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Almost fixed point property for digital spaces associated with Marcus-Wyse topological spaces

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Abstract

The present paper studies almost fixed point property for digital spaces whose structures are induced by Marcus-Wyse (M-, for brevity) topology. In this paper we mainly deal with spaces X which are connected M-*topological spaces with* M-*adjacency* (MA-spaces or M-topological graphs for short) whose cardinalities are greater than 1. Let MAC be a category whose objects, denoted by Ob(MAC), are MA-spaces and morphisms are MA-maps between MA-spaces (for more details, see Section 3), and *MTC* a category of M-topological spaces as Ob(MTC) and M-continuous maps as morphisms of *MTC* (for more details, see Section 3). We prove that whereas any MA-space does not have the fixed point property (*FPP* for short) for any MA-maps, a bounded simple MA-path has the almost fixed point property (*AFPP* for short). Finally, we refer the topological invariant of the *FPP* for M-topological spaces from the viewpoint of MTC. ©2017 all rights reserved.

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1. Introduction

Let $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{Z}^n$ represent the sets of integers, natural numbers and points in the Euclidean nD space with integer coordinates, respectively. Digital topology focuses on studying digital topological properties of nD digital spaces [7, 26, 27], which has contributed to some areas of computer sciences such as computer graphics, image processing, mathematical morphology and so forth. In digital topology, although there are several approaches of studying digital images [1, 7, 11, 15, 16, 20, 22, 26, 29], the present paper follows an M-topological approach [13, 19, 29]. Since M-topology is considered in the Euclidean 2D space with integer coordinates, it plays an important role in digital topology and digital geometry. The present paper studies the almost fixed point property (*AFPP* for short) for spaces in *MAC* and further, refers the topological invariant of the fixed point property (*FPP* for brevity) for M-topological spaces from the viewpoint of MTC.

Since every singleton obviously has the *FPP*, in studying the *FPP* of spaces, all spaces X (resp. digital images (X, k)) are assumed to be connected (resp. k-connected) and $|X| \geqslant 2$.

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The well-known Lefschetz fixed point theorem [24] plays an important role in fixed point theory. Thus, it has been often used to study the *FPP* of a certain space. Since the ordinary Lefschetz fixed point theorem for a topological space X is formulated in terms of homology groups of X [24, 25], it is obvious that the theorem implies that a contractible topological space has the *FPP*.

Let (X, R) be a digital space (see Definition 2.1 of the current paper). Then we may pose the following query [15]:

$$\{\text{Is there any relation between its contractibility and the existence of the } FPP \text{ of } X\,?\} \tag{1.1}$$

In digital topology, it turns out that [12, 14] both the ordinary Lefschetz fixed point theorem and its digital version [5] cannot be helpful to study the issue of (1.1) [12] (for more details, [14, 15]). The recent paper [15] studied the issue of (1.1) from the viewpoint of Khalimsky topology. Thus it turns out that in Khalimsky topology not every Khalimsky topological space with Khalimsky contractibility has the *FPP*. Thus we need to study the question (1.1) by using only various properties of digital continuity (see Proposition 5.13). To study the issue of (1.1), let us recall basic notions and terminology on digital topology [7, 8, 23, 26, 27]. Since the present paper mainly studies the *AFPP* of digital spaces associated with Marcus-Wyse (for brevity M-) topology, let us further recall basic notions and terms of M-topology and its related areas. As mentioned in the paper [13], an M-continuous map is so rigid that some geometric transformations are not even M-continuous maps (see Remark 3.4 in the present paper). Hence the recent paper [13] established an MA-map which is broader than an M-continuous map and it is an M-connectedness preserving map.

In relation to the study of contractibility of spaces $X \in Ob(MAC)$ relevant to (1.1), the present paper will use the MA-homotopy developed in [19] to study MA-contractibility of X (see Section 4). Besides, the well-known T_0-Alexandroff topological structure of M-topology [1, 4] will be often used in the present paper.

Rosenfeld [27] was first come up with the fixed point theorem for digital images (X, k) in a graph theoretical approach (for more details, see [12, 14]). Indeed, this digital image (X, k) is one of the digital spaces because (X, k) is a kind of relation set (X, R) in such a way: for $x, y \in X$ with $x \neq y$ we say that $(x, y) \in R$ if and only if x and y are k-adjacent (for more details, see Section 2). Finally, the paper [27] proved that (see Theorems 3.3 and 4.1 of [27], for more details, see [12, 14])

$$\text{a digital image } (X, k) \text{ with } |X| \geqslant 2 \text{ does not have the } FPP. \tag{1.2}$$

This means that only a singleton has the *FPP* in digital topology in a graph theoretical approach followed by Rosenfeld model [27].

Owing to the *non-FPP* of digital images [27], Rosenfeld [27] studied the *AFPP* of digital images. Thus the present paper will mainly study the *AFPP* for digital spaces $X \in Ob(MAC)$ and further, refer the topological invariant of the *FPP* for M-topological spaces from the viewpoint of MTC.

The rest of the paper proceeds as follows: Section 2 provides some basic notions on digital topology. Section 3 recalls some properties of an MA-map and its various properties. Section 4 studies MA-contractibility and its properties. Section 5 proposes both the *AFPP* of a bounded MA-path and a relation between contractibility of an MA-space X and the existence of the *FPP* of X in MAC. Section 6 refers the topological invariant of the *FPP* for M-topological spaces from the viewpoint of MTC. Section 7 concludes the paper with a summary.

2. Preliminaries

The study of 2D digital spaces plays an important role in digital geometry related to mathematical morphology, computer graphics, image analysis, image processing and so forth. Thus let us now recall some basic facts and terminology on digital topology such as $\mathbf{Z}^2$ with k-adjacency relations, where $k \in \{4, 8\}$ and M-topology. First of all, we need to recall a digital space which was defined by Herman [20].

Definition 2.1 ([20]). A *digital space* is a relation set (X, R) where X is a nonempty set and R is a binary symmetric relation on X such that X is R-connected.

In Definition 2.1, we say that the set X is R-*connected* if for any two elements x and y of X there is a finite sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in R$ for $j \in [0, l-1]_{\mathbf{Z}}$. Besides, we should remind that the relation set (X, R) in Definition 2.1 need not be either a preordered set or a partially ordered set. In view of Definition 2.1, we see that not every topological spaces satisfying T_i-separation axiom is a digital space such as $i \in \{1, 2, 3, 4\}$. Besides, a relation set without any topological structure can be a digital space. For instance, owing to the Alexandroff topological structure of M-topology, an M-topological space X is a digital space because it has a digital space structure (X, R) in such a way: for any two elements $x, y \in X \in Ob(MTC)$ we say that $(x, y) \in R$ if $x \neq y$ and $x \in SN_M(y)$ or $y \in SN_M(x)$, where $SN_M(x)$ means the smallest open neighborhood of x in the M-topological space X (for more details, see Sections 5 and 6).

