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The viscosity approximation forward-backward splitting method for solving quasi inclusion problems in Banach spaces

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Abstract

In this paper, we introduce viscosity approximation forward-backward splitting method for an accretive operator and an m-accretive operator in Banach spaces. The strong convergence of this viscosity method is proved under certain assumptions imposed on the sequence of parameters. Applications to the minimization optimization problem and the linear inverse problem are included. The results presented in the paper extend and improve some recent results announced in the current literature.

Keywords: Accretive operator, viscosity approximation, Banach space, splitting method, forward-backward algorithm.
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1. Introduction

Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two (possibly simpler) nonlinear operators. The central problem is to iteratively find a zero of the sum of two monotone operators, namely, let X be a real Banach space, find $x^* \in X$ such that

$$0 \in Ax^* + Bx^*, \tag{1.1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem, linear inverse problem and minimization problem.

A classical method for solving problem (1.1) is the forward-backward splitting method [11, 16, 23, 29] which is defined by the following manner: $x_1 \in X$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \quad n \geqslant 1,$$

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where $r > 0$. We see that each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of B. This method includes, in particular, the proximal point algorithm [4, 6, 14, 20, 27] and the gradient method [3, 13]. Lions-Mercier [16] introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n, \quad n \geqslant 1,$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n, \quad n \geqslant 1,$$

where $J_r^T = (I + rT)^{-1}$. The first one is often called Peaceman-Rachford algorithm [24] and the second one is called Douglas-Rachford algorithm [12]. We note that both algorithms can be weakly convergent in general [23].

In 2012, López et al. [17] introduced the following Halpern-type forward-backward method: $x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n), \tag{1.2}$$

where J_r^B is the resolvent of B, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\}$, $\{b_n\}$ are error sequences in X. It was proved that the sequence $\{x_n\}$ generated by (1.2) strongly converges to a zero point of the sum of A and B under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces), see [10, 28, 29, 32].

In 2016, Cholamjiak [8] studied a generalized forward-backward method for solving the inclusion problem (1.1) for an accretive and an m-accretive operator in Banach spaces. They then proved its strong convergence under some mild conditions.

The viscosity approximation method for nonexpansive mapping in Hilbert spaces was introduced by Moudafi [22], following the ideas of Attouch [1]. Refinements in Hilbert spaces and extensions to Banach spaces were obtained by Xu [31].

Let $T : X \to X$ be a nonexpansive mapping and $f : X \to X$ be a contraction. Explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n)Tx_n, \quad n \geqslant 0,$$

where I is the identity of X. It is well-known [22, 31] that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point q of T.

Motivated and inspired by the research going on in this direction. The purpose of this paper is to introduce viscosity approximation forward-backward splitting method for an accretive operator and an m-accretive operator in the framework of Banach spaces. More precisely, we consider the following iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}^B(x_n - r_n A x_n) + e_n, \quad n \geqslant 1.$$

Under certain assumptions imposed on the sequence of parameters, the strong convergence of this viscosity method is proved.

2. Preliminaries

Throughout the paper, X is a real Banach space with norm $\|\cdot\|$ and dual space X^*. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively.

The modulus of convexity of X is the function $\delta(\epsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \epsilon\}.$$

A Banach space X is said to be uniformly convex, if $\delta(\epsilon) > 0$ for every $\epsilon \in (0, 2]$.

Let $\rho : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of X defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : x, y \in X, \|x\| = \|y\| = 1\}.$$

A Banach space X is said to be *uniformly smooth* if $\frac{\rho(t)}{t} \to 0$ as $t \to 0$. Let q be a fixed real number with $q > 1$. Then a Banach space E is said to be q-*uniformly smooth* if there exists a constant $b > 0$ such that $\rho(t) \leqslant bt^q$ for all $t > 0$. It is well-known that every q-uniformly smooth Banach space is uniformly smooth.

Let $J_q (q > 1)$ denote the *generalized duality mapping* from X into 2^{X^*} given by

$$J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. In particular, $J_2 := J$ is called the normalized duality mapping on X. It is also known (e.g., [30, p.1128]) that

$$j_q(x) = \|x\|^{q-2}J(x), \quad x \neq 0.$$

We next provide some properties of the duality mapping.

