Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 175–185
Research Article

ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

On alternating direction method for solving variational inequality problems with separable structure

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Communicated by Y. H. Yao

Abstract

We present an alternating direction scheme for the separable constrained convex programming problem. The predictor is obtained via solving two sub-variational inequalities in a parallel wise at each iteration. The new iterate is obtained by a projection type method along a new descent direction. The new direction is obtained by combining the descent directions using by He [B.-S. He, Comput. Optim. Appl., **42** (2009), 195–212] and Jiang and Yuan [Z.-K. Jiang, X.-M. Yuan, J. Optim. Theory Appl., **145** (2010), 311–323]. Global convergence of the proposed method is proved under certain assumptions. We also report some numerical results to illustrate the efficiency of the proposed method.

Keywords: Variational inequalities, monotone operator, projection method, alternating direction method.
2010 MSC: 49J40, 65N30.

1. Introduction

Let $A \in \mathcal{R}^{l\times n}$, $B \in \mathcal{R}^{l\times m}$ be given matrices, $b \in \mathcal{R}^{l}$ be a given vector, and $f : \mathcal{X} \to \mathcal{R}^{n}$, $g : \mathcal{Y} \to \mathcal{R}^{m}$ be given monotone operators. We consider the following variational inequality problem: Find $u \in \Omega$ such that

$$(u' - u)^{\top} F(u) \geqslant 0, \quad \forall\, u' \in \Omega, \tag{1.1}$$

with block-separated structure

$$\begin{aligned} u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} \text{ and} \\ \Omega = \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}, Ax + By = b\}. \end{aligned} \tag{1.2}$$

It has several applications in network economics, transportation equilibrium problems and regional science, see, for example, [8, 11–14, 20] and the references therein.

By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^{l}$ to the linear constraint $Ax + By = b$, the problem

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doi:10.22436/jnsa.010.01.17

Received 2016-03-16

(1.1)-(1.2) can be written in terms of finding $w \in \mathcal{W}$ such that

$$(w'-w)^\top Q(w) \geqslant 0, \quad \forall\, w' \in \mathcal{W}, \tag{1.3}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \quad Q(w) = \begin{pmatrix} f(x) - A^\top \lambda \\ g(y) - B^\top \lambda \\ Ax + By - b \end{pmatrix}, \quad \mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l. \tag{1.4}$$

The problem (1.3)-(1.4) is referred to as *structured variational inequalities* (in short, SVI).

The alternating direction method (ADM) for solving the structured problem (1.3)-(1.4) was proposed by Gabay and Mercier [13] and Gabay [12]. They decomposed the original problem into a series of subproblems with lower scale. This method appears to be one of the most powerful methods. For ADM with logarithmic-quadratic proximal regularization we quoted references [1–5, 15, 24, 25, 28]. To make the ADM more efficient and practical some strategies have been studied, for more details, one can refer to [6, 7, 9, 16, 17, 22, 23, 25, 27].

He et al. [17] proposed a modified PADM as follows: For given $(x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$, the new iterative $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is obtained via the following steps.

Step 1. Solve the following variational inequality to obtain x^{k+1}:

$$(x' - x^{k+1})^\top \left\{ f(x^{k+1}) - A^\top \left[\lambda^k - H_k(Ax^{k+1} + By^k - b)\right] + R_k(x^{k+1} - x^k) \right\} \geqslant 0, \tag{1.5}$$

for all $x' \in \mathcal{X}$.

Step 2. Solve the following variational inequality to obtain y^{k+1}:

$$(y' - y^{k+1})^\top \{ g(y^{k+1}) - B^\top [\lambda^k - H_k(Ax^{k+1} + By^{k+1} - b)] + S_k(y^{k+1} - y^k) \} \geqslant 0, \tag{1.6}$$

for all $y' \in \mathcal{Y}$.