Before studying fixed point theory for digital spaces, first of all, we need to recall the *FPP* for digital spaces as follows:

Remark 2.2. We say that a digital space (X, R) has the *FPP* if every relation preserving self-map of (X, R) has at least a point $x \in X$ such that $f(x) = x$, where we say that a self-map f of (X, R) is a relation preserving map if for any $x, y \in X$ with $(x, y) \in R$ and $x \neq y$, $f(x) = f(y)$ or $(f(x), f(y)) \in R$.

In case a digital space (X, R) is a topological space, we say that (X, R) has the *FPP* if every continuous self-map of (X, R) has at least a point $x \in X$ such that $f(x) = x$, as usual.

To study 2D digital spaces, we have often used M-topology on $\mathbf{Z}^2$ [29], denoted by $(\mathbf{Z}^2, \gamma)$. For a set $X \subset \mathbf{Z}^2$ consider the subspace induced by $(\mathbf{Z}^2, \gamma)$ and denoted by (X, γ_X). Indeed, the study of (X, γ_X) is so related to that of (X, k) in $\mathbf{Z}^2, k \in \{4, 8\}$.

Meanwhile, in relation to the study of nD digital images in a graph theoretical approach, we have often used the k (or $k(m, n)$)-adjacency relations of $\mathbf{Z}^n$ as follows: for a natural number m with $1 \leqslant m \leqslant n$, two distinct points $p = (p_i)_{i\in[1,n]_{\mathbf{Z}}}$ and $q = (q_i)_{i\in[1,n]_{\mathbf{Z}}}$ in $\mathbf{Z}^n$ are called $k(m, n)$- (for short k-)adjacent if

$$\text{at most } m \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.}$$

Concretely, these $k(m, n)$-adjacency relations of $\mathbf{Z}^n$ are determined according to the two numbers $m, n \in \mathbf{N}$ [7] (see also [10, 11]).

Using the above operator, we can obtain the k-adjacency relations of $\mathbf{Z}^n$ [10] as follows:

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n,$$

where $C_i^n = \frac{n!}{(n-i)!\, i!}$.

Rosenfeld [26] called a set $X \subset \mathbf{Z}^n$ with a k-adjacency a *digital image* denoted by (X, k). Indeed, to follow a graph theoretical approach of studying 2D digital images, both the k-adjacency relations of $\mathbf{Z}^2$, where $k \in \{4, 8\}$ and a digital k-neighborhood have been often used. More precisely, using the k-adjacency of $\mathbf{Z}^2, k \in \{4, 8\}$, in the paper we say that a digital k-neighborhood of p in $\mathbf{Z}^2$ is the set [26]

$$N_k(p) := \{q \in \mathbf{Z}^2 | p \text{ is } k\text{-adjacent to } q\} \cup \{p\}.$$

Furthermore, we often use the notation [23]

$$N_k^*(p) := N_k(p) \setminus \{p\}, k \in \{4, 8\}.$$

For a k-adjacency relation of $\mathbf{Z}^2, k \in \{4, 8\}$, a simple k-path with $l + 1$ elements in $\mathbf{Z}^2$ is assumed to be an injective finite sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^2$ such that x_i and x_j are k-adjacent if and only if $|i - j| = 1$ [23]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^2$, denoted by $SC_k^{2,l}$ [6], is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i-j| = 1(\mathrm{mod}\, l)$ [23].

For a digital image (X, k), as a generalization of $N_k(p)$ [6] the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset [7] of X

$$N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \leqslant \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$. Concretely, for $X \subset \mathbf{Z}^n$ we obtain [9]

$$N_k(x, 1) = N_k(x) \cap X.$$

Let us now recall basic concepts on M-topology. The M-topology on $\mathbf{Z}^2$, denoted by $(\mathbf{Z}^2, \gamma)$, is induced by the set $\{SN_M(p)\}$ in (2.1) as a base [29], where for each point $p = (x, y) \in \mathbf{Z}^2$

$$SN_M(p) := \left\{ \begin{array}{l} N_4(p) \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ else.} \end{array} \right\} \tag{2.1}$$

In relation to the further statement of a point in $\mathbf{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [28].

In all subspaces of $(\mathbf{Z}^2, \gamma)$ of Figures 1, 2, 3, and 4 a black jumbo dot means an *even point* and further, the symbols $\Diamond$ and $\bullet$ mean a *double even point* and an *odd point*, respectively. In view of (2.1), we can obviously obtain the following: under $(\mathbf{Z}^2, \gamma)$ the singleton with either a double even point or an even point is a closed set. In addition, the singleton with an odd point is an open set.

3. Some properties of categories associated with the M-topological structure

This section studies several categories associated with a Rosenfeld's digital topological structure and the M-topological structure. To map every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [26] established the notion of digital continuity. Motivated by this approach, the digital continuity of maps between digital images was represented with the following version, which can be substantially used to study digital spaces X in $\mathbf{Z}^n$.

Proposition 3.1 ([7, 9])**.** *Let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^2, k_i \in \{4, 8\}$ and $i \in \{0, 1\}$. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 3.1, in case $k_0 = k_1$, the map f is called a k_0-continuous map.

Using this concept, we establish a digital topological category, denoted by *DTC*, consisting of two sets [7] (see also [13]):

- For any set $X \subset \mathbf{Z}^2$, the set of digital images (X, k) in $\mathbf{Z}^2, k \in \{4, 8\}$ as objects of *DTC*;
- For every ordered pair of objects (X, k_0) and (Y, k_1), the set of all (k_0, k_1)-continuous maps as morphisms of *DTC*, $k_i \in \{4, 8\}, i \in \{0, 1\}$. In *DTC*, in case $k_0 = k_1 := k$, we will particularly use the notation *DTC(k)* [13].

Since a 2D digital image (X, k) is viewed as a set $X \subset \mathbf{Z}^2$ with one of the k-adjacency relations, where $k \in \{4, 8\}$, in relation to the classification of 2D digital images, we use the term a (k_0, k_1)-isomorphism as in [8] (see also [18]) rather than a (k_0, k_1)-homeomorphism as in [2].

Definition 3.2 ([8, 17, 18])**.** For two digital images (X, k_0) and (Y, k_1) in $\mathbf{Z}^2$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.