Lemma 2.1 ([9])**.** *Let* $1 < q < \infty$.

(i) *The Banach space* X *is smooth, if and only if the duality mapping* J_q *is single-valued.*

(ii) *The Banach space* X *is uniformly smooth, if and only if the duality mapping* J_q *is single-valued and norm-to-norm uniformly continuous on bounded subsets of* X.

By using the concept of sub-differentials, we know the following inequality:

Lemma 2.2 ([7, p. 33])**.** *Let* $q > 1$ *and* X *be a real normed space with the generalized duality mapping* J_q. *Then, for any* $x, y \in X$, *we have*

$$\|x + y\|^q \leqslant \|x\|^q + q\langle y, j_q(x + y)\rangle,$$

for all $j_q(x + y) \in J_q(x + y)$.

We define the domain and the range of an operator $A : X \to 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. The inverse of A, denoted by A^{-1}, is defined by $x \in A^{-1}y$, if and only if $y \in Ax$. A set-valued operator A is said to be accretive, if for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geqslant 0, \quad u \in Ax, \quad v \in Ay.$$

An accretive operator A is said to be m-accretive, if $R(I + rA) = X$ for all $r > 0$.

Given $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is β-inverse strongly accretive (β-isa) of order q, if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geqslant \beta\|u - v\|^q, \quad u \in Ax, \quad v \in Ay.$$

Let C be a nonempty subset of a real Banach space X. Let $T : C \to C$ be a nonlinear mapping. We denote the fixed point set of T by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$.

Lemma 2.3 ([25, Corollary 1])**.** *Let* C *be a closed convex subset of a uniformly smooth Banach space* X *and let* $T : C \to C$ *be a nonexpansive mapping with a fixed point. For each fixed* $u \in C$ *and every* $t \in (0, 1)$, *the unique fixed point* $x_t \in C$ *of the contraction* $C \ni x \mapsto tu + (1 - t)Tx$ *converges strongly as* $t \to 0$ *to a fixed point of* T .

In what follows, we shall use the following notation:

$$T_r^{A,B} = J_r^B(I - rA) = (I + rB)^{-1}(I - rA), \quad r > 0.$$

Lemma 2.4 ([17, Lemma 3.1 and Lemma 3.2]). *Let X be a Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator. Then we have*

(i) *For $r > 0$, $Fix(T_r^{A,B}) = (A+B)^{-1}(0)$.*

(ii) *For $0 < s \leqslant r$ and $x \in X$, $\|x - T_s^{A,B}x\| \leqslant 2\|x - T_r^{A,B}x\|$.*

Lemma 2.5 ([17, Lemma 3.3]). *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1,2]$. Assume that A is a single-valued α-isa of order q in X. Then, given $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : R^+ \to R^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in B_r$,*

$$\begin{aligned}\|T_r^{A,B}x - T_r^{A,B}y\|^q \leqslant & \|x-y\|^q - r(\alpha q - r^{q-1}k_q)\|Ax - Ay\|^q \\ & - \phi_q(\|(I-J_r^B)(I-rA)x - (I-J_r^B)(I-rA)y\|),\end{aligned}$$

where k_q is the q-uniform smoothness coefficient of X.

Lemma 2.6 ([18, Lemma 3.1]). *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-\delta_n)a_n + b_n + c_n, \quad n \geqslant 1,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume $\sum\limits_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n \leqslant \delta_n M$ for some $M \geqslant 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum\limits_{n=1}^{\infty} \delta_n = \infty$ and $\limsup\limits_{n\to\infty} \frac{b_n}{\delta_n} \leqslant 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

By employing the technique of Maingé [19], He and Yang [15] proved the following lemma.

Lemma 2.7 ([15, Lemma 8]). *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leqslant (1-\gamma_n)s_n + \gamma_n \tau_n, \quad n \geqslant 1,$$

and

$$s_{n+1} \leqslant s_n - \eta_n + \rho_n, \quad n \geqslant 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that

(i) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$.