Step 3. Update λ^k via

$$\lambda^{k+1} = \lambda^k - H_k(Ax^{k+1} + By^{k+1} - b),$$

where $\{R_k\}$, $\{H_k\}$, $\{S_k\}$ are sequences of both lower and upper bounded symmetric positive matrices. A sequence of positive matrices $\{H_k\}$ is said to be both lower and upper bounded if

$$\inf_k \{\lambda_k : \lambda_k \text{ is the smallest eignevalue of matrix } H_k\} = \lambda_{\min} > 0$$

and

$$\sup_k \{\lambda_k : \lambda_k \text{ is the largest eignevalue of matrix } H_k\} = \lambda_{\max} < +\infty.$$

The main disadvantage of the method in [17] is that solving (1.6) requires the solution of (1.5). To overcome this difficulty, He [18] proposed the following algorithm: For a given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$, the predictor $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is obtained via solving the following variational inequalities:

$$\begin{aligned}
(x' - x)^\top (f(x) - A^\top [\lambda^k - H(Ax + By^k - b)]) &\geqslant 0, \\
(y' - y)^\top (g(y) - B^\top [\lambda^k - H(Ax^k + By - b)]) &\geqslant 0, \\
\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b)&,
\end{aligned}$$

where $H \in \mathcal{R}^{l \times l}$ is symmetric positive definite. And the new iterate $w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by:

$$w^{k+1}(\alpha_k) = w^k - \alpha_k G^{-1} M (w^k - \tilde{w}^k),$$

where

$$G = \begin{pmatrix} A^\top HA & 0 & 0 \\ 0 & B^\top HB & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}.$$

In 2010, Jiang and Yuan [23] proposed a new parallel descent-like method for solving a class of variational inequalities with separate structures by using the same predictor as He's method [18] and the new iterate $w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by:

$$w^{k+1}(\alpha_k) = P_{\mathcal{W}}[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)],$$

where

$$d(w^k, \tilde{w}^k) = \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix}.$$

Inspired by the above cited works and by the recent work going on this direction, we propose a descent alternating direction method for SVI. Each iteration of the above method contains a prediction and a correction, the predictor is obtained via solving two subvariational inequalities at each iteration and the new iterate is obtained by a projection type method along a new descent direction. The new direction is obtained by combining the descent directions using by He [18] and Jiang and Yuan [23]. Our results can be viewed as significant extensions of the previously known results.

2. Iterative method

This section states some preliminaries that are useful later. The first lemma provides some basic properties of projection onto Ω.

Lemma 2.1. *Let* G *be a symmetry positive definite matrix and* Ω *be a nonempty closed convex subset of* R^l, *we denote by* $P_{\Omega,G}(.)$ *the projection under the* G*-norm, that is,*

$$P_{\Omega,G}(v) = \operatorname{argmin}\{\|v - u\|_G \ : \ u \in \Omega\}.$$

Then, we have the following inequalities.

$$\begin{aligned} (z - P_{\Omega,G}[z])^\top G(P_{\Omega,G}[z] - v) \geqslant 0, \quad \forall\, z \in R^l, v \in \Omega, \\ \|P_{\Omega,G}[u] - P_{\Omega,G}[v]\|_G \leqslant \|u - v\|_G, \quad \forall\, u, v \in R^l, \\ \|u - P_{\Omega,G}[z]\|_G^2 \leqslant \|z - u\|_G^2 - \|z - P_{\Omega,G}[z]\|_G^2, \quad \forall\, z \in R^l, u \in \Omega. \end{aligned} \tag{2.1}$$

We make the following standard assumptions.

Assumption A. f is monotone with respect to $\mathcal{X}$ and g is monotone with respect to $\mathcal{Y}$,

Assumption B. The solution set of SVI, denoted by $\mathcal{W}^*$, is nonempty.

We propose the following alternating direction method for solving SVI:

Algorithm 2.2.

Step 0. *The initial step: Given* $\varepsilon > 0$, $\beta_1 \geqslant 0, \beta_2 \geqslant 0$ $(\beta_1 + \beta_2 > 0)$ *and* $w^0 = (x^0, y^0, \lambda^0) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$. *Set* $k = 0$.

