the main results in this paper are still new since the sum of spectral radius or the sum of restricted constants may be greater than or equal to 1.

Let E be a topological vector space. A nonempty closed subset P of E is a cone if it is such that $ax + by \in P$, for each $x, y \in P$ and each $a, b \geqslant 0$, and $P \cap (-P) = \{\theta\}$, where θ is the zero element of E. Each cone P of E determines a partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$ for each $x, y \in X$. For each $u_0, v_0 \in E$ with $u_0 \preceq v_0$, we set $[u_0, v_0] = \{u \in E : u_0 \preceq u \preceq v_0\}$, $[u_0, +\infty) = \{x \in E : u_0 \preceq x\}$ and $(-\infty, v_0] = \{x \in E : x \preceq v_0\}$. A cone P of E is solid [4] if $\mathrm{int}(P)$ is nonempty, where $\mathrm{int}(P)$ denotes the interior of P. For each $x, y \in E$ with $y - x \in \mathrm{int}(P)$, we write $x \ll y$.

Let $(E, \|\cdot\|)$ be a normed vector space. A cone P of E is normal [4] if there is a positive real number N such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leqslant N\|y\|$, and the minimal N is called a normal constant of P. Note that if P is non-normal then the sandwich theorem does not hold in the sense of norm-convergence.

Definition 1.3. Let P be a solid cone of a topological space E, $\{x_n\} \subset E$ and $D \subset E$.

(i) The sequence $\{x_n\}$ is w-convergent [5, 9] if for each $\epsilon \in \mathrm{int}(P)$, there is a positive integer n_0 and $x \in E$ such that $x - \epsilon \ll x_n \ll x + \epsilon$ for each $n \geqslant n_0$ (denote $x_n \xrightarrow{w} x$ and x is called a w-limit of $\{x_n\}$).

(ii) The sequence $\{x_n\}$ is w-Cauchy [6] if for each $\epsilon \in \mathrm{int}(P)$, there is a positive integer n_0 such that $-\epsilon \ll x_n - x_m \ll \epsilon$ for each $m, n \geqslant n_0$, i.e., $x_n - x_m \xrightarrow{w} \theta$ $(m, n \to \infty)$.

(iii) The space E is w-complete if each w-Cauchy sequence in E is w-convergent;

(iv) The subset D is w-closed [6] if for each $\{x_n\} \subset D$, $x_n \xrightarrow{w} x$ implies $x \in D$.

Lemma 1.4 ([6, 9]). *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Then*

(i) *each sequence $\{x_n\} \subset E$ has a unique w-limit;*

(ii) *the partial order intervals* $[u_0, v_0]$, $[u_0, +\infty)$ *and* $(-\infty, v_0]$ *are* w*-closed;*

(iii) *for each* $\{x_n\}, \{y_n\}, \{z_n\} \subset E$ *with* $x_n \preceq y_n \preceq z_n$ *for each* n, $x_n \xrightarrow{w} z$ *and* $z_n \xrightarrow{w} z$ *imply* $y_n \xrightarrow{w} z$, *where* $z \in E$.

Lemma 1.5 ([5, 9])**.** *Let* P *be a solid cone of a normed vector space* $(E, \|\cdot\|)$ *and* $x_n \subset E$. *Then* $x_n \xrightarrow{\|\cdot\|} x$ *implies* $x_n \xrightarrow{w} x$. *Moreover, if* P *is normal then* $x_n \xrightarrow{w} x \Leftrightarrow x_n \xrightarrow{\|\cdot\|} x$.

Proposition 1.6. *Let* P *be a normal solid cone of a normed vector space* $(E, \|\cdot\|)$. *Then* E *is* w*-complete if and only if it is complete.*

Proof. Suppose that E is complete and $\{x_n\}$ is a w-Cauchy sequence of E. By Lemma 1.5 and the normality of P, $\{x_n\}$ is a Cauchy sequence of E, and hence it is convergent by the completeness of E. Moreover by Lemma 1.5, $\{x_n\}$ is w-convergent. This shows E is w-complete.