(ii) $\lim\limits_{n\to\infty} \rho_n = 0$.

(iii) $\lim\limits_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup\limits_{k\to\infty} \tau_{n_k} \leqslant 0$, *for any subsequence $\{n_k\} \subset \{n\}$.*

Then $\lim\limits_{n\to\infty} s_n = 0$.

Lemma 2.8 ([21, p. 63]). *Let $q > 1$. Then the following inequality holds:*

$$ab \leqslant \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a and b.

Lemma 2.9 ([8, Proposition 3.1]). *Let $q > 1$ and let X be a real smooth Banach space with the generalized duality mapping j_q. Let $m \in \mathbb{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geqslant 0$, for all $i = 1, 2, ..., m$ with $\sum_{i=1}^m t_i \leqslant 1$. Then we have*

$$\|\sum_{i=1}^m t_i x_i\|^q \leqslant \frac{\sum_{i=1}^m t_i \|x_i\|^q}{q-(q-1)\sum_{i=1}^m t_i}.$$

3. Main results

In this section, we first establish a crucial proposition and then prove our main theorem.

Proposition 3.1. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an β-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A + B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X and f be a contraction on X with coefficient $\alpha \in [0, 1)$. Let $\{x_n\}$ be generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}^B(x_n - r_n A x_n) + e_n, \quad n \geqslant 1,$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $0 < r_n < (\beta q/k_q)^{1/(q-1)}$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. If $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$, then $\{x_n\}$ is bounded.

Proof. For each $n \in \mathbb{N}$, we put $T_n = J_{r_n}^B(I - r_n A)$ and let $\{y_n\}$ be defined by

$$y_{n+1} = \alpha_n f(y_n) + \lambda_n y_n + \delta_n T_n y_n. \tag{3.1}$$

Firstly, we compute the following:

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &= \|\alpha_n(f(x_n) - f(y_n)) + \lambda_n(x_n - y_n) + \delta_n(T_n x_n - T_n y_n) + e_n\|,\\ &\leqslant \alpha_n\|f(x_n) - f(y_n)\| + \lambda_n\|x_n - y_n\| + \delta_n\|T_n x_n - T_n y_n\| + \|e_n\|\\ &\leqslant \alpha_n\alpha\|x_n - y_n\| + \lambda_n\|x_n - y_n\| + \delta_n\|x_n - y_n\| + \|e_n\|\\ &= (1 - \alpha_n(1 - \alpha))\|x_n - y_n\| + \|e_n\|.\end{aligned}$$

By the assumptions and Lemma 2.6 (ii), we conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Let $z \in \mathrm{Fix}(T_n)$. We next show that $\{y_n\}$ is bounded. Indeed

$$\begin{aligned}\|y_{n+1} - z\| &= \|\alpha_n(f(y_n) - z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|,\\ &\leqslant \alpha_n\|f(y_n) - z\| + \lambda_n\|y_n - z\| + \delta_n\|T_n y_n - z\|\\ &\leqslant \alpha_n(\|f(y_n) - f(z)\| + \|f(z) - z\|) + \lambda_n\|y_n - z\| + \delta_n\|y_n - z\|\\ &\leqslant \alpha_n\alpha\|y_n - z\| + \alpha_n\|f(z) - z\| + \lambda_n\|y_n - z\| + \delta_n\|y_n - z\|\\ &= (1 - \alpha_n(1 - \alpha))\|y_n - z\| + \alpha_n\|f(z) - z\|.\end{aligned}$$

This shows that $\{y_n\}$ is bounded by Lemma 2.6 (i) and hence $\{x_n\}$ is also bounded. □

Theorem 3.2. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an β-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A + B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X and f be a contraction on X with coefficient $\alpha \in [0, 1)$. Let $\{x_n\}$ be generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}^B(x_n - r_n A x_n) + e_n, \quad n \geqslant 1,$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < (\beta q/k_q)^{1/(q-1)}$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ *strongly converges to some* $z \in \Omega$.