Step 1. *Prediction step: Compute* $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$ *by solving the following variational inequalities:*

$$(x' - x)^T(f(x) - A^T[\lambda^k - H(Ax + By^k - b)] + R(x - x^k)) \geqslant 0, \quad \forall x' \in \mathcal{X}, \tag{2.2}$$

$$(y' - y)^T(g(y) - B^T[\lambda^k - H(Ax^k + By - b)] + S(y - y^k)) \geqslant 0, \quad \forall y' \in \mathcal{Y}, \tag{2.3}$$

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b). \tag{2.4}$$

Step 2. *Convergence verification: If* $\max\{\|x^k - \tilde{x}^k\|_\infty, \|y^k - \tilde{y}^k\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \epsilon$, *then stop.*

Step 3. *Correction step: The new iterate* $w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ *is given by:*

$$w^{k+1}(\alpha_k) = P_{\mathcal{W}}[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)], \tag{2.5}$$

where

$$\begin{aligned} \alpha_k &= \frac{\varphi_k}{(\beta_1+\beta_2)\|w^k - \tilde{w}^k\|_G^2}, &(2.6)\\ \varphi_k &= \|w^k - \tilde{w}^k\|_G^2 + (\lambda^k - \tilde{\lambda}^k)^T (A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)), &(2.7)\\ d(w^k, \tilde{w}^k) &= \beta_1 D(w^k, \tilde{w}^k) + \beta_2 G(w^k - \tilde{w}^k),\\ D(w^k, \tilde{w}^k) &= \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix} \end{aligned}$$

and

$$G = \begin{pmatrix} R + A^T HA & 0 & 0 \\ 0 & S + B^T HB & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}.$$

Set $k := k+1$ *and go to Step 1.*

Remark 2.3. By using as special case of our method, we can obtain some alternating direction methods, for example:

- If $\beta_1 = 0, \beta_2 = 1$ and $R = S = 0$, we obtain the method proposed by He [18].
- If $\beta_1 = 1, \beta_2 = 0$ and $R = S = 0$, we obtain the method proposed by Jiang and yuan [23].

Remark 2.4. It is easy to check that $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is solution of SVI if and only if

$$\begin{cases} x^k - \tilde{x}^k = 0, \\ y^k - \tilde{y}^k = 0, \\ \lambda^k - \tilde{\lambda}^k = 0. \end{cases}$$

Hence, the stopping criterion adopted here is reasonable: if it is satisfied with a small ϵ, we can regard the current iterate as an approximate solution.

In the next theorem, we show that α_k is lower bounded away from zero and it is useful for the convergence analysis.

Theorem 2.5. *For given* $w^k \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$, *let* $\tilde{w}^k$ *be generated by* (2.2)-(2.4), *then we have the following*

$$\varphi_k \geqslant \frac{2-\sqrt{2}}{2}\|w^k - \tilde{w}^k\|_G^2 \tag{2.8}$$

and

$$\alpha_k \geqslant \frac{2-\sqrt{2}}{2}. \tag{2.9}$$

Proof. It follows from (2.7) that

$$\begin{aligned} \varphi_k &= \|w^k - \tilde{w}^k\|_G^2 + (\lambda^k - \tilde{\lambda}^k)^T (A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ &= \|x^k - \tilde{x}^k\|_R^2 + \|Ax^k - A\tilde{x}^k\|_H^2 + \|y^k - \tilde{y}^k\|_S^2 + \|By^k - B\tilde{y}^k\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ &\quad + (\lambda^k - \tilde{\lambda}^k)^T (A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)). \end{aligned} \tag{2.10}$$

By using the Cauchy-Schwarz inequality, we have

$$(\lambda^k - \tilde{\lambda}^k)^T(A(x^k - \tilde{x}^k)) \geqslant -\frac{1}{2}\left(\sqrt{2}\|A(x^k - \tilde{x}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \tag{2.11}$$

and

$$(\lambda^k - \tilde{\lambda}^k)^T(B(y^k - \tilde{y}^k)) \geqslant -\frac{1}{2}\left(\sqrt{2}\|B(y^k - \tilde{y}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right). \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10), we get

$$\begin{aligned}
\varphi_k &\geqslant \frac{2-\sqrt{2}}{2}\left(\|Ax^k - A\tilde{x}^k\|_H^2 + \|By^k - B\tilde{y}^k\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) + \|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2 \\
&\geqslant \frac{2-\sqrt{2}}{2}\left(\|Ax^k - A\tilde{x}^k\|_H^2 + \|By^k - B\tilde{y}^k\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) + \frac{2-\sqrt{2}}{2}\left(\|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2\right) \\
&\geqslant \frac{2-\sqrt{2}}{2}\|w^k - \tilde{w}^k\|_G^2.
\end{aligned}$$

Therefore, it follows from (2.6) and (2.8) that

$$\alpha_k \geqslant \frac{2-\sqrt{2}}{2}$$

and this completes the proof. □

3. Basic results

In this section, we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method.