Suppose that E is w-complete and $\{x_n\}$ is a Cauchy sequence of E. By Lemma 1.5, $\{x_n\}$ is a w-Cauchy sequence of E, and hence it is convergent by the w-completeness of E. Moreover by Lemma 1.5 and the normality of P, $\{x_n\}$ is convergent. This shows E is complete. □

Definition 1.7 ([6])**.** Let P be a solid cone of a topological vector space $(E, \|\cdot\|)$, $D \subset E$, $x_0 \in D$ and $T : D \to E$. If the Picard iteration sequence $O(T, x_0)$ is w-convergent provided that it is w-Cauchy, then T is said to be Picard-complete at x_0. Moreover, if T is Picard-complete at each $x \in D$, then T is said to be Picard-complete on D.

Remark 1.8. If E is a w-complete topological vector space then the mapping $T : E \to E$ is Picard-complete on E. Consequently, if P is a normal solid cone of a Banach space $(E, \|\cdot\|)$ then each mapping $T : E \to E$ is Picard-complete on E by Proposition 1.6.

Definition 1.9. Let P be a solid cone of a topological vector space E and $D \subset E$. A mapping $T : D \to E$ is w-continuous at $x_0 \in E$ if for each $\{x_n\} \subset E$, $x_n \xrightarrow{w} x_0$ implies $Tx_n \xrightarrow{w} Tx_0$. If T is w-continuous at each $x \in D$, then T is w-continuous on D.

Lemma 1.10. *Let* P *be a solid cone of a normed vector space* $(E, \|\cdot\|)$ *and* $A : E \to E$ *a linear bounded mapping with* $A(P) \subset P$. *Then* A *is* w*-continuous on* E.

Proof. Let $x \in E$ and $\{x_n\}$ be a sequence in E such that $x_n \xrightarrow{w} x$. For each $\epsilon \in \mathrm{int}(P)$, it is clear that $\frac{\epsilon}{m} \in \mathrm{int}(P)$ for each m, and hence there exists n_m such that $-\frac{\epsilon}{m} \ll x_n - x \ll \frac{\epsilon}{m}$ for each $n \geqslant n_m$. Note that A is a linear mapping with $A(P) \subset P$, then $-\frac{A\epsilon}{m} \preceq Ax_n - Ax \preceq \frac{A\epsilon}{m}$ for each $n \geqslant n_m$. It is clear that $\frac{A\epsilon}{m} \xrightarrow{\|\cdot\|} \theta (m \to \infty)$ since A is bounded, and hence $\frac{A\epsilon}{m} \xrightarrow{w} \theta (m \to \infty)$ by Lemma 1.5. Moreover, by Lemma 1.4 (iii), we obtain $Ax_n - Ax \xrightarrow{w} \theta$, i.e., A is continuous at x. □

2. Main results

In this section, we first consider the existence of fixed points of order-Ćirić-Lipschitz mappings.

Theorem 2.1. *Let* P *be a solid cone of a normed vector space* $(E, \|\cdot\|)$, $x_0 \in E$ *and* $T : D = [x_0, +\infty) \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with linear bounded mappings* $A_i : P \to P$ $(i = 1, 2, 3, 4, 5)$. *Assume that* $x_0 \preceq Tx_0$, T *is Picard-complete at* x_0 *and*

$$r(A_2 + A_5) < 1, \;\; r((I - A_2 - A_5)^{-1}(A_1 + A_3 + A_5)) < 1. \tag{2.1}$$

Then T *has a fixed point in* $[x_0, +\infty)$.

Proof. Let $B : P \to P$ be an arbitrary linear bounded mapping with $r(B) < 1$, then $I - B$ is invertible, denote the inverse of $I - B$ by $(I - B)^{-1}$. Moreover, it follows from Neumann's formula that

$$(I - B)^{-1} = \sum_{n=0}^{\infty} B^n = I + B + B^2 + \cdots + B^n + \cdots, \tag{2.2}$$

which implies that $(I - B)^{-1} : P \to P$ is a linear bounded mapping. It follows from $r(B) < 1$ and Gelfand's formula that there exist a positive integer n_1 and $\beta \in (r(B), 1)$ such that

$$\|B^n\| \leqslant \beta^n, \ \ \forall\, n \geqslant n_1.$$

Thus for each $u \in P$, we get

$$\|B^n u\| \leqslant \|B^n\| \|u| \leqslant \beta^n \|u\|, \ \ \forall\, n \geqslant n_1,$$

which implies $B^n u \overset{\|\cdot\|}{\to} \theta$ for each $u \in P$, and hence by Lemma 1.5,

$$B^n u \overset{w}{\to} \theta, \ \ \forall\, u \in P. \tag{2.3}$$

Let $x_n = T^n x_0$ for each n. Note that T is nondecreasing on $[x_0, +\infty)$, then it follows from $x_0 \preceq Tx_0$ that

