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Quadratic ρ-functional inequalities in β-homogeneous normed spaces

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Abstract

In this paper, we solve the quadratic ρ-functional inequalities

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leqslant \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\|, \tag{1}$$

where ρ is a fixed complex number with $|\rho|<1$, and

$$\left\|4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right\| \leqslant \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \tag{2}$$

where ρ is a fixed complex number with $|\rho|<1$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (1) and (2) in β-homogeneous complex Banach spaces.

Keywords: Hyers-Ulam stability, β-homogeneous space, quadratic ρ-functional inequality.
2010 MSC: 39B62, 39B72, 39B52, 39B82.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [4] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

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The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [15] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [5–8, 11–14]).

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;

(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;

(FN_3) $\|x+y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$;

(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;

(FN_5) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [10]). A β-homogeneous F-space is called a β-*homogeneous complex Banach space*.

In Section 2, we solve the quadratic ρ-functional inequality (1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (1) in β_2-homogeneous complex Banach space.

In Section 3, we solve the quadratic ρ-functional inequality (2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (2) in β_2-homogeneous complex Banach space.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leqslant 1$ and $\beta_2 \leqslant 1$. Assume that X is a β_1-homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with norm $\|\cdot\|$.

2. Quadratic ρ-functional inequality (1) in β-homogeneous complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the quadratic ρ-functional inequality (1) in complex normed spaces.

Lemma 2.1. *If a mapping $f : G \to Y$ satisfies*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leqslant \left\| \rho \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right) \right\|, \tag{2.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leqslant |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leqslant 0$ and so $f(2x) = 4f(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x), \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y)+f(x-y)-2f(x)-2f(y)\| &\leqslant \left\| \rho \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right) \right\| \\ &= |\rho| \|f(x+y)+f(x-y)-2f(x)-2f(y)\| \end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$

for all $x, y \in G$. $\square$

Now, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (2.1) in β-homogeneous complex Banach spaces.

Theorem 2.2. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leqslant &\left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\| \\ &+\theta(\|x\|^r+\|y\|^r),\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leqslant \frac{2\theta}{2^{\beta_1 r}-4^{\beta_2}}\|x\|^r, \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$\|f(2x)-4f(x)\| \leqslant 2\theta\|x\|^r, \tag{2.5}$$

for all $x \in X$. So

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\| \leqslant \frac{2\theta}{2^{\beta_1 r}}\|x\|^r,$$

for all $x \in X$. Hence

$$\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| \leqslant \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right)-4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leqslant \frac{2}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r, \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right),$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned}\|Q\left(x+y\right)+Q\left(x-y\right)-2Q(x)-2Q(y)\| &= \lim_{n\to\infty}\left\|4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\leqslant \lim_{n\to\infty}\left\|4^n\rho\left(4f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\quad+\lim_{n\to\infty}\frac{4^{\beta_2 n}}{2^{\beta_1 r n}}\theta(\|x\|^r+\|y\|^r) \\ &= \left\|\rho\left(4Q\left(\frac{x+y}{2}\right)+Q\left(x-y\right)-2Q(x)-2Q(y)\right)\right\|,\end{aligned}$$

for all $x, y \in X$. So

$$\left\|Q\left(\frac{x+y}{2}\right)+Q\left(\frac{x-y}{2}\right)-2Q(x)-2Q(y)\right\| \leqslant \left\|\rho\left(4Q\left(\frac{x+y}{2}\right)+Q(x-y)-2Q(x)-2Q(y)\right)\right\|,$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned}\|Q(x)-T(x)\| = \left\|4^q Q\left(\frac{x}{2^q}\right)-4^q T\left(\frac{x}{2^q}\right)\right\| &\leqslant \left\|4^q Q\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leqslant \frac{2\theta}{2^{\beta_1 r}-4^{\beta_2}}\frac{4^{\beta_2 q}}{2^{\beta_1 q r}}\|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q, as desired. □

Theorem 2.3. *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \leqslant \frac{2\theta}{4^{\beta_2}-2^{\beta_1 r}}\|x\|^r, \tag{2.7}$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\| \leqslant \frac{2\theta}{4^{\beta_2}}\|x\|^r,$$

for all $x \in X$. Hence

$$\left\|\frac{1}{4^l}f(2^l x)-\frac{1}{4^m}f(2^m x)\right\| \leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right)-\frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \leqslant \frac{2\theta}{4^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r}}{4^{\beta_2 j}}\|x\|^r, \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x),$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of proof is similar to the proof of Theorem 2.2. □

Remark 2.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β-homogeneous real Banach space, then all the assertions in this section remain valid.

3. Quadratic ρ-functional inequality (2) in β-homogeneous complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the quadratic ρ-functional inequality (2) in β-homogeneous complex normed spaces.

Lemma 3.1. *If a mapping $f : G \to Y$ satisfies*

$$\left\|4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right\| \leqslant \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \tag{3.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leqslant |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leqslant 0$ and so

$$4f\left(\frac{x}{2}\right) = f(x), \tag{3.2}$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &= \left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\leqslant |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in β-homogeneous complex Banach spaces.

Theorem 3.2. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}\left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| &\leqslant \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \\ &\quad + \theta(\|x\|^r + \|y\|^r),\end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 4^{\beta_2}}\|x\|^r, \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leqslant \theta\|x\|^r, \tag{3.5}$$

for all $x \in X$. So

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leqslant \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leqslant \sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r, \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right),$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let* $r < \frac{2\beta_2}{\beta_1}$ *and* θ *be nonnegative real numbers and let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and* (3.3)*. Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leqslant \frac{2^{\beta_1 r}\theta}{4^{\beta_2} - 2^{\beta_1 r}}\|x\|^r, \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leqslant \frac{2^{\beta_1 r}}{4^{\beta_2}}\theta\|x\|^r,$$

for all $x \in X$. Hence

$$\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \leqslant \sum_{j=l+1}^{m}\frac{2^{\beta_1 rj}}{4^{\beta_2 j}}\theta\|x\|^r, \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is convergent. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x),$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of proof is similar to the proof of Theorem 2.2. □

Remark 3.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β-homogeneous real Banach space, then all the assertions in this section remain valid.

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