Lemma 3.1. *For given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$, let $\tilde{w}^k$ be generated by (2.2)-(2.4). Then for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, we have*

$$(w^k - w^*)^\top G(w^k - \tilde{w}^k) \geqslant \|w^k - \tilde{w}^k\|_G^2 + (\lambda^k - \tilde{\lambda}^k)^\top\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \tag{3.1}$$

and

$$(w^{k+1}(\alpha_k) - \tilde{w}^k)^\top D(w^k, \tilde{w}^k) \geqslant (w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) + \|w^k - \tilde{w}^k\|_G^2. \tag{3.2}$$

Proof. By setting $x' = x^*$ in (2.2), we get

$$(x^* - \tilde{x}^k)^\top\left\{f(\tilde{x}^k) - A^\top\tilde{\lambda}^k - A^T HA(x^k - \tilde{x}^k) + A^T H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) - R(x^k - \tilde{x}^k)\right\} \geqslant 0. \tag{3.3}$$

Similarly, substituting $y' = y^*$ in (2.3), we obtain

$$(y^* - \tilde{y}^k)^\top\left\{g(\tilde{y}^k) - B^\top\tilde{\lambda}^k - B^T HB(y^k - \tilde{y}^k) + B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) - S(y^k - \tilde{y}^k)\right\} \geqslant 0. \tag{3.4}$$

Since (x^*, y^*, λ^*) is a solution of SVI, $\tilde{x}^k \in \mathcal{X}$ and $\tilde{y}^k \in \mathcal{Y}$, we have

$$(\tilde{x}^k - x^*)^\top(f(x^*) - A^\top\lambda^*) \geqslant 0,$$
$$(\tilde{y}^k - y^*)^\top(g(y^*) - B^\top\lambda^*) \geqslant 0,$$

and

$$Ax^* + By^* - b = 0.$$

Using the monotonicity of f and g, we obtain

$$\begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^* \end{pmatrix}^\top \begin{pmatrix} f(\tilde{x}^k) - A^\top \tilde{\lambda}^k \\ g(\tilde{y}^k) - B^\top \tilde{\lambda}^k \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix} \geqslant \begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^* \end{pmatrix}^\top \begin{pmatrix} f(x^*) - A^\top \lambda^* \\ g(y^*) - B^\top \lambda^* \\ Ax^* + By^* - b \end{pmatrix} \geqslant 0. \tag{3.5}$$

Adding (3.3), (3.4) and (3.5), we get

$$\begin{aligned}(w^* - \tilde{w}^k)^\top G(w^k - \tilde{w}^k) &= (x^* - \tilde{x}^k)^\top (R(x^k - \tilde{x}^k) + A^\top HA(x^k - \tilde{x}^k)) + (y^* - \tilde{y}^k)^\top (S(y^k - \tilde{y}^k) \\ &\quad + B^\top HB(y^k - \tilde{y}^k)) + (\lambda^* - \tilde{\lambda}^k)^\top (A\tilde{x}^k + B\tilde{y}^k - b) \\ &\leqslant (x^* - \tilde{x}^k)^\top A^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\ &\quad + (y^* - \tilde{y}^k)^\top B^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\ &= -(A\tilde{x}^k + B\tilde{y}^k - b)^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\ &= -(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right),\end{aligned} \tag{3.6}$$

where the last equality follows from (2.4). It follows from (3.6) that

$$(w^k - w^*)^\top G(w^k - \tilde{w}^k) \geqslant \|w^k - \tilde{w}^k\|_G^2 + (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)$$

and the first assertion of this lemma is proved.