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1}, \ \ \forall\, n. \tag{2.4}$$

By (1.1) and (2.4),

$$\begin{aligned}\theta \preceq x_{n+1} - x_n &= Tx_n - Tx_{n-1}\\ &\preceq A_1(x_n - x_{n-1}) + A_2(x_{n+1} - x_n) + A_3(x_{n+1} - x_n) + A_4(x_n - x_n) + A_5(x_{n+1} - x_{n-1})\\ &= A_1(x_n - x_{n-1}) + A_2(x_{n+1} - x_n) + A_3(x_n - x_{n-1}) + A_5(x_{n+1} - x_n) + A_5(x_n - x_{n-1}), \ \ \forall\, n,\end{aligned}$$

and so

$$\theta \preceq (I - A_2 - A_5)(x_{n+1} - x_n) \preceq (A_1 + A_3 + A_5)(x_n - x_{n-1}), \ \ \forall\, n. \tag{2.5}$$

Note that $(I - A_2 - A_5)^{-1} : P \to P$ is a linear bounded mapping by $r(A_2 + A_5) < 1$ and taking $B = A_2 + A_5$ in (2.2). Acting (2.5) with $(I - A_2 - A_5)^{-1}$, we get

$$\theta \preceq x_{n+1} - x_n \preceq A(x_n - x_{n-1}), \ \ \forall\, n, \tag{2.6}$$

where $A = (I - A_2 - A_5)^{-1}(A_1 + A_3 + A_5)$. Clearly, $(I - A)^{-1}$ is also a linear bounded mapping since $r(A) < 1$ by (2.1). Taking $B = A$ in (2.2), by (2.6), we have

$$\begin{aligned}\theta \preceq x_m - x_n = \sum_{i=n}^{m-1} (x_{i+1} - x_i) &\preceq \sum_{i=n}^{m-1} A^i (x_1 - x_0)\\ &= A^n \sum_{i=0}^{m-1} A^i (x_1 - x_0) \preceq A^n (I - A)^{-1}(x_1 - x_0), \ \ \forall\, m > n.\end{aligned} \tag{2.7}$$

Taking $B = A$ and $u = (I - A)^{-1}(x_1 - x_0)$ in (2.3), it follows that

$$A^n (I - A)^{-1}(x_1 - x_0) \overset{w}{\to} \theta \ (n \to \infty),$$

which together with (2.7) and Lemma 1.4 (iii) implies that

$$x_m - x_n \overset{w}{\to} \theta \ (m > n \to \infty), \tag{2.8}$$

i.e., $\{x_n\}$ is a w-Cauchy sequence. Thus by the Picard-completeness of T at x_0, there exists $x^* \in E$ such that

$$x_n \xrightarrow{w} x^*. \tag{2.9}$$

It is clear that $[x_n, +\infty)$ is w-closed for each n by Lemma 1.4 (ii), and so it follows from (2.4) and (2.9) that

$$x_n \preceq x^*, \ n = 0, 1, 2, \cdots . \tag{2.10}$$

Moreover, by the nondecreasing property of T on $[x_0, +\infty)$,

$$x_{n+1} = Tx_n \preceq Tx^*, \ n = 0, 1, 2, \cdots ,$$

which together with (2.9) and Lemma 1.4 (ii) implies that

$$x^* \preceq Tx^*. \tag{2.11}$$

By (1.1), (2.4), (2.10), and (2),

$$\begin{aligned}
\theta &\preceq Tx^* - x_{n+1} \\
&= Tx^* - Tx_n \\
&\preceq A_1(x^* - x_n) + A_2(Tx^* - x^*) + A_3(x_{n+1} - x_n) + A_4(x^* - x_{n+1}) + A_5(Tx^* - x_n), \ \ \forall\, n.
\end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in (2.12), by (2.8), (2.9), Lemma 1.4 (iii), and Lemma 1.10 we get

$$\theta \preceq Tx^* - x^* \preceq A_2(Tx^* - x^*) + A_5(Tx^* - x^*),$$

and so

$$(I - A_2 - A_5)(Tx^* - x^*) \preceq \theta.$$

Acting the above inequality with $(I - A_2 - A_5)^{-1}$, we have $\theta \preceq Tx^* - x^* \preceq \theta$, i.e., $x^* = Tx^*$. □