In Definition 3.2, in case $k_0 = k_1$, we call it a k_0-*isomorphism* [9].

Let us now recall the M-topological category and an M-homeomorphism [16] as follows: Owing to the Alexandroff topological structure of M-topology, M-continuity of a map between M-topological spaces is defined as follows:

Definition 3.3 ([13, 29]). For two M-topological spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a function $f : X \to Y$ is said to be M-continuous at a point $x \in X$ if $f(SN_M(x)) \subset SN_M(f(x))$. Furthermore, we say that a map $f : X \to Y$ is M-continuous if it is M-continuous at every point $x \in X$.

Using M-continuous maps, we establish an M-*topological category*, denoted by *MTC*, consisting of two sets [13].

(1) For any set $X \subset \mathbf{Z}^2$ the set of spaces (X, γ_X) denoted by $Ob(MTC)$,
(2) for every ordered pair of objects (X, γ_X) and (Y, γ_Y), the set of all M-continuous maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms of *MTC*.

Besides, in MTC, for two spaces (X, γ_X) and (Y, γ_Y), we say that a map $f : X \to Y$ is an M-homeomorphism [29] if f is an M-continuous bijection and that $f^{-1} : Y \to X$ is M-continuous.

Although the concepts of both an M-continuous map and an M-homeomorphism play important roles in studying M-topological spaces, as referred in the paper [13] they are so rigid that we have some difficulty in proceeding some geometric transformations using them (see Remark 3.4 below).

Remark 3.4 ([13, 19]). Let us consider the space $(X := (x_i)_{i\in[0,7]_Z}, \gamma_X)$ in Figure 1. Consider the self-map $f : X \to X$ given by $f(x_i) = x_{i+1\,(mod\,8)}, i \in [0, 7]_Z$ which means just one click transformation of the given set X. Then we observe that the map f is not an M-continuous map.

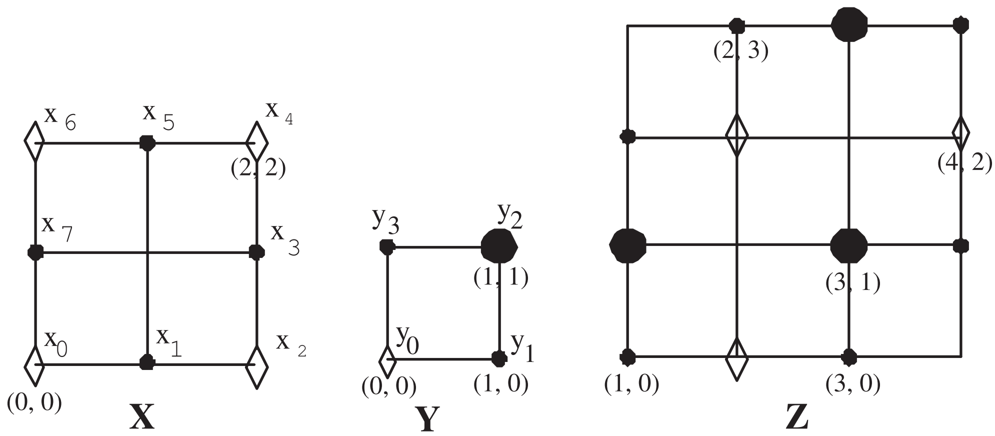

Figure 1: Some limitations of an M-continuous map [13] and an explanation of SC^l_M such as $Y := SC^4_M$ [13] and $Z := SC^{12}_M$ [19].

Let us now recall the following terminology which has been used to study M-topological spaces.

Definition 3.5 ([13, 16]). Let $(X, \gamma_X) := X$ be an M-topological space. Then we define the following:

(1) Two distinct points $x, y \in X$ are called M-*path connected* if there is a path $(x_i)_{i\in[0,m]_Z}$ on X with $\{x_0 = x, x_1, ..., x_m = y\}$ such that $\{x_i, x_{i+1}\}$ is M-connected, $i \in [0, m-1]_Z, m \geqslant 1$. Besides, the number m is called the *length* of this M-path. Furthermore, an M-path is called a closed M-curve if $x_0 = x_m$.
(2) A *simple* M-*path* in X is an M-path $(x_i)_{i\in[0,m]_Z}$ such that the set $\{x_i, x_j\}$ is M-connected if and only if $|i - j| = 1$.
Furthermore, we say that a simple closed M-curve with l elements $(x_i)_{i\in[0,l]_Z} \subset \mathbf{Z}^2$ and the number l is an even number, denoted by SC^l_M, is a simple M-path with $x_0 = x_l$ if and only if $|i - j| = 1(\mathrm{mod}\, l)$.

For instance, let us consider the spaces X, Y and Z in Figure 1. Then we see that [19] X, Y and Z are kinds of SC^8_M, SC^4_M and SC^{12}_M, respectively. Let us recall the notions of an M-adjacent relation for any two points in $\mathbf{Z}^2$ and an MA-neighborhood of a point $x \in \mathbf{Z}^2$ and further, we investigate their properties.

Definition 3.6 ([13])**.** In $(\mathbf{Z}^2, \gamma)$, we say that two distinct points x,y in $\mathbf{Z}^2$ are M-adjacent if $y \in SN_M(x)$ or $x \in SN_M(y)$, $SN_M(q)$ means the smallest open set containing the point $q \in \mathbf{Z}^2, q \in \{x, y\}$.

Definition 3.7 ([16])**.** We say that a space X is MA-connected (or M-connected) if for any two distinct points $x, y \in X$ there is an *M-path* in X connecting these two points.

According to the conditional logic, we see that a singleton set is an MA-connected set.

Remark 3.8 ([13]).

(1) Under $(\mathbf{Z}^2, \gamma)$ the notions of M-adjacency and M-connectedness are equivalent.
(2) Under $(\mathbf{Z}^2, \gamma)$ take a point $p \in \mathbf{Z}^2$. For any point $q \in N^*_4(p)$ the subspace $(\{p, q\} := X_1, \gamma_{X_1})$ is both M-connected and MA-connected.

To overcome the limitations referred in Remark 3.4, the paper [13] developed an *MA-map* which can be substantially used to study geometric transformations of M-topological spaces. Besides, for a point $p \in (\mathbf{Z}^2, \gamma)$ we define

$$MA(p) := \{q \in \mathbf{Z}^2 \,|\, p \text{ and } q \text{ are M-adjacent to each other}\}.$$

For a space $(X, \gamma_X) := X$ we now recall an *MA-relation* of a point $p \in X$ as follows.