Similarly as in (3.3) and (3.4), we have

$$(x^{k+1} - \tilde{x}^k)^\top \left\{R(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^\top \tilde{\lambda}^k + A^T HA(x^k - \tilde{x}^k) - A^T H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\right\} \leqslant 0 \tag{3.7}$$

and

$$(y^{k+1} - \tilde{y}^k)^T \left\{S(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^\top \tilde{\lambda}^k + B^T HB(y^k - \tilde{y}^k) - B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\right\} \leqslant 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{pmatrix} x^{k+1} - \tilde{x}^k \\ y^{k+1} - \tilde{y}^k \\ \lambda^{k+1} - \tilde{\lambda}^k \end{pmatrix}^\top \begin{pmatrix} (R + A^T HA)(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^\top \tilde{\lambda}^k - A^T H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\ (S + B^\top HB)(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^\top \tilde{\lambda}^k - B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ H^{-1}(\lambda^k - \tilde{\lambda}^k) - (A\tilde{x}^k + B\tilde{y}^k - b) \end{pmatrix} \leqslant 0,$$

which implies

$$(w^{k+1}(\alpha_k) - \tilde{w}^k)^\top (G(w^k - \tilde{w}^k) - D(w^k, \tilde{w}^k)) \leqslant 0.$$

By simple manipulation, we obtain

$$\begin{aligned}(w^{k+1}(\alpha_k) - \tilde{w}^k)^\top D(w^k, \tilde{w}^k) &\geqslant (w^{k+1}(\alpha_k) - \tilde{w}^k)^\top G(w^k - \tilde{w}^k) \\ &= (w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) + \|w^k - \tilde{w}^k\|_G^2\end{aligned}$$

and the second assertion of this lemma is proved. □

The following theorem provides a unified framework for proving the convergence of the new algorithm.

Theorem 3.2. *Let $w^* \in \mathcal{W}^*, w^{k+1}(\alpha_k)$ be defined by (2.5), and*

$$\Theta(\alpha_k) := \|w^k - w^*\|_G^2 - \|w^{k+1}(\alpha_k) - w^*\|_G^2,$$

then

$$\begin{aligned}\Theta(\alpha_k) \geqslant\ & \|w^k - w^{k+1}(\alpha_k) - \alpha_k(\beta_1 + \beta_2)(w^k - \tilde{w}^k)\|_G^2 \\ & + 2\alpha_k(\beta_1 + \beta_2)\varphi_k - \alpha_k^2(\beta_1 + \beta_2)^2\|w^k - \tilde{w}^k\|_G^2.\end{aligned} \tag{3.9}$$

Proof. Since $w^* \in \mathcal{W}^*$ and $w^{k+1}(\alpha_k) = P_{\mathcal{W}}[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)]$, it follows from (2.1) that

$$\|w^{k+1}(\alpha_k) - w^*\|_G^2 \leqslant \|w^k - \alpha_k G^{-1} d(w^k,\tilde{w}^k) - w^*\|_G^2 - \|w^k - \alpha_k G^{-1} d(w^k,\tilde{w}^k) - w^{k+1}(\alpha_k)\|_G^2. \quad (3.10)$$

Using the definition of $\Theta(\alpha_k)$ and (3.10), we get

$$\Theta(\alpha_k) \geqslant \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k (w^{k+1}(\alpha_k) - w^k)^T d(w^k,\tilde{w}^k) + 2\alpha_k (w^k - w^*)^T d(w^k,\tilde{w}^k). \quad (3.11)$$

It follows from (3.5) that

$$\begin{aligned}
(\tilde{w}^k - w^*)^\top D(w^k,\tilde{w}^k) &\geqslant (\tilde{w}^k - w^*)^\top \begin{pmatrix} A^\top H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\ B^\top H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\ 0 \end{pmatrix} \\
&= (A\tilde{x}^k + B\tilde{y}^k - b)^\top H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\
&= (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right).
\end{aligned}$$

Thus,

$$(w^k - w^*)^\top D(w^k,\tilde{w}^k) \geqslant (w^k - \tilde{w}^k)^\top D(w^k,\tilde{w}^k) + (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right). \quad (3.12)$$

Applying (3.1) and (3.12) to the last term on the right side of (3.11), we obtain

$$\begin{aligned}
\Theta(\alpha_k) \geqslant\ & \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k (w^{k+1}(\alpha_k) - w^k)^\top d(w^k,\tilde{w}^k) \\
& + 2\alpha_k\{\beta_1 (w^k - \tilde{w}^k)^\top D(w^k,\tilde{w}^k) + (\beta_1+\beta_2)(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\
& + \beta_2 \|w^k - \tilde{w}^k\|_G^2\} \\
=\ & \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k\beta_1 (w^{k+1}(\alpha_k) - \tilde{w}^k)^\top D(w^k,\tilde{w}^k) \\
& + 2\alpha_k\beta_2 (w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) \\
& + 2\alpha_k(\beta_1+\beta_2)(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) + 2\alpha_k\beta_2\|w^k - \tilde{w}^k\|_G^2.
\end{aligned} \quad (3.13)$$