Proof. Let $z \in \mathrm{Fix}(T_n)$, from Lemma 2.2 and Lemma 2.8, we have

$$\begin{aligned}
\|y_{n+1} - z\|^q &= \|\alpha_n(f(y_n) - z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q, \\
&\leqslant \|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q + q\alpha_n\langle (f(y_n) - z), j_q(y_{n+1} - z)\rangle \\
&\leqslant \|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\
&\quad + q\alpha_n(\langle f(y_n) - f(z), j_q(y_{n+1} - z)\rangle + \langle f(z) - z, j_q(y_{n+1} - z)\rangle) \\
&\leqslant \|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\
&\quad + q\alpha_n\alpha\|y_n - z\|\|y_{n+1} - z\|^{q-1} + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle) \\
&\leqslant \|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\
&\quad + q\alpha_n\alpha(\frac{1}{q}\|y_n - z\|^q + \frac{q-1}{q}\|y_{n+1} - z\|^q) + q\alpha_n\langle f(z) - z, j_q(y_{n+1} - z)\rangle.
\end{aligned}$$

After simplifying, it follows that

$$\begin{aligned}
\|y_{n+1} - z\|^q \leqslant{}& \frac{1}{1-(q-1)\alpha_n\alpha}\|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\
&+ \frac{\alpha_n\alpha}{1-(q-1)\alpha_n\alpha}\|y_n - z\|^q + \frac{q\alpha_n}{1-(q-1)\alpha_n\alpha}\langle f(z) - z, j_q(y_{n+1} - z)\rangle.
\end{aligned} \tag{3.2}$$

On the other hand, by Lemma 2.9 and Lemma 2.5, we obtain

$$\begin{aligned}
&\|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\
&\leqslant \frac{1}{\alpha_n q + 1 - \alpha_n}(\lambda_n\|y_n - z\|^q + \delta_n\|T_n y_n - z\|^q) \\
&\leqslant \frac{1}{\alpha_n q + 1 - \alpha_n}(\lambda_n\|y_n - z\|^q + \delta_n(\|y_n - z\|^q - r_n(\beta q - r_n^{q-1}k_q)\|Ay_n - Az\|^q \\
&\quad - \phi_q(\|y_n - r_n Ay_n - T_n y_n + r_n Az\|))) \\
&\leqslant \frac{1-\alpha_n}{\alpha_n q + 1 - \alpha_n}\|y_n - z\|^q - \frac{\delta_n r_n(\beta q - r_n^{q-1}k_q)}{\alpha_n q + 1 - \alpha_n}\|Ay_n - Az\|^q \\
&\quad - \frac{\delta_n}{\alpha_n q + 1 - \alpha_n}\phi_q(\|y_n - r_n Ay_n - T_n y_n + r_n Az\|).
\end{aligned} \tag{3.3}$$

By replacing (3.3) into (3.2), it follows that

$$\begin{aligned}
\|y_{n+1} - z\|^q \leqslant{}& (1 - \frac{\alpha_n q(1 - \alpha - (q-1)\alpha_n\alpha)}{(1-(q-1)\alpha_n\alpha)(\alpha_n q + 1 - \alpha_n)})\|y_n - z\|^q \\
&- \frac{\delta_n r_n(\beta q - r_n^{q-1}k_q)}{(1-(q-1)\alpha_n\alpha)(\alpha_n q + 1 - \alpha_n)}\|Ay_n - Az\|^q \\
&- \frac{\delta_n}{(1-(q-1)\alpha_n\alpha)(\alpha_n q + 1 - \alpha_n)}\phi_q(\|y_n - r_n Ay_n - T_n y_n + r_n Az\|) \\
&+ \frac{q\alpha_n}{1-(q-1)\alpha_n\alpha}\langle f(z) - z, j_q(y_{n+1} - z)\rangle.
\end{aligned} \tag{3.4}$$

We can check that $\frac{\alpha_n q(1-\alpha-(q-1)\alpha_n\alpha)}{(1-(q-1)\alpha_n\alpha)(\alpha_n q+1-\alpha_n)}$ is in $(0,1)$, since $1<q\leqslant 2$, $\{\alpha_n\}\subset(0,1)$ and $\lim\limits_{n\to\infty}\alpha_n=0$. Moreover, by condition (ii), $\frac{\delta_n r_n(\beta q-r_n^{q-1}k_q)}{(1-(q-1)\alpha_n\alpha)(\alpha_n q+1-\alpha_n)}$ and $\frac{\delta_n}{(1-(q-1)\alpha_n\alpha)(\alpha_n q+1-\alpha_n)}$ are positive.