Definition 3.9 ([13])**.** For $(X, \gamma_X) := X$ put $MA_X(p) := MA(p) \cap X$. We say that for two distinct points $p, q \in X$ they are M-adjacent to each other if $q \in MA_X(p)$ or $p \in MA_X(q)$.

In Definition 3.9 we say that the two points p, q have an MA-relation or p is MA-related to q. In view of Definition 3.9, we see that an MA-relation is an irreflexive symmetric relation [13]. The following MA-neighborhood of a point $p \in X$ is substantially used to establish an MA-map.

Definition 3.10 ([13])**.** For a space $(X, \gamma_X) := X$ and a point $p \in X$ we define an MA-neighborhood of p in X to be the set $MA_X(p) \cup \{p\} := MN_X(p)$.

Hereafter, in (X, γ_X) we use the notation $MN(p)$ instead of $MN_X(p)$ if there is no danger of ambiguity.

As referred in Remark 3.8 (1) and Definition 3.6, it is obvious that an M-topological space (X, γ_X) induces an M-adjacency on the space. Thus we may use the notation (X, γ_X) again for an *M-topological space* (X, γ_X) *with an M-adjacency* if there is no ambiguity. Hereafter, we call the space an *MA-space* for brevity.

For an M-topological space $(X, \gamma_X) := X$ and each point $x \in X$, owing to the Alexandroff topological structure of (X, γ_X), it is clear that each point $x \in X$ always has $MN(x) \subset X$ so that we now establish a map sending $MN(x)$ into $MN(f(x))$ as follows:

Definition 3.11 ([13])**.** For two MA-spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, we say that a function $f : X \to Y$ is an MA-map at a point $x \in X$ if

$$f(MN(x)) \subset MN(f(x)).$$

Furthermore, we say that a map $f : X \to Y$ is an MA-map if the map f is an MA-map at every point $x \in X$.

In view of Definition 3.11, we observe the following:

Remark 3.12.

(1) An M-continuous map implies an MA-map. But the converse does not hold [13].
(2) An MA-map is an M-connectedness preserving map [13].

(3) For a given bijective MA-map, its inverse map need not be an MA-map [19].

Using MA-maps, we establish an MA-*category* [13], denoted by *MAC*, consisting of two sets.

(1) For any set $X \subset \mathbf{Z}^2$, the set of MA-spaces (X, γ_X) as objects of *MAC*,
(2) for every ordered pair of MA-spaces (X, γ_X) and (Y, γ_Y), the set of all MA-maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms of *MAC*.

As referred in Remark 3.12 (3), since the inverse of an MA-map (resp. M-continuous map) does not need to be an MA-map (resp. M-continuous map), we need to establish the following notion.

Definition 3.13 ([13])**.** For two MA-spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a map $h : X \to Y$ is called an MA-isomorphism if h is a bijective MA-map (for short MA-bijection) and further, $h^{-1} : Y \to X$ is an MA-map.

In Definition 3.13, we denote by $X \approx_{MA} Y$ an MA-isomorphism from X to Y. In view of Remark 3.12, we obtain the following:

Remark 3.14. Both an MA-map and an MA-isomorphism are generalizations of an M-continuous map and an M-homeomorphism [13] so that these maps have strong merits of studying geometric transformations of M-topological spaces.

Definition 3.15 ([13])**.**

(1) Two distinct points $x, y \in X \in Ob(MAC)$ are called MA-*path connected* if there is a path $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ on X with $\{x_0 = x, x_1, ..., x_m = y\}$ such that $\{x_i, x_{i+1}\}$ is MA-connected, $i \in [0, m-1]_{\mathbf{Z}}, m \geqslant 1$. Besides, the number m is called the *length* of this MA-path. Furthermore, an MA-path is called a closed MA-curve if $x_0 = x_m$.
(2) A *simple* MA-*path* in X is the finite sequence $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ such that x_i and x_j are M-adjacent to each other if and only if $|i - j| = 1$.
(3) We say that a simple closed MA-curve with l elements $(x_i)_{i \in [0,l]_{\mathbf{Z}}}$ in $\mathbf{Z}^n$, denoted by SC^l_{MA}, is a simple MA-path with $x_0 = x_l$ (or MA-loop) and that x_i and x_j are M-adjacent if and only if $|i - j| = 1 (\mathrm{mod}\, l)$.

For instance, for the spaces X, Y and Z in Figure 1, we see that [19] X, Y and Z are sorts of SC^8_{MA}, SC^4_{MA} and SC^{12}_{MA}, respectively. Besides, we see that [13] $SC^{l_1}_{MA}$ is MA-isomorphic to $SC^{l_2}_{MA}$ if and only if $l_1 = l_2$. In view of Definition 3.15, we see that [19] for SC^l_{MA}, the number l is an even number such that $l \in \{2n | n \in \mathbf{N} \setminus \{1, 3\}\}$.

4. MA-contractibility and its properties in MAC

For an MA-space X let B be a subset of X. Then (X, B) is called a MA-space pair. Furthermore, if B is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed MA-space.

Remark 4.1. Hereafter, assume the set $\mathbf{Z}$ or $[a, b]_{\mathbf{Z}}$ as the subspace of $(\mathbf{Z}^2, \gamma)$. Thus each point p of these sets $\mathbf{Z}$ and $[a, b]_{\mathbf{Z}}$ has an MA-neighborhood $MN(p)$ which is equal to $N_2(p, 1)$. Owing to Remark 3.8, since MAC is equivalent to DTC(4), we have the following homotopy for MAC (see Definition 4.2). In particular, we need to refer the set $X \times [0, m]_{\mathbf{Z}}$ for the function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ in Definition 4.2. Indeed, at the moment the Cartesian product $X \times [0, m]_{\mathbf{Z}}$ does not have any topological structure related to M-topology because MAC deals with only 2 dimensional digital images. In other words, the set $X \times [0, m]_{\mathbf{Z}}$ can be considered to be a disjoint union such as $\cup_{i \in [0,m]_{\mathbf{Z}}} (X \times \{i\})$ like a labeled set indexed by $i \in [0, m]_{\mathbf{Z}}$.

Motivated by a pointed digital homotopy in [2] and a digital relative homotopy in [7, 9], we will establish the notions of an MA-homotopy relative to a subset $B \subset X$, MA-contractibility and an MA-homotopy equivalence, which will be used to study spaces in MAC.

Definition 4.2 ([19]). Let (X, B) and Y be an MA-space pair and an MA-space, respectively. Let $f, g : X \to Y$ be MA-maps. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

($\bullet$1) for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
($\bullet$2) for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is an MA-map;
($\bullet$3) for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is an MA-map.

Then we say that F is an MA-homotopy between f and g.