Applying (3.2) to the second term in the right side of (3.13) and using the notation of φ_k in (2.7), we get

$$\begin{aligned}
\Theta(\alpha_k) &\geqslant \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k(\beta_1+\beta_2)(w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) \\
&\quad + 2\alpha_k(\beta_1+\beta_2)\left[\|w^k - \tilde{w}^k\|_G^2 + (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\right] \\
&= \|w^k - w^{k+1}(\alpha_k) - \alpha_k(\beta_1+\beta_2)(w^k - \tilde{w}^k)\|_G^2 - \alpha_k^2(\beta_1+\beta_2)^2\|w^k - \tilde{w}^k\|_G^2 + 2\alpha_k(\beta_1+\beta_2)\varphi_k
\end{aligned}$$

and the theorem is proved. □

From the computational point of view, a relaxation factor $\gamma \in (0,2)$ is preferable in the correction. We are now in a position to prove the contractive property of the iterative sequence.

Theorem 3.3. *Let $w^* \in \mathcal{W}^*$ be a solution of SVI and let $w^{k+1}(\gamma\alpha_k)$ be generated by (2.5). Then w^k and $\tilde{w}^k$ are bounded, and*

$$\|w^{k+1}(\gamma\alpha_k) - w^*\|_G^2 \leqslant \|w^k - w^*\|_G^2 - c\|w^k - \tilde{w}^k\|_G^2, \quad (3.14)$$

where

$$c := \tfrac{\gamma(2-\gamma)(2-\sqrt{2})^2}{4} > 0.$$

Proof. It follows from (3.9), (2.8), and (2.9) that

$$\begin{aligned}
\|w^{k+1}(\gamma\alpha_k) - w^*\|_G^2 &\leqslant \|w^k - w^*\|_G^2 - 2\gamma\alpha_k(\beta_1+\beta_2)\varphi_k + \gamma^2\alpha_k^2(\beta_1+\beta_2)^2\|w^k - \tilde{w}^k\|_G^2 \\
&= \|w^k - w^*\|_G^2 - \gamma(2-\gamma)(\beta_1+\beta_2)\alpha_k\varphi_k \\
&\leqslant \|w^k - w^*\|_G^2 - \tfrac{\gamma(2-\gamma)(2-\sqrt{2})^2}{4}\|w^k - \tilde{w}^k\|_G^2.
\end{aligned}$$

Since $\gamma \in (0,2)$, we have

$$\|w^{k+1}(\alpha_k) - w^*\|_G \leqslant \|w^k - w^*\|_G \leqslant \cdots \leqslant \|w^0 - w^*\|_G,$$

and thus, $\{w^k\}$ is a bounded sequence.

It follows from (3.14) that

$$\sum_{k=0}^{\infty} c\|w^k - \tilde{w}^k\|_G^2 < +\infty,$$

which means that

$$\lim_{k\to\infty} \|w^k - \tilde{w}^k\|_G = 0. \tag{3.15}$$

Since $\{w^k\}$ is a bounded sequence, we conclude that $\{\tilde{w}^k\}$ is also bounded. □

Now, we are ready to prove the convergence of the proposed method.

Theorem 3.4. *The sequence $\{w^k\}$ generated by the proposed method converges to some w^∞ which is a solution of SVI.*

Proof. It follows from (3.15) that

$$\lim_{k\to\infty} \|x^k - \tilde{x}^k\|_R = 0, \qquad \lim_{k\to\infty} \|y^k - \tilde{y}^k\|_S = 0 \tag{3.16}$$

and

$$\lim_{k\to\infty} \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}} = \lim_{k\to\infty} \|A\tilde{x}^k + B\tilde{y}^k - b\|_H = 0. \tag{3.17}$$