For each $n\geqslant 1$, we set

$$\begin{aligned}
s_n &= \|y_n-z\|^q,\quad \gamma_n=\frac{\alpha_n q(1-\alpha-(q-1)\alpha_n\alpha)}{(1-(q-1)\alpha_n\alpha)(\alpha_n q+1-\alpha_n)},\\
\tau_n &= \frac{\alpha_n q+1-\alpha_n}{1-\alpha-(q-1)\alpha_n\alpha}\langle f(z)-z, j_q(y_{n+1}-z)\rangle,\\
\eta_n &= \frac{\delta_n r_n(\beta q-r_n^{q-1}k_q)}{(1-(q-1)\alpha_n\alpha)(\alpha_n q+1-\alpha_n)}\|Ay_n-Az\|^q\\
&\quad+\frac{\delta_n}{(1-(q-1)\alpha_n\alpha)(\alpha_n q+1-\alpha_n)}\phi_q(\|y_n-r_nAy_n-T_ny_n+r_nAz\|),\\
\rho_n &= \frac{q\alpha_n}{1-(q-1)\alpha_n\alpha}\langle f(z)-z, j_q(y_{n+1}-z)\rangle.
\end{aligned}$$

From (3.4), we have

$$s_{n+1}\leqslant(1-\gamma_n)s_n+\gamma_n\tau_n,\quad n\geqslant 1,$$

and

$$s_{n+1}\leqslant s_n-\eta_n+\rho_n,\quad n\geqslant 1.$$

Since $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$, it follows that $\sum\limits_{n=1}^{\infty}\gamma_n=\infty$. By the boundedness of $\{y_n\}$ and $\lim\limits_{n\to\infty}\alpha_n=0$, we see that $\lim\limits_{n\to\infty}\rho_n=0$. In order to complete the proof, by using Lemma 2.7, it remains to show that $\lim\limits_{k\to\infty}\eta_{n_k}=0$ implies $\limsup\limits_{k\to\infty}\tau_{n_k}\leqslant 0$, for any subsequence $\{n_k\}\subset\{n\}$.

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim\limits_{k\to\infty}\eta_{n_k}=0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$\lim_{k\to\infty}\|Ay_{n_k}-Az\|=\lim_{k\to\infty}\|y_{n_k}-\gamma_{n_k}Ay_{n_k}-T_{n_k}y_{n_k}+r_{n_k}Az\|=0.$$

This gives, by the triangle inequality, that

$$\lim_{k\to\infty}\|T_{n_k}y_{n_k}-y_{n_k}\|=0. \tag{3.5}$$

Since $\liminf\limits_{k\to\infty}r_n>0$, there is $r>0$ such that $r_n\geqslant r$, for all $n\geqslant 1$. In particular, $r_{n_k}\geqslant r$ for all $k\geqslant 1$. Lemma 2.4 (ii) yields that

$$\|T_r^{A,B}y_{n_k}-y_{n_k}\|\leqslant 2\|T_{n_k}y_{n_k}-y_{n_k}\|.$$

Then, by (3.5), we obtain

$$\limsup_{k\to\infty}\|T_r^{A,B}y_{n_k}-y_{n_k}\|\leqslant 2\lim_{k\to\infty}\|T_{n_k}y_{n_k}-y_{n_k}\|.$$