($\bullet$4) Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, assume that $F_t(x) = f(x) = g(x)$ for all $x \in B$.

Then we call F an MA-homotopy relative to B between f and g, and we say that f and g are MA-homotopic relative to B in Y, $f \simeq_{MArel.B} g$ in symbol.

In Definition 4.2, if $B = \{x_0\} \subset X$, then we say that F is a pointed MA-homotopy at $\{x_0\}$. When f and g are pointed MA-homotopic in Y, we use the notation $f \simeq_{MA} g$ and $f \in [g]$ which denotes the MA-homotopy class of g. If, for some $x_0 \in X$, 1_X is MA-homotopic to the constant map in the space $\{x_0\}$ relative to $\{x_0\}$, then we say that (X, x_0) is *pointed MA-contractible* (for brevity *MA-contractible* if there is no danger of ambiguity).

Let us investigate some properties of MA-contractibility.

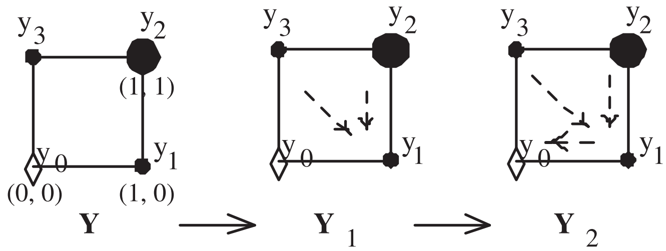

Figure 2: MA-contractibility of SC^4_{MA}.

Motivated by the notion of a digital homotopy equivalence [6, 17], we develop the following:

Definition 4.3 ([19]). For two MA-spaces (X, γ_X) and (Y, γ_Y) in MAC, if there are MA-maps $h : X \to Y$ and $l : Y \to X$ such that $l \circ h$ is MA-homotopic to 1_X and $h \circ l$ is MA-homotopic to 1_Y, then the map $h : X \to Y$ is called an MA-homotopy equivalence and denote it by $X \simeq_{MA \cdot h \cdot e} Y$.

If a space $X \in Ob(MAC)$ is MA-homotopy equivalent to a singleton in *MAC*, then we say that the space is MA-contractible.

By using the MA-homotopy $F : SC^4_{MA} \times [0, 2]_{\mathbf{Z}} \to SC^4_{MA}$ described in Figure 2, where $SC^4_{MA} := Y$, we obtain the following:

Lemma 4.4 ([19]). SC^4_{MA} *is MA-contractible.*

5. Almost fixed point property for digital spaces in MAC

This section studies a relation between MA-contractibility of an MA-space X and the existence of the *FPP* of X from the viewpoint of MAC. Hence this work addresses the issue (1.1) from the viewpoint of *MAC*.

Proposition 5.1. *Let $X \in Ob(MAC)$ be an MA-connected space. Then we obtain the followings:*

(1) (X, R) *is a digital space, where the relation* R *is an* M*-adjacency of* X.
(2) *An* MA*-map as a morphism of MAC is an* M*-adjacency preserving map.*

Proof.

(1) Consider a relation R on $X \in Ob(MAC)$ in such a way: for two distinct points $x, y \in X$ we say that $(x, y) \in R$ if and only if x and y are M-adjacent. Then we see that the relation set (X, R) is a binary symmetric relation. Besides, since X is MA-connected, the relation set (X, R) is R-connected.
(2) Consider an MA-map between two MA-spaces. According to Definition 3.11, the proof is completed.

□

Let us now study some properties of MA-spaces from the viewpoint of fixed point theory.

Theorem 5.2. *Any space* $X \in Ob(MAC)$ *does not have the FPP for any MA-maps in MAC, where* X *is MA-connected and* $|X| \geqslant 2$.

Proof. Owing to the hypothesis, we can take two distinct points x, y in X such that $y \in SN_M(x)$. Then it is obvious that $SN_M(y)$ is the singleton $\{y\}$ and $|SN_M(x)| \geqslant 2$. Let us consider the self-map of X given by

$$\left\{ \begin{array}{l} f(z) = x, \ z \in X, z \neq x \text{ and} \\ f(x) = y. \end{array} \right\} \tag{5.1}$$

Then it is obvious that the map f in (5.1) is an MA-map which does not have any fixed point on X. □

In (5.1), we need to point out that the map f cannot be an M-continuous map, which will be used in Section 6.

Example 5.3. Consider the spaces X and Y in Figure 3 from the viewpoint of *MAC*.

(1) Consider the self-map f of X such that $f(x_0) = x_1$ and $f(x_1) = x_0$. Then it is obvious that f is an MA-map which does not have any fixed point.
(2) Consider the self-map g of Y such that $g(\{y_1, y_2\}) = \{y_0\}$ and $f(y_0) = y_1$ or y_2. Then it is obvious that g is an MA-map which does not have any fixed point.

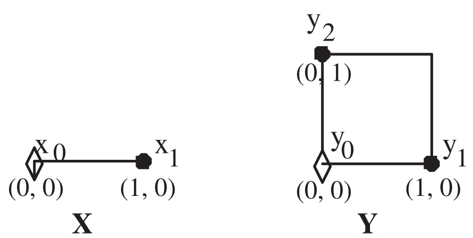

Figure 3: Explanation of the non-*FPP* of the given MA-spaces (X, γ_X) and (Y, γ_Y) in *MAC*.

In *MAC*, we say that an MA-*isomorphic invariant* is a property of an MA-space which is invariable under MA-isomorphism. Motivated by the paper [15], we obtain the following:

Proposition 5.4. *In MAC, the FPP is an MA-isomorphic invariant.*

Proof. By the hypothesis, assume that an MA-space X has the *FPP* from the viewpoint of *MAC* and there exists an MA-isomorphism $h : X \to Y$. Then we prove that the MA-space Y has the *FPP* in *MAC*. To do this work, let g be any self-MA-map of Y in *MAC*. Then consider the composition $h \circ f \circ h^{-1} := g : Y \to Y$, where f is a self-MA-map of X. Owing to the hypothesis, assume that $x \in X$ is a fixed point for the self-MA-map f of X. Since h is an MA-isomorphism, there is a point $y \in Y$ such that $h(x) = y$. Let us consider the mapping

$$f(x) = h^{-1} \circ g \circ h(x) = h^{-1}(g(h(x))) = h^{-1}(g(y)). \tag{5.2}$$

From (5.2) we see $h(f(x)) = g(y)$ and further, owing to the hypothesis of the *FPP* of X and the MA-isomorphism between X and Y, we obtain

$$h(f(x)) = h(x) = y = g(y), \tag{5.3}$$

which implies that the point $h(x)$ is a fixed point of the MA-map g, which implies that Y also has the *FPP*. □

Remark 5.5. According to Proposition 5.4, although the *FPP* is an MA-isomorphic invariant, its utility is very limited because of Theorem 5.2.