Theorem 2.2. *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $y_0 \in E$ and $T : D = (-\infty, y_0] \to E$ a nondecreasing order-Ćirić-Lipschitz mapping restricted with linear bounded mappings $A_i : P \to P$ $(i = 1, 2, 3, 4, 5)$. Assume that $Ty_0 \preceq y_0$, T is Picard-complete at y_0 and*

$$r(A_3 + A_4) < 1, \ \ r((I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)) < 1. \tag{2.13}$$

Then T has a fixed point in $(-\infty, y_0]$.

Proof. Let $y_n = T^n y_0$ for each n. Note that T is nondecreasing on $(-\infty, y_0]$, then it follows from $Ty_0 \preceq y_0$ that

$$y_{n+1} \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0, \ \ \forall\, n. \tag{2.14}$$

Set $\overline{A} = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$, then $r(\overline{A}) < 1$ by (2.13). In analogy to (2.7), by (1.1) and (2.14), we obtain

$$\begin{aligned}
\theta \preceq y_n - y_m = \sum_{i=n}^{m-1} (y_i - y_{i+1}) &\preceq \sum_{i=n}^{m-1} \overline{A}^i (y_0 - y_1) \\
&= \overline{A}^n \sum_{i=0}^{m-1} \overline{A}^i (y_0 - y_1) \preceq \overline{A}^n (I - \overline{A})^{-1}(y_0 - y_1), \ \ \forall\, m > n.
\end{aligned} \tag{2.15}$$

Taking $B = \overline{A}$ and $u = (I - \overline{A})^{-1}(x_1 - x_0)$ in (2.3), it follows that

$$\overline{A}^n (I - \overline{A})^{-1}(x_1 - x_0) \xrightarrow{w} \theta \ (n \to \infty),$$

which together with (2.15) and Lemma 1.4 (iii) implies that

$$y_n - y_m \xrightarrow{w} \theta \ (m > n \to \infty). \tag{2.16}$$

Thus by the Picard-completeness of T at y_0, there exists $y^* \in E$ such that

$$y_n \xrightarrow{w} y^*. \tag{2.17}$$

In analogy to (2.9), (2.10), and (2), we obtain

$$Ty^* \preceq y^* \preceq y_n, \ n = 0, 1, 2, 3, \cdots . \tag{2.18}$$

By (1.1), (2.14), and (2.18),

$$\begin{aligned}\theta \preceq y_{n+1} - Ty^* &= Ty_n - Tx^* \\ &\preceq A_1(y_n - y^*) + A_2(y_n - y_{n+1}) + A_3(y^* - Ty^*) + A_4(y_n - Ty^*) + A_5(y_{n+1} - y^*), \quad \forall\, n.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by (2.16), (2.17), Lemma 1.4 (iii), and Lemma 1.10 we get

$$\theta \preceq y^* - Ty^* \preceq A_3(y^* - Ty^*) + A_4(y^* - Ty^*),$$

and so

$$(I - A_2 - A_5)(y^* - T^*y^*) \preceq \theta.$$

Acting the above inequality with $(I - A_2 - A_5)^{-1}$, we have $\theta \preceq y^* - Ty^* \preceq \theta$, i.e., $y^* = Ty^*$. □

Remark 2.3. Theorems 2.1 and 2.2 are still valid without assumption of the Picard-completeness of T at the expense that E is w-complete by Remark 1.8.

Proposition 2.4. *Let P be a cone of a normed vector space $(E, \|\cdot\|)$ and $A_i : P \to P$ $(i = 1, 2, 3, 4, 5)$ be linear bounded mappings. If one of the following conditions is satisfied:*

(H$_1$) $A_i (i = 1, 2, 3, 4, 5)$ *are mutually commutative (i.e., $A_iA_j = A_jA_i$ for each $1 \leqslant i, j \leqslant 5$), $r(A_4) = r(A_5)$ and $\sum_{i=1}^{5} r(A_i) < 1$;*

(H$_2$) $A_i (i = 1, 2, 3, 4, 5)$ *are mutually commutative, there exists $\varepsilon \geqslant 0$ such that $r(A_4) - r(A_5) > \varepsilon$ and $\sum_{i=1}^{5} r(A_i) = 1 + \varepsilon$,*

then $A_i (i = 1, 2, 3, 4, 5)$ satisfy the condition (2.1).

