Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 186–191
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Revisit of identities for Apostol-Euler and Frobenius-Euler numbers arising from differential equation

Taekyun Kim[a,b,*], Gwan-Woo Jang[b], Jong Jin Seo[b,c]

[a] *Department of Mathematics, College of Science Tianjin Polytechnic University, Tianjin 300160, China.*
[b] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.*
[c] *Department of Mathematics, Pukyung National University, Busan 608-737, Republic of Korea.*

Communicated by Y. J. Cho

Abstract

In this paper, we study differential equation arising from the generating function of Apostol-Euler and Frobenius-Euler numbers. In addition, we revisit some identities of Apostol-Euler and Frobenius-Euler numbers which are derived from differential equations.

Keywords: Nonlinear differential equations, Apostol-Euler numbers, Frobenius-Euler numbers.
2010 MSC: 11B68, 11S80.

1. Introduction

The Apostol-Euler numbers are defined by the generating function to be

$$\frac{2}{\lambda e^t+1}=\sum_{n=0}^{\infty} E_{n,\lambda}\frac{t^n}{n!},\ (\lambda\neq 0)\quad (\text{see [1, 2]}). \tag{1.1}$$

From (1.1), we note that

$$\lambda(E_\lambda+1)^n+E_{n,\lambda}\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases}$$

with the usual convention about replacing E_λ^n by $E_{n,\lambda}$. For $u\in\mathbb{C}$ with $u\neq 1$, the Frobenius-Euler numbers are defined by the generating function to be

$$\frac{1-u}{e^t-u}=\sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!},\quad (\text{see [3–14]}). \tag{1.2}$$

*Corresponding author
Email addresses: tkkim@kw.ac.kr (Taekyun Kim), jgw5687@naver.com (Gwan-Woo Jang), seo2011@pknu.ac.kr (Jong Jin Seo)

doi:10.22436/jnsa.010.01.18

Received 2016-10-13

Thus, we note that $H_n(-1) = E_n$ are ordinary Euler numbers which are defined by the generating function to be

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{(see [6–11])}.$$

From (1.1) and (1.2), we note that

$$\frac{2}{\lambda e^t+1} = \left(\frac{2}{\lambda(1+\lambda^{-1})}\right)\left(\frac{1+\lambda^{-1}}{e^t+\lambda^{-1}}\right) = \frac{2}{\lambda+1}\sum_{n=0}^{\infty} H_n(-\lambda^{-1})\frac{t^n}{n!}. \tag{1.3}$$

By (1.1) and (1.3), we get

$$E_{n,\lambda} = \frac{2}{\lambda+1} H_n(-\lambda^{-1}), \quad (n \geqslant 0).$$

Let k be the positive integer. Then the higher-order Apostol-Euler numbers are defined by the generating function as follows:

$$\left(\frac{2}{\lambda e^t+1}\right)^k = \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!}, \quad \text{(see [1, 2])}. \tag{1.4}$$

For $u \in \mathbb{C}$ with $u \neq 1$, the higher-order Frobenius-Euler numbers are also given by

$$\left(\frac{1-u}{e^t-u}\right)^k = \underbrace{\left(\frac{1-u}{e^t-u}\right) \times \cdots \times \left(\frac{1-u}{e^t-u}\right)}_{k-\text{times}} = \sum_{n=0}^{\infty} H_n^{(k)}(u)\frac{t^n}{n!}. \tag{1.5}$$

From (1.1), (1.2), (1.4), and (1.5), we have

$$\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, l_2, \cdots, l_r} E_{l_1,\lambda} E_{l_2,\lambda} \cdots E_{l_r,\lambda} = E_{n,\lambda}^{(r)}$$

and

$$\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, l_2, \cdots, l_r} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_r}(u) = H_n^{(r)}(u),$$

where $n \geqslant 0$ and $r \in \mathbb{N}$.

In this paper, we study some differential equations which are derived from the generating function of Apostol-Euler and Frobenius-Euler numbers and we revisit some identities of Apostol-Euler and Frobenius-Euler numbers arising from differential equations.