Owing to the property (1.2), Rosenfeld [27] studied the "almost fixed point property" (for short *AFPP*) [27]. Before studying this issue, we need to recall the notion of the *AFPP* in *DTC*. We say that [27]

$$\text{a digital image } (X, k) \text{ in } \mathbf{Z}^n \text{ has the } AFPP$$

if for every k-continuous map $f : (X, k) \to (X, k)$ there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x.

Then Rosenfeld [27] studied the *AFPP* of a digital image (X, k) for k-continuous self-maps f of (X, k). Finally, it turns out that not every digital image (X, k) for k-continuous self-maps f of (X, k) satisfies the *AFPP* [27], as follows:

Example 5.6. Consider the map $f : (\mathbf{Z}, 2) \to (\mathbf{Z}, 2)$ given by $f(i) = i + t, |t| \geqslant 2, t \in \mathbf{Z}$. While it is a 2-continuous map, $(\mathbf{Z}, 2)$ does not have the *AFPP*.

To study the *AFPP* in *MAC*, we need to introduce the notion of the *AFPP* from the viewpoint of *MAC*.

Definition 5.7. We say that an MA-space X has the *AFPP* if for every self-MA-map f of X there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is M-adjacent to x.

Definition 5.8. We say that an MA-connected space X is bounded if any MA-path in X has a finite length.

As proved in Theorem 5.2, while an MA-space X with $|X| \geqslant 2$ does not have the *FPP*, unlike Example 5.6, we have the following *AFPP* of a bounded (or finite) simple MA-path.

Theorem 5.9. *In MAC, a bounded simple MA-path has the AFPP.*

Proof. Let (X, γ_X) be an MA-path, denoted by $X := (x_i)_{i \in [0,l]_Z}$ in $\mathbf{Z}^2$. Further consider any self-MA-maps f of (X, γ_X). Then it is obvious $|f(X)| \leqslant |X|$, where $|\cdot|$ means the cardinality of the given set. Let us now use the notation for brevity

$$\max\{|f(X)| : f \text{ is any self MA-map of } X\} := \max\{|f(X)|\}.$$

Based on the number l of $X := (x_i)_{i \in [0,l]_Z}$, we apply the principle of mathematical induction (the induction for short) for the proof of this assertion. Without loss of generality, we may consider the following two cases.

(Case 1) Assume that the point x_0 is an double even point or an even point.

By the mathematical induction, in case $l = 1$, for any self-MA-maps f of X, we see $\max\{|f(X)|\} \leqslant 2$. Then it is obvious that f has an almost fixed point (see the MA-spaces (X, γ_X) in Figure 3).

In case $l = n$, consider any self-MA-maps f of X. Then we see $\max\{|f(X)|\} \leqslant n + 1$. Further assume that f has an almost fixed point.

Let us now prove the case $l = n+1$. Thus let us further consider any self-MA-maps f of $X := (x_i)_{i\in[0,n+1]_Z}$ relevant to the mapping of the element x_{n+1} by the map f. Then we see $max\{|f(X)|\} \leqslant n+2$. Now we can consider the following four cases.

First, in case $f(x_{n+1}) = x_{n+1}$, the proof is completed.

Second, in case $f(x_{n+1}) = x_n$, regardless of the MA-mapping f of the other points $x_i \in X$ with $x_i \neq x_{n+1}$, the proof is completed because the point x_{n+1} is at least an almost fixed point.

Third, in case $f(x_{n+1}) = x_i, i \in [1, n-1]_Z$, the MA-maps f have their images $f(X)$ such that $max\{|f(X)|\} \leqslant n+1$ regardless of the mapping $f(x_j), j \in [0, n]_Z$ including the case $f(x_0) = x_{n+1}$ because

$$|MN(x_{n+1})| = 2 \text{ and } |MN(x_i)| = 3,$$

and further the properties of the MA-map f, which completes the proof.

Fourth, in case $f(x_{n+1}) = x_0$, we have the following two cases:

$$max|f(X)| \leqslant n+1 \text{ or } |f(X)| = n+2.$$

The former completes the proof by the induction, the latter implies that the MA-map f should be the case $f(x_i) = x_{n+1-i}, i \in [0, n+1]_Z$, we obtain at least an almost fixed point in X. More precisely, if n is even, then both $x_{\frac{n}{2}+1}$ and $x_{\frac{n}{2}}$ are almost fixed points which are the approximate intermediate points of $(x_i)_{i\in[0,n+1]_Z}$, if n is odd, then $x_{\frac{n+1}{2}}$ is a fixed point.

(Case 2) Assume that the point x_0 is an odd point. Then the proof is similar to that of Case 1. □

Motivated by Proposition 5.4 and the paper [15], we obtain the following:

Proposition 5.10. *In MAC, the AFPP is an* MA*-isomorphic invariant.*

Proof. Suppose that (X, γ_X) in $Ob(MAC)$ has the *AFPP* and there exists an MA-isomorphism $h : (X, \gamma_X) \to (Y, \gamma_Y)$. Then we prove that (Y, γ_Y) has the *AFPP*. To do this work, let g be any self-MA-map of (Y, γ_Y). Then consider the composition $h \circ f \circ h^{-1} := g : (Y, \gamma_Y) \to (Y, \gamma_Y)$, where f is a self-MA-map of (X, γ_X). Owing to the hypothesis, assume that there is $x \in X$ such that $f(x) = x$ or $f(x) = y$ where y is M-adjacent to x.

In case $f(x) = x$, by using the methods similar to those of (5.2) and (5.3), we see that the point $h(x) \in Y$ has the property $g(h(x)) = h(x)$.

In case $f(x) = y$, where y is M-adjacent to x, by using the methods similar to those of (5.2) and (5.3), we obtain that the point $h(x) \in Y$ has the property that $g(h(x))$ is M-adjacent to $h(x)$.

Therefore, we see that the point $h(x)$ is an almost fixed point of the map g, which implies that (Y, γ_Y) has the *AFPP*. □

To study the *AFPP* of X in *MAC*, we will use an MA-retract in [13] (see Definition 5.11 below).

Definition 5.11 ([13])**.** In *MAC*, we say that an MA-map $r : (X', \gamma_{X'}) \to (X, \gamma_X)$ is an MA-retraction if
(1) (X, γ_X) is a subspace of $(X', \gamma_{X'})$ and
(2) $r(a) = a$ for all $a \in (X, \gamma_X)$.