Moreover, (2.2) and (2.3) imply that

$$\begin{aligned}(x-\tilde{x}^k)^T(f(\tilde{x}^k) - A^T\tilde{\lambda}^k) &\geqslant (x^k - \tilde{x}^k)^T R(x - \tilde{x}^k)\\ &\quad + (x-\tilde{x}^k)^T\left(A^THA(x^k-\tilde{x}^k) - A^TH\left(A(x^k-\tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\right)\end{aligned}$$

and

$$\begin{aligned}(y-\tilde{y}^k)^T(g(\tilde{y}^k) - B^T\tilde{\lambda}^k) &\geqslant (y^k - \tilde{y}^k)^T S(y - \tilde{y}^k)\\ &\quad + (y-\tilde{y}^k)^T\left(B^THB(y^k-\tilde{y}^k) - B^TH\left(A(x^k-\tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\right).\end{aligned}$$

We deduce from (3.16) that

$$\begin{cases} \lim\limits_{k\to\infty} (x-\tilde{x}^k)^T\{f(\tilde{x}^k) - A^T\tilde{\lambda}^k\} \geqslant 0, & \forall x \in \mathcal{X},\\ \lim\limits_{k\to\infty} (y-\tilde{y}^k)^T\{g(\tilde{y}^k) - B^T\tilde{\lambda}^k\} \geqslant 0, & \forall y \in \mathcal{Y}. \end{cases} \tag{3.18}$$

Since $\{w^k\}$ is bounded, it has at least one cluster point. Let w^∞ be a cluster point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to w^∞, since $\mathcal{W}$ is closed set, we have $w^\infty \in \mathcal{W}$. It follows from (3.17) and (3.18) that

$$\begin{cases} \lim\limits_{j\to\infty} (x-x^{k_j})^T\{f(x^{k_j}) - A^T\lambda^{k_j}\} \geqslant 0, & \forall x \in \mathcal{X},\\ \lim\limits_{j\to\infty} (y-y^{k_j})^T\{g(y^{k_j}) - B^T\lambda^{k_j}\} \geqslant 0, & \forall y \in \mathcal{Y},\\ \lim\limits_{j\to\infty} (Ax^{k_j} + By^{k_j} - b) = 0 \end{cases}$$

and consequently

$$\begin{cases} (x-x^\infty)^T\{f(x^\infty) - A^T\lambda^\infty\} \geqslant 0, & \forall x \in \mathcal{X},\\ (y-y^\infty)^T\{g(y^\infty) - B^T\lambda^\infty\} \geqslant 0, & \forall y \in \mathcal{Y},\\ Ax^\infty + By^\infty - b = 0, \end{cases}$$

which means that w^∞ is a solution of SVI.

Now we prove that the sequence $\{w^k\}$ converges to w^∞. Since

$$\lim_{k\to\infty} \|w^k - \tilde{w}^k\|_G = 0, \quad \text{and} \quad \{\tilde{w}^{k_j}\} \to w^\infty,$$

for any $\epsilon > 0$, there exists an $l > 0$ such that

$$\left\|\tilde{w}^{k_l} - w^\infty\right\|_G < \frac{\epsilon}{2} \quad \text{and} \quad \left\|w^{k_l} - \tilde{w}^{k_l}\right\|_G < \frac{\epsilon}{2}. \tag{3.19}$$

Therefore, for any $k \geqslant k_l$, it follows from (3.14) and (3.19) that

$$\|w^k - w^\infty\|_G \leqslant \|w^{k_l} - w^\infty\|_G \leqslant \|w^{k_l} - \tilde{w}^{k_l}\|_G + \|\tilde{w}^{k_l} - w^\infty\|_G < \epsilon.$$

This implies that the sequence $\{w^k\}$ converges to w^∞ which is a solution of SVI. □

4. Preliminary computational results

Let H_L, H_U, and C be given $n \times n$ symmetric matrices. In order to verify the theoretical assertions, we consider the following optimization problem with matrix variables:

$$\min\left\{\frac{1}{2}\|X - C\|_F^2 : X \in S_+^n \cap \mathcal{B}\right\}, \tag{4.1}$$

where

$$S_+^n = \left\{H \in \mathcal{R}^{n\times n} : H^\top = H,\ H \succeq 0\right\}$$

and

$$\mathcal{B} = \left\{H \in \mathcal{R}^{n\times n} : H^\top = H,\ H_L \leqslant H \leqslant H_U\right\}.$$