2. Revisit some identities for Apostol-Euler and Frobenius-Euler numbers

Let

$$F = F(t, \lambda) = \frac{1}{e^t+\lambda}, \quad (\lambda \neq 0).$$

Then we have

$$F^{(1)} = \frac{d}{dt}F(t,\lambda) = -\frac{1}{e^t+\lambda} + \frac{\lambda}{(e^t+\lambda)^2} = -F + F^2. \tag{2.1}$$

From (2.1), we have

$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left(\frac{d}{dt}\right)^2 F(t,\lambda) = F - 3\lambda F^2 + 2\lambda^2 F^3,$$

and

$$F^{(3)} = \left(\frac{d}{dt}\right)^3 F(t,\lambda) = -F + 7\lambda F^2 - 12\lambda^2 F^3 + 6\lambda^3 F^4.$$

Continuing this process, we have

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,\lambda) = \sum_{k=0}^{N} (-1)^{N-k}\lambda^k b_k(N,\lambda)F^{k+1}, \ (n \in \mathbb{N}). \tag{2.2}$$

From (2.2), we note that

$$\begin{aligned} F^{(N+1)} &= \left(\frac{d}{dt}\right) F^{(N)} = \frac{d}{dt}\sum_{k=0}^{N}(-1)^{N-k} b_k(N,\lambda)\lambda^k F^{k+1} \\ &= \sum_{k=1}^{N+1}(-1)^{N-k-1} b_{k-1}(N,\lambda)k\lambda^k F^{k+1} + \sum_{k=0}^{N}(-1)^{N-k-1} b_k(N,\lambda)(k+1)\lambda^k F^{k+1}. \end{aligned} \tag{2.3}$$

By replacing N by N + 1 in (2.2), we get

$$F^{(N+1)} = \sum_{k=0}^{N+1}(-1)^{N-k-1} b_k(N+1,\lambda)\lambda^k F^{k+1}. \tag{2.4}$$

Comparing the coefficients on the both sides of (2.3) and (2.4), we obtain

$$b_0(N,\lambda) = b_0(N+1,\lambda), \ \lambda^{N+1}(N+1)b_N(N,\lambda) = \lambda^{N+1} b_{N+1}(N+1,\lambda), \tag{2.5}$$

and

$$\lambda^k b_k(N+1,\lambda) = \lambda^k k b_{k-1}(N,\lambda) + \lambda^k(k+1)b_k(N,\lambda), \text{ where } 1 \leqslant k \leqslant N. \tag{2.6}$$

By (2.5) and (2.6), we get

$$b_0(N+1,\lambda) = b_0(N,\lambda) = b_0(N-1,\lambda) = \cdots = b_1(1,\lambda) = 1,$$

and

$$b_{N+1}(N+1,\lambda) = (N+1)b_N(N,\lambda) = (N+1)Nb_{N-1}(N-1,\lambda) = \cdots = (N+1)N\cdots 2b_1(1,\lambda) = (N+1)!.$$

Since

$$-F + \lambda F^2 = -b_0(1,\lambda) + \lambda b_1(1,\lambda)F^2.$$

Thus, $b_0(1,\lambda) = 1$ and $b_1(1,\lambda) = 1$. From (2.6), we note that

$$\begin{aligned} b_1(N+1,\lambda) &= 2b_1(N,\lambda) + b_0(N,\lambda) \\ &\vdots \\ &= 2^N b_1(1,\lambda) + 2^{N-1} b_0(2,\lambda) + \cdots + 2b_0(N-1,\lambda) + b_0(N,\lambda) \\ &= \sum_{i_1=0}^{N} 2^{i_1}, \end{aligned}$$

$$\begin{aligned} b_2(N+1,\lambda) &= 3b_2(N,\lambda)+2b_1(N,\lambda) \\ &= 3^2 b_2(N-1,\lambda)+3\cdot 2b_1(N-1,\lambda)+2b_1(N,\lambda) \\ &\vdots \\ &= 3^{N-1}b_2(2,\lambda)+2\cdot 3^{N-2}b_1(2,\lambda)+2\cdot 3^{N-3}b_1(3,\lambda)+\cdots+2\cdot 3b_1(N-1,\lambda)+2b_1(N,\lambda) \\ &= 2!\sum_{i_2=0}^{N-1}\sum_{i_1=0}^{N-i_2-1} 3^{i_2}2^{i_1}, \end{aligned}$$

and

$$b_3(N+1,\lambda) = 3!\sum_{i_3=0}^{N-2}\sum_{i_2=0}^{N-i_3-2}\sum_{i_1=0}^{N-i_3-i_2-2} 2^{i_1}3^{i_2}4^{i_3}.$$

Continuing this process, we have

$$b_k(N+1,\lambda) = k!\sum_{i_k=0}^{N-k+1}\sum_{i_{k-1}=0}^{N-i_k-k+1}\cdots\sum_{i_2=0}^{N-i_k-\cdots-i_3-k+1}\sum_{i_1=0}^{N-i_k-\cdots-i_2-k+1} 2^{i_1}3^{i_2}\cdots(k+1)^{i_k},$$

where $1 \leqslant k \leqslant N$. Therefore, we obtain the following theorem.