Then we say that (X, γ_X) is an MA-retract of $(X', \gamma_{X'})$. Furthermore, we say that the point $a \in X' \setminus X$ is MA-retractable.

In view of Definition 5.11, it is clear that an MA-retract holds the reflexivity and the transitivity.

Lemma 5.12. *If* $X \in Ob(MAC)$ *has the AFPP, then its* MA*-retract* $A \in Ob(MAC)$ *also has the AFPP.*

Proof. Let $r : (X, \gamma_X) \to (A, \gamma_A)$ be an MA-retraction and $i : (A, \gamma_A) \to (X, \gamma_X)$ is the inclusion map. Then, consider any self-MA-map f of (A, γ_A). By the hypothesis and the above MA-retraction r, we obtain $r \circ i = 1_{(A,\gamma_A)}$. Since we have the composition $i \circ f \circ r$ as a self-MA-map having an almost fixed point $p \in X$, the point $r(p)$ is proved an almost fixed point of (A, γ_A). □

Proposition 5.13. *Not every MA-contractible space in* $\mathrm{Ob}(MAC)$ *has the AFPP.*

Proof. By Lemma 4.4, while $SC^4_{MA} := (x_i)_{i\in[0,3]_Z}$ is MA-contractible, we have the following MA-map f of SC^4_{MA}: $f(x_i) = x_{i+t(\mathrm{mod}\, l)}, t \in [2,3]_Z$. Then we see that the map f does not have the *AFPP*. □

6. An M-topological invariant of the *FPP*

This section studies the *FPP* and the *AFPP* of M-topological spaces from the viewpoint of *MTC*. Comparing these works with those of digital spaces in MAC, we need to point out that we have some difficulties in establishing a homotopy for M-topological spaces which can be used to study the *FPP* of M-topological spaces. Thus we cannot study the issue (1.1) by using the notion of M-contractibility. Hence we study both the *FPP* and the *AFPP* of M-topological spaces in terms of only M-continuous maps.

Proposition 6.1.

(1) *Let* $X \in \mathrm{Ob}(MTC)$ *be an M-connected space. Then* (X, γ_X) *is a digital space.*
(2) *An M-continuous map is a relation preserving map.*

Proof.

(1) Consider two distinct points $x, y \in X$. Then the M-connectedness of these two points in *MTC* is equivalent to the M-adjacency of them in *MAC*. Thus consider a relation R on X in such a way: for two distinct points $x, y \in X$ we say that $(x, y) \in R$ if and only if $x \in SN_M(y)$ or $y \in SN_M(x)$. Then we see that the relation set (X, R) is a binary symmetric relation. Besides, since X is M-connected, the set (X, R) is R-connected.
(2) Consider an M-continuous map between two M-topological spaces. According to Definition 3.3, the proof is completed.

□

Unlike Theorem 5.2, the *FPP* in *MTC* has its own feature, as follows:

Theorem 6.2. *Not every space* $X \in Ob(MTC)$ *has the FPP, where* X *is M-connected and* $|X| \geqslant 2$.

Proof. By using the following three examples, we prove the assertion.

(Case 1) Let us consider the M-topological space (I, γ_I) with $I := \{x_0 := (0,0), x_1 := (0,1)\}$. Then we see that the space (I, γ_I) has the *FPP* from the viewpoint of *MTC*. To be specific, we see that there are only three M-continuous maps among four self-maps of (I, γ_I). To be specific, consider the following self-maps f_i of (I, γ_I) in such a way, $i \in \{1, 2, 3\}$:

$$f_1 := 1_I, f_2(x_i) = x_0, i \in \{0,1\} \text{ and } f_3(x_i) = x_1, i \in \{0,1\}.$$

Then we see that these three maps are all M-continuous. Furthermore, it is obvious that each of these three M-continuous maps has a fixed point. At the moment we need to point out that the self-map of (I, γ_I) such that $f(x_0) = x_1, f(x_1) = x_0$ is not an M-continuous map.

(Case 2) Let us consider the M-topological space (X, γ_X) with $X := \{x_0 := (0,0), x_1 := (1,0), x_2 := (0,1)\}$ in Figure 4. Then we see that the space X has the *FPP* in *MTC*. To be specific, we see that there are only seven M-continuous maps among nine self-maps of (X, γ_X). Then it is obvious that each of these seven M-continuous maps has a fixed point. Indeed, among the nine self-maps of (X, γ_X), only these two self-mappings

$$x_0 \to x_1 \to x_2 \to x_0 \text{ and } x_0 \to x_2 \to x_1 \to x_0$$

cannot be M-continuous self-maps of (X, γ_X).

(Case 3) Let us consider the M-topological space (Y, γ_Y) with $Y := \{y_0 := (0,0), y_1 := (1,0), y_2 := (1,1), y_3 := (0,1)\}$ in Figure 4. Then we see that the space Y does not have the *FPP* in *MTC*. To be specific, consider the self-map h of Y such that

$$h(\{y_0\}) = \{y_2\}, h(\{y_2\}) = \{y_0\}, h(\{y_1\}) = \{y_3\} \text{ and } h(\{y_3\}) = \{y_1\}.$$

Then it is obvious that h is an M-continuous map which does not have any fixed point.

In view of these three cases above, the proof is completed. □

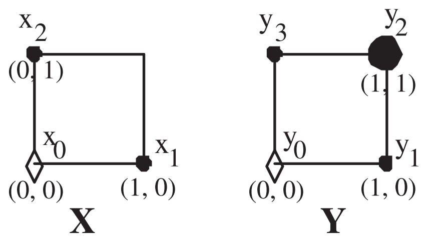

Figure 4: Explanation of the *FPP* or the non-*FPP* of the given spaces in *MTC*: (X, γ_X) has the *FPP* and (Y, γ_Y) has the non-*FPP*.

Example 6.3. Consider the map $f : SC_M^l \to SC_M^l$ given by $f(x_i) = x_{i+2(\bmod l)}$. While it is an M-continuous map, SC_M^l does not have the *FPP*.