Proof. Suppose that (H$_1$) is satisfied. Since $A_i (i = 1, 2, 3, 4, 5)$ are mutually commutative, it follows from (H$_1$) that $r(A_2 + A_5) \leqslant r(A_2) + r(A_5) < 1$ and

$$\begin{aligned} r((I - A_2 - A_5)^{-1}(A_1 + A_3 + A_5)) \leqslant \frac{r(A_1 + A_3 + A_5)}{1 - r(A_2 + A_5)} &\leqslant \frac{r(A_1) + r(A_3) + r(A_5)}{1 - r(A_2) - r(A_5)} \\ &= \frac{r(A_1) + r(A_3) + r(A_4)}{1 - r(A_2) - r(A_5)} < 1, \end{aligned}$$

and hence (2.1) holds.

Suppose that (H$_2$) is satisfied. Since $A_i (i = 1, 2, 3, 4, 5)$ are mutually commutative, it follows from (H$_2$) that

$$r(A_2 + A_5) \leqslant r(A_2) + r(A_5) \leqslant \sum_{i=1}^{3} r(A_i) + 2r(A_5) < \sum_{i=1}^{3} r(A_i) + (r(A_4) - \varepsilon + r(A_5) \leqslant 1,$$

and

$$r((I - A_2 - A_5)^{-1}(A_1 + A_3 + A_5)) \leqslant \frac{r(A_1 + A_3 + A_5)}{1 - r(A_2 + A_5)} \leqslant \frac{r(A_1) + r(A_3) + r(A_5)}{1 - r(A_2) - r(A_5)} < 1,$$

and hence (2.1) holds. □

In analogy to Proposition 2.4, we have the following result.

Proposition 2.5. *Let* P *be a cone of a normed vector space* $(E, \|\cdot\|)$ *and* $A_i : P \to P$ $(i = 1, 2, 3, 4, 5)$ *be linear bounded mappings. If either* (H_1)*, or the following condition is satisfied:*

(H_3) A_i $(i = 1, 2, 3, 4, 5)$ *are mutually commutative, there exists* $\varepsilon \geqslant 0$ *such that* $r(A_5) - r(A_4) > \varepsilon$ *and* $\sum_{i=1}^{5} r(A_i) = 1 + \varepsilon$,

then A_i $(i = 1, 2, 3, 4, 5)$ *satisfy condition* (2.13).

By Remark 1.8, Propositions 2.4 and 2.5, we have the following two corollaries corresponding to Theorems 2.1 and 2.2.

Corollary 2.6. *Let* P *be a solid cone of a normed vector space* $(E, \|\cdot\|)$, $x_0 \in E$ *and* $T : D = [x_0, +\infty) \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with linear bounded mappings* $A_i : P \to P$ $(i = 1, 2, 3, 4, 5)$*. Assume that* $x_0 \preceq Tx_0$ *and* (H_2) *is satisfied. If* E *is* w*-complete, or* T *is Picard-complete at* x_0*, then* T *has a fixed point in* $[x_0, +\infty)$.

Corollary 2.7. *Let* P *be a solid cone of a* w*-complete normed vector space* $(E, \|\cdot\|)$, $y_0 \in E$ *and* $T : D = (-\infty, y_0] \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with linear bounded mappings* $A_i : P \to P$ $(i = 1, 2, 3, 4, 5)$*. Assume that* $Ty_0 \preceq y_0$ *and* (H_3) *is satisfied. If* E *is* w*-complete, or* T *is Picard-complete at* y_0*, then* T *has a fixed point in* $(-\infty, y_0]$.

In particular when A_i, B_i $(i = 1, 2, 3, 4, 5)$ are nonnegative real numbers, we have the following two fixed point results.

Corollary 2.8. *Let* P *be a solid cone of a normed vector space* $(E, \|\cdot\|)$, $x_0 \in E$ *and* $T : D = [x_0, +\infty) \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with nonnegative real numbers* $A_i \geqslant 0$ $(i = 1, 2, 3, 4, 5)$*. Assume that* $x_0 \preceq Tx_0$ *and the following condition is satisfied:*

(H_2') *there exists* $\varepsilon \geqslant 0$ *such that* $\sum_{i=1}^{5} A_i = 1 + \varepsilon$ *and* $A_4 - A_5 > \varepsilon$.