The matrices H_L and H_U are given by:

$$(H_U)_{jj} = (H_L)_{jj} = 1, \text{ and } (H_U)_{ij} = -(H_L)_{ij} = 0.1, \quad \forall i \neq j, \quad i, j = 1, 2, \cdots, n.$$

Note that the problem (4.1) is equivalent to the following:

$$\begin{aligned} &\min\left\{\frac{1}{2}\|X - C\|^2 + \frac{1}{2}\|Y - C\|^2\right\}, \\ &\text{s.t.} \quad X - Y = 0, \\ &\qquad\ \ X \in S_+^n, Y \in \mathcal{B}, \end{aligned} \tag{4.2}$$

by attaching a Lagrange multiplier $Z \in \mathcal{R}^{n\times n}$ to the linear constraint $X - Y = 0$, the Lagrange function of (4.2) is

$$L(X, Y, Z) = \frac{1}{2}\|X - C\|^2 + \frac{1}{2}\|Y - C\|^2 - \langle Z, X - Y\rangle,$$

which is defined on $S_+^n \times \mathcal{B} \times \mathcal{R}^{n\times n}$. If $(X^*, Y^*, Z^*) \in S_+^n \times \mathcal{B} \times \mathcal{R}^{n\times n}$ is a KKT point of (4.2), then (4.2) can be converted to the following variational inequality: find $u^* = (X^*, Y^*, Z^*) \in \mathcal{W} = S_+^n \times \mathcal{B} \times \mathcal{R}^{n\times n}$ such that

$$\begin{cases} \langle X - X^*, (X^* - C) - Z^*\rangle \geqslant 0, \\ \langle Y - Y^*, (Y^* - C) + Z^*\rangle \geqslant 0, \quad \forall\, u = (X, Y, Z) \in \mathcal{W}, \\ X^* - Y^* = 0. \end{cases} \tag{4.3}$$

Problem (4.3) is a special case of (1.3)-(1.4) with matrix variables, where $A = I_{n\times n}$, $B = -I_{n\times n}$, $b = 0$, $f(X) = X - C$, $g(Y) = Y - C$, and $\mathcal{W} = S_+^n \times \mathcal{B} \times \mathcal{R}^{n\times n}$.

For simplification, we take $R = rI_{n\times n}, S = sI_{n\times n}$ and $H = I_{n\times n}$ where $r > 0$ and $s > 0$ are scalars. In all tests we take $\gamma = 1.8$, $\beta_1 = 0.01$, $\beta_2 = 0.01$, $C = \text{rand}(n)$, and $(X^0, Y^0, Z^0) = (I_{n\times n}, I_{n\times n}, 0_{n\times n})$ as the initial point in the test, and $r = 0.5$, $s = 5$. The iteration is stopped as soon as

$$\max\left\{\|X^k - \tilde{X}^k\|, \|Y^k - \tilde{Y}^k\|, \|Z^k - \tilde{Z}^k\|\right\} \leqslant 10^{-6}.$$

All codes were written in Matlab. We compare the proposed method with those in [18] and [23]. The numerical results for problem (4.1) with different dimensions are given in Table 1, which demonstrate that the proposed algorithm is effective and reliable in practice.

Table 1: Numerical results for the problem (4.1).

Dimension of the problem	The proposed method		The method in [23]		The method in [18]	
	k	CPU (Sec.)	k	CPU (Sec.)	k	CPU (Sec.)
100	37	0.63	80	0.95	83	0.81
200	66	3.51	117	6.24	128	6.04
300	100	15.67	178	37.26	183	19.29
400	138	43.19	244	84.21	246	51.71
500	184	100.44	309	188.21	313	159.26
600	224	214.66	384	507.21	397	366.45

5. Conclusions

In this paper, we proposed a new modified parallel alternating direction method for solving structured variational inequalities. Each iteration of the proposed method includes a prediction step where a prediction point is obtained by solving two sub-variational inequalities in a parallel wise, and a correction step where the new iterate is generated by searching the optimal step size along a new descent direction. Global convergence of the proposed method is proved under mild assumptions.

Acknowledgment

The authors are very grateful to the referees for their careful reading, comments, and suggestions, which help us improve the presentation of this paper.

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