Let us now study some properties of M-topological spaces from the viewpoint of fixed point theory. In *MTC*, we say that an M-homeomorphic invariant is a property of an M-topological space which is invariable under M-homeomorphism. Then, by using the method similar to Proposition 5.4 and motivated by [3], we obtain the following:

Proposition 6.4. *The FPP from the viewpoint of MTC is an M-homeomorphic invariant.*

Proof. Suppose that (X, γ_X) has the *FPP* and there exists an M-homeomorphism $h : (X, \gamma_X) \to (Y, \gamma_Y)$. Then we prove that (Y, γ_Y) has the *FPP*. Assume that g is any M-continuous self-map of (Y, γ_Y). Then consider the composition $h \circ f \circ h^{-1} := g : (Y, \gamma_Y) \to (Y, \gamma_Y)$, where f is an M-continuous self-map of (X, γ_X). Owing to the hypothesis, assume that $x \in X$ is a fixed point for an M-continuous self-map f of (X, γ_X). Since h is an M-homeomorphism, there is a point $y \in Y$ such that $h(x) = y$. Let us consider the mapping

$$f(x) = h^{-1} \circ g \circ h(x) = h^{-1}(g(h(x))) = h^{-1}(g(y)). \tag{6.1}$$

Thus, owing to the property of (6.1), we see $h(f(x)) = g(y)$ and further, owing to the hypothesis of the *FPP* of (X, γ_X) and the M-homeomorphism between (X, γ_X) and (Y, γ_Y),

$$h(f(x)) = h(x) = y = g(y),$$

which implies that the point $h(x)$ is a fixed point of the map g, which implies that (Y, γ_Y) has the *FPP*. □

7. Concluding Remarks

Motivated by the study of the *FPP* and the *AFPP* in *DTC* in [27] and the *FPP* for Khalimsky topological spaces [15], we have studied the *FPP* and the *AFPP* from the viewpoint of *MAC* or *MTC*. It turns out that these properties are quite different from those of *DTC*. As a further work, we can study fixed point theory for another digital spaces.

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References

[1] P. Alexandroff, *Diskrete Räume*, Mat. Sb., **2** (1937), 501–518. 1, 1
[2] L. Boxer, *A classical construction for the digital fundamental group*, J. Math. Imaging Vision, **10** (1999), 51–62. 3, 4
[3] R. F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Co., Glenview, Ill.-London, (1971). 6
[4] V. A. Chatyrko, S.-E. Han, Y. Hattori, *Some remarks concerning semi-*$T_{\frac{1}{2}}$ *spaces*, Filomat, **28** (2014), 21–25. 1
[5] O. Ege, I. Karaca, *Lefschetz fixed point theorem for digital images*, Fixed Point Theorey Appl., **2013** (2013), 13 pages. 1
[6] S.-E. Han, *On the classication of the digital images up to a digital homotopy equivalence*, J. Comput. Commun. Res., **10** (2000), 194–207. 2, 4
[7] S.-E. Han, *Non-product property of the digital fundamental group*, Inform. Sci., **171** (2005), 73–91. 1, 1, 2, 3.1, 3, 4
[8] S.-E. Han, *On the simplicial complex stemmed from a digital graph*, Honam Math. J., **27** (2005), 115–129. 1, 3, 3.2
[9] S.-E. Han, *Equivalent* (k_0, k_1)*-covering and generalized digital lifting*, Inform. Sci., **178** (2008), 550–561. 2, 3.1, 3, 4
[10] S.-E. Han, *The* k*-homotopic thinning and a torus-like digital image in* Z^n, J. Math. Imaging Vision, **31** (2008), 1–16. 2
[11] S.-E. Han, *KD-*(k_0, k_1)*-homotopy equivalence and its applications*, J. Korean Math. Soc., **47** (2010), 1031–1054. 1, 2
[12] S.-E. Han, *Fixed point theorems for digital images*, Honam Math. J., **37** (2015), 595–608. 1
[13] S.-E. Han, *Generalizations of continuity of maps and homeomorphisms for studying 2D digital topological spaces and their applications*, Topology Appl., **196** (2015), 468–482. 1, 1, 3, 3.3, 3, 3.4, 1, 3.5, 3.6, 3.8, 3, 3.9, 3, 3.10, 3.11, 3.12, 3, 3.13, 3.14, 3.15, 3, 5, 5.11
[14] S.-E. Han, *Banach fixed point theorem from the viewpoint of digital topology*, J. Nonlinear Sci. Appl., **9** (2016), 895–905. 1
[15] S.-E. Han, *Contractibility and fixed point property: the case of Khalimsky topological spaces*, Fixed Point Theory Appl., **2016** (2016), 20 pages. 1, 1, 5, 5, 7
[16] S.-E. Han, W.-J. Chun, *Classification of spaces in terms of both a digitization and a Marcus Wyse topological structure*, Honam Math. J., **33** (2011), 575–589. 1, 3, 3.5, 3.7
[17] S.-E. Han, B. G. Park, *Digital graph* (k_0, k_1)*-homotopy equivalence and its applications*, `http://atlas-conferences.com/c/a/k/b/35.htm`, (2003). 3.2, 4
[18] S.-E. Han, B. G. Park, *Digital graph* (k_0, k_1)*-isomorphism and its applications*, `http://atlas-conferences.com/c/a/k/b/36.htm`, (2003). 3, 3.2
[19] S.-E. Han, W. Yao, *Homotopy based on Marcus-Wyse topology and its applications*, Topology Appl., **201** (2016), 358–371. 1, 1, 3.4, 1, 3, 3.12, 3, 4.2, 4.3, 4.4
[20] G. T. Herman, *Oriented surfaces in digital spaces*, CVGIP: Graph. Models Image Proc., **55** (1993), 381–396. 1, 2, 2.1
[21] E. Khalimsky, *Motion, deformation and homotopy in finite spaces*, Proceedings of the IEEE International Conference on Systems, Man and Cybernetics, (1987), 227–234.
[22] E. Khalimsky, R. Kopperman, P. R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology Appl., **36** (1990), 1–17. 1
[23] T. Y. Kong, A. Rosenfeld, *Topological algorithms for digital image processing*, Elsevier Science, Amsterdam, (1996). 1, 2
[24] S. Lefschetz, *Topology*, Amer. Math. Soc., New York, (1930). 1
[25] S. Lefschetz, *On the fixed point formula*, Ann. of Math., **38** (1937), 819–822. 1
[26] A. Rosenfeld, *Digital topology*, Amer. Math. Monthly, **86** (1979), 621–630. 1, 1, 2, 3
[27] A. Rosenfeld, *Continuous functions on digital pictures*, Pattern Recogn. Lett., **4** (1986), 177–184. 1, 1, 1, 5, 5, 7
[28] J. Šlapal, *Topological structuring of the digital plane*, Discrete Math. Theor. Comput. Sci., **15** (2013), 165–176. 2
[29] F. Wyse, D. Marcus, *Solution to problem 5712*, Amer. Math. Monthly, **77** (1970), 1119. 1, 2, 3.3, 3