If E *is* w*-complete, or* T *is Picard-complete at* x_0*, then* T *has a fixed point in* $[x_0, +\infty)$.

Corollary 2.9. *Let* P *be a solid cone of a normed vector space* $(E, \|\cdot\|)$, $y_0 \in E$ *and* $T : D = (-\infty, y_0] \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with nonnegative real numbers* $A_i \geqslant 0$ $(i = 1, 2, 3, 4, 5)$*. Assume that* $Ty_0 \preceq y_0$ *and the following condition is satisfied:*

(H_3') *there exists* $\varepsilon \geqslant 0$ *such that* $\sum_{i=1}^{5} A_i = 1 + \varepsilon$ *and* $A_5 - A_4 > \varepsilon$.

If E *is* w*-complete, or* T *is Picard-complete at* y_0*, then* T *has a fixed point in* $(-\infty, y_0]$.

Remark 2.10. Corollaries 2.8 and 2.9 are still valid in Hausdorff topological vector spaces.

Remark 2.11. In Corollaries 2.6–2.9, the sum $\sum_{i=1}^{5} r(A_i)$ or $\sum_{i=1}^{5} A_i$ may be even greater than or equal to 1. Therefore, even in the setting of normal cones, Corollaries 2.6–2.9 are still new.

In particular when P is normal cone of a Banach space $(E, \|\cdot\|)$, we have the following two corollaries by Proposition 1.6 and Remark 1.8.

Corollary 2.12. *Let* P *be a normal cone of a Banach space* $(E, \|\cdot\|)$, $x_0 \in E$ *and* $T : D = [x_0, +\infty) \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with nonnegative real numbers* $A_i \geqslant 0$ $(i = 1, 2, 3, 4, 5)$*. Assume that* $x_0 \preceq Tx_0$ *and* (H_2') *is satisfied. Then* T *has a fixed point in* $[x_0, +\infty)$.

Corollary 2.13. *Let* P *be a normal cone of a Banach space* $(E, \|\cdot\|)$, $y_0 \in E$ *and* $T : D = (-\infty, y_0] \to E$ *a nondecreasing order-Ćirić-Lipschitz mapping restricted with nonnegative real numbers* $A_i \geqslant 0$ $(i = 1, 2, 3, 4, 5)$*. Assume that* $Ty_0 \preceq y_0$ *and* (H_3') *is satisfied. Then* T *has a fixed point in* $(-\infty, y_0]$.

Example 2.14. Let $E = \{1,2,3\}$ be endowed with the usual norm $|\cdot|$ and $P = \{x \in \mathbb{R} : x \geqslant 0\}$. Then $(E, |\cdot|)$ is a Banach space and P is a normal cone of E and the partial order $\preceq$ induced by P is the usual total order $\leqslant$.

Let $T1 = T2 = 1$ and $T3 = 2$, and set $x_0 = 1$. Clearly, $x_0 \preceq Tx_0$, $[x_0, +\infty) = E$ and $T : [x_0, +\infty) \to E$ is nondecreasing. Note that $T1 - T1 = T2 - T2 = T3 - T3 = T2 - T1 = 0$ and $T3 - T1 = T3 - T2 = 1$, then it suffices to verify that (1.1) is satisfied with $x = 3, y = 1$ and $x = 3, y = 2$. Set $B_i = 0$ $(i = 1,2,3,4,5)$, $A_1 = A_2 = A_3 = A_5 = \frac{1}{6}$ and $A_4 \geqslant \frac{1}{3}$. Clearly, $\sum_{i=1}^{5} A_i \geqslant 1$ but (H_2') is satisfied with $\varepsilon = A_4 - \frac{1}{3}$ since $A_4 - A_5 = A_4 - \frac{1}{6} > \varepsilon$. Direct calculation gives that

$$\begin{aligned}
0 < T3 - T1 = 1 &< \frac{4}{3} \leqslant \frac{2}{3} + 2A_4 = 2A_1 + A_2 + 2A_4 + A_5 \\
&= A_1(1-3)_+ + A_2(3 - T3)_+ + A_3(1 - T1)_+ + A_4(3 - T1)_+ + A_5(1 - T3)_+, \\
0 < T3 - T2 = 1 &< \frac{7}{6} \leqslant A_1 + A_2 + A_3 + 2A_4 \\
&= A_1(2-3)_+ + A_2(3 - T3)_+ + A_3(2 - T2)_+ + A_4(3 - T2)_+ + A_5(2 - T3)_+.
\end{aligned}$$

Thus T has a fixed point in E by Corollary 2.12 (in fact, $x = 1$ is a fixed point of T).