It follows that

$$\lim_{k\to\infty}\|T_r^{A,B}y_{n_k}-y_{n_k}\|=0. \tag{3.6}$$

Let $z_t=tf(z_t)+(1-t)T_r^{A,B}z_t, t\in(0,1)$. By employing Lemma 2.3, we have $z_t\to z\in\Omega$ as $t\to 0$. From Lemma 2.2 we have that

$$\begin{aligned}
\|z_t-y_{n_k}\|^q &= \|t(f(z_t)-y_{n_k})+(1-t)(T_r^{A,B}z_t-y_{n_k})\|^q\\
&\leqslant(1-t)^q\|T_r^{A,B}z_t-y_{n_k}\|^q+qt\langle f(z_t)-y_{n_k}, j_q(z_t-y_{n_k})\rangle
\end{aligned}$$

$$\begin{aligned}&\leqslant (1-t)^q(\|T_r^{A,B}z_t - T_r^{A,B}y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q\\&\quad + qt\langle f(z_t) - z_t, j_q(z_t - y_{n_k})\rangle + qt\langle z_t - y_{n_k}, j_q(z_t - y_{n_k})\rangle\\&\leqslant (1-t)^q(\|z_t - y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q\\&\quad + qt\langle f(z_t) - z_t, j_q(z_t - y_{n_k})\rangle + qt\|z_t - y_{n_k}\|^q.\end{aligned}$$

This shows that

$$\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle \leqslant \frac{(1-t)^q}{qt}(\|z_t - y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q + \frac{qt-1}{qt}\|z_t - y_{n_k}\|^q. \tag{3.7}$$

From (3.7) and (3.6), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle &\leqslant \frac{(1-t)^q}{qt}M^q + \frac{qt-1}{qt}M^q\\&= \frac{(1-t)^q + qt - 1}{qt}M^q,\end{aligned} \tag{3.8}$$

where $M = \limsup_{k\to\infty}\|z_t - y_{n_k}\|$, $t \in (0,1)$. We see that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Lemma 2.1 (ii), we know that j_q is norm-to-norm uniformly continuous on bounded subsets of X. Since $z_t \to z$ as $t \to 0$, we have $\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\| \to 0$ as $t \to 0$. Observe that

$$\begin{aligned}&|\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle - \langle z - f(z), j_q(z - y_{n_k})\rangle|\\&\quad\leqslant |\langle z_t - z + z - f(z) + f(z) - f(z_t), j_q(z_t - y_{n_k})\rangle - \langle z - f(z), j_q(z - y_{n_k})\rangle|\\&\quad\leqslant |\langle z_t - z, j_q(z_t - y_{n_k})\rangle| + |\langle z - f(z), j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\rangle|\\&\qquad + |\langle f(z) - f(z_t), j_q(z_t - y_{n_k})\rangle|\\&\quad\leqslant (1+\alpha)\|z_t - z\|\|z_t - y_{n_k}\|^{q-1} + \|z - f(z)\|\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\|^{q-1}.\end{aligned}$$

So as $t \to 0$, we get

$$\langle z_t - f(z_t), j_q(z_t - y_{n_k})\rangle \to \langle z - f(z), j_q(z - y_{n_k})\rangle.$$

From (3.8), as $t \to 0$, it follows that

$$\limsup_{k\to\infty}\langle z - f(z), j_q(z - y_{n_k})\rangle \leqslant 0. \tag{3.9}$$

By Proposition 3.1, $\{y_n\}$ is bounded, and so is $\{f(x_n)\}$, by condition (i) and (3.1), (3.5), we have

$$\begin{aligned}\|y_{n_k+1} - y_{n_k}\| &= \|\alpha_{n_k}f(y_{n_k}) + \lambda_{n_k}y_{n_k} + \delta_{n_k}T_{n_k}y_{n_k} - y_{n_k}\|\\&\leqslant \alpha_{n_k}\|f(y_{n_k}) - y_{n_k}\| + \delta_{n_k}\|T_{n_k}y_{n_k} - y_{n_k}\| \to 0,\end{aligned} \tag{3.10}$$

as $k \to \infty$. By combining (3.9) and (3.10), we get that

$$\limsup_{k\to\infty}\langle z - f(z), j_q(z - y_{n_k+1})\rangle \leqslant 0.$$

It also follows that $\limsup_{k\to\infty}\tau_{n_k} \leqslant 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.7. Hence $y_n \to z$ as $n \to \infty$, by Proposition 3.1, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, so $\lim_{n\to\infty} x_n = z \in \Omega$. We thus complete the proof. □