However, for each $A \in [0, 1)$ we get $T3 - T2 = 1 > A = A(3 - 2)$, and hence none of the results of [6, 8, 10, 11] concerned with order-Lipschitz mappings is applicable here. On the other hand, for each $q, r, s, t \geqslant 0$ with $q + r + s + 2t < 1$, we have

$$q|3-2| + r|3 - T3| + s|2 - T2| + t(|3 - T2| + |2 - T3|) = q + r + s + 2t < 1 = |T3 - T2|,$$

i.e., T is not a Ćirić's contraction and hence none of the results in [1–3, 7] is applicable here.

Now we come to consider the unique existence of fixed points of order-Ćirić-Lipschitz mappings.

Theorem 2.15. *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $x_0, y_0 \in E$ with $x_0 \preceq y_0$ and $T : D = [x_0, y_0] \to E$ a nondecreasing order-Ćirić-Lipschitz mapping restricted with linear bounded mappings $A_i : P \to P$ $(i = 1,2,3,4,5)$. Assume that $x_0 \preceq Tx_0, Ty_0 \preceq y_0$, T is Picard-complete at x_0 and y_0, and (H_1) is satisfied. Then T has a unique fixed point $x^* \in [x_0, y_0]$, and for each $z_0 \in [x_0, y_0]$, $z_n \xrightarrow{w} x^*$, where $z_n = O(T, z_0)$.*

Proof. The existence of fixed points immediately follows from Theorems 2.1 and 2.2 and Propositions 2.4 and 2.5. Note that T is nondecreasing on $[x_0, y_0]$, thus it follows from $x_0 \preceq Tx_0, Ty_0 \preceq y_0$ that

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0, \quad \forall\, n. \tag{2.19}$$

Letting $n \to \infty$ in (2.19), by (2.9), (2.17), and Lemma 1.4 (ii), we get

$$x^* \preceq y^*. \tag{2.20}$$

Thus by (1.1) and (2.19),

$$\begin{aligned}
\theta &\preceq y_{n+1} - x_{n+1} \\
&= Ty_n - Tx_n \\
&\preceq A_1(y_n - x_n) + A_2(y_n - y_{n+1}) + A_3(x_{n+1} - x_n) + A_4(y_n - x_{n+1}) + A_5(y_{n+1} - x_n), \quad \forall\, n.
\end{aligned} \tag{2.21}$$

Letting $n \to \infty$ in (2.21), by (2.8), (2.9), (2.17), (2.17), (2.20), Lemma 1.4 (iii), and Lemma 1.10, we get

$$\theta \preceq y^* - x^* \preceq A_1(y^* - x^*) + A_4(y^* - x^*) + A_5(y^* - x^*),$$

and so

$$(I - A_1 - A_4 - A_5)(y^* - x^*) \preceq \theta. \tag{2.22}$$

Note that $(I - A_1 - A_4 - A_5)^{-1} : P \to P$ is a linear bounded mapping by $\sum_{i=1}^{5} r(A_i) < 1$ and taking $B = A_1 + A_4 + A_5$ in (2.2). Acting (2.22) with $(I - A_1 - A_4 - A_5)^{-1}$, we obtain $y^* = x^*$.

For each $z_0 \in [x_0, y_0]$, we have

$$x_n \preceq z_n \preceq y_n,$$

which together with (2.9), (2.17), Lemma 1.4 (iii), and $x^* = y^*$ implies that $z_n \xrightarrow{w} x^*$.

Suppose that $x \in [x_0, y_0]$ is another fixed point of T. Clearly, $T^n x = x$ for each n and $T^n x \xrightarrow{w} x^*$. Then by Lemma 1.4 (i), $x = x^*$. This shows x^* is the unique fixed point of T in $[x_0, y_0]$. □

Remark 2.16. Theorem 1.2 is a particular case of Theorem 2.15 in Banach algebras with $A_1 \in P$ and $A_i = \theta$ $(i = 2, 3, 4, 5)$.

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