By setting $\lambda_n = 0$ for all $n \geqslant 1$, we obtain the following result:

Corollary 3.3. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an β-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X and f be a contraction on X with coefficient $\alpha \in [0,1)$. Let $\{x_n\}$ be generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}^B(x_n - r_nAx_n) + e_n, \quad n \geqslant 1,$$

where $J_{r_n}^B = (I + r_nB)^{-1}$, $\{r_n\} \subset (0,\infty)$ and $\{\alpha_n\}$ is a sequences in $[0,1]$. Assume that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0;$

(ii) $0 < \liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < (\beta q/k_q)^{1/(q-1)};$

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$, *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0.$

Then $\{x_n\}$ *strongly converges to some* $z \in \Omega$.

4. Applications

4.1. Minimization problem

In this subsection, we apply Theorem 3.2 to the convex minimization problem. Let H be a real Hilbert space. Let $F : H \to R$ be a convex smooth function and $G : H \to R$ be a convex, lower-semicontinuous and nonsmooth function. We consider the problem of finding $x^* \in H$ such that

$$F(x^*) + G(x^*) \leqslant F(x) + G(x), \tag{4.1}$$

for all $x \in H$. This problem (4.1) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla F(x^*) + \partial G(x^*),$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. In this point of view, we can set $A = \nabla F$ and $B = \partial G$ in Theorem 3.2. This is because if ∇F is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone [2, Corollary 10]. Moreover, ∂G is maximal monotone [26, Theorem A]. So we obtain the following result.

Theorem 4.1. *Let* H *be real Hilbert space. Let* $F : H \to R$ *be a convex and differentiable function with* $(1/L)$*-Lipschitz continuous gradient* ∇F *and* $G : H \to R$ *be a convex and lower semi-continuous function which* $F + G$ *attains a minimizer. Let* $\{e_n\}$ *be a sequence in* H *and* f *be a contraction on* X *with coefficient* $\alpha \in [0, 1)$. *Let* $\{x_n\}$ *be generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}(x_n - r_n \nabla F(x_n)) + e_n, \quad n \geqslant 1,$$

where $J_{r_n} = (I + r_n \partial G)^{-1}, \{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}$, $\{\lambda_n\}$, *and* $\{\delta_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \lambda_n + \delta_n = 1$. *Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0;$

(ii) $0 < \liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < 2L;$

(iii) $\liminf_{n\to\infty} \delta_n > 0;$

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$, *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0.$

Then $\{x_n\}$ *strongly converges to a minimizer of* $F + G$.

4.2. Linear inverse problem

In this subsection, we apply Theorem 3.2 to solve the unconstrained linear system

$$Cx = d, \tag{4.2}$$

where C is a bounded linear operator on H and $d \in H$. For each $x \in H$, we define $F : H \to R$ by

$$F(x) = \frac{1}{2}\|Cx - d\|^2.$$

From [5] we know that $\nabla F(x) = C^T(Cx - d)$ and ∇F is K-Lipschitz continuous with K the largest eigenvalue of C^TC. So we obtain the following result.

Theorem 4.2. *Let* H *be real Hilbert space. Let* $C : H \to H$ *be a bounded linear operator and* $d \in H$ *with* K *the largest eigenvalue of* C^TC. *Let* $\{e_n\}$ *be a sequence in* H *and* f *be a contraction on* X *with coefficient* $\alpha \in [0, 1)$. *Let* $\{x_n\}$ *be generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n(x_n - r_n C^T(C(x_n) - d)) + e_n, \quad n \geqslant 1,$$

where $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}$, $\{\lambda_n\}$, *and* $\{\delta_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \lambda_n + \delta_n = 1$. *Assume that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0;$

(ii) $0 < \liminf_{n\to\infty} r_n \leqslant \limsup_{n\to\infty} r_n < 2/K;$

(iii) $\liminf_{n\to\infty} \delta_n > 0;$

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$, *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

If (4.2) *is consistent, then* $\{x_n\}$ *strongly converges to a solution of a linear system.*

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