Theorem 2.1. *For $N \in \mathbb{N}$, the following differential equation*

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,\lambda) = \sum_{k=0}^{N}(-1)^{N-k}b_k(N,\lambda)\lambda^k F^{k+1}$$

has a solution $F = F(t,\lambda) = \frac{1}{e^t+\lambda}$, where

$$b_0(N,\lambda)=1,\ b_N(N,\lambda)=N!$$

and

$$b_k(N,\lambda) = k!\sum_{i_k=0}^{N-k}\sum_{i_{k-1}=0}^{N-i_k-k}\cdots\sum_{i_2=0}^{N-i_k-\cdots-i_3-k}\sum_{i_1=0}^{N-i_k-\cdots-i_2-k} 2^{i_1}3^{i_2}\cdots(k+1)^{i_k},\ (1\leqslant k\leqslant N).$$

Now, we observe that

$$\begin{aligned} F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,\lambda) = \left(\frac{d}{dt}\right)^N\left(\frac{1}{e^t+\lambda}\right) = \frac{1}{2\lambda}\left(\frac{d}{dt}\right)^N\left(\frac{2}{\lambda^{-1}e^t+1}\right) &= \frac{1}{2\lambda}\left(\frac{d}{dt}\right)^N\sum_{n=0}^{\infty}E_{n,\lambda^{-1}}\frac{t^n}{n!} \\ &= \frac{1}{2\lambda}\sum_{n=0}^{\infty}E_{n+N,\lambda^{-1}}\frac{t^n}{n!}, \end{aligned}$$

and

$$\begin{aligned} F^{k+1} &= \underbrace{\left(\frac{1}{e^t+\lambda}\right)\times\left(\frac{1}{e^t+\lambda}\right)\times\cdots\times\left(\frac{1}{e^t+\lambda}\right)}_{k+1-\text{times}} \\ &= \frac{1}{2^{k+1}\lambda^{k+1}}\underbrace{\left(\frac{2}{\lambda^{-1}e^t+1}\right)\times\left(\frac{2}{\lambda^{-1}e^t+1}\right)\times\cdots\times\left(\frac{2}{\lambda^{-1}e^t+1}\right)}_{k+1-\text{times}} = \frac{1}{2^{k+1}\lambda^{k+1}}\sum_{n=0}^{\infty}E^{(k+1)}_{n,\lambda^{-1}}\frac{t^n}{n!}. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n \geqslant 0$, $N \in \mathbb{N}$, we have*

$$E_{n+N,\lambda^{-1}} = \sum_{k=1}^{N} (-1)^{N-k} k! 2^{-k} E_{n,\lambda^{-1}}^{(k+1)} \sum_{i_k=0}^{N-k} \sum_{i_{k-1}=0}^{N-i_k-k} \cdots \sum_{i_1=0}^{N-i_k-\cdots-i_2-k} 2^{i_1} 3^{i_2} \cdots (k+1)^{i_k} + (-1)^N E_{n,\lambda^{-1}}.$$

For $\lambda \neq 0, -1$, by (1.2), we get

$$\begin{aligned} F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,\lambda) = \frac{1}{1+\lambda}\left(\frac{d}{dt}\right)^N \left(\frac{1+\lambda}{e^t+\lambda}\right) &= \frac{1}{1+\lambda}\left(\frac{d}{dt}\right)^N \sum_{n=0}^{\infty} H_n(-\lambda)\frac{t^n}{n!} \\ &= \frac{1}{1+\lambda}\sum_{n=0}^{\infty} H_{n+N}(-\lambda)\frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

From (1.5), we can easily derive the following equation:

$$F^{k+1} = \underbrace{\left(\frac{1}{e^t+\lambda}\right) \times \left(\frac{1}{e^t+\lambda}\right) \times \cdots \times \left(\frac{1}{e^t+\lambda}\right)}_{k+1-\text{times}} = \left(\frac{1}{1+\lambda}\right)^{k+1} \sum_{n=0}^{\infty} H_n^{(k+1)}(-\lambda)\frac{t^n}{n!}. \tag{2.8}$$

Therefore, by Theorem 2.1, (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. *For $n \geqslant 0$, $N \in \mathbb{N}$, we have*

$$\begin{aligned} H_{n+N}(-\lambda) = &\sum_{k=1}^{N} (-1)^{N-k} \left(\frac{\lambda}{1+\lambda}\right)^k H_n^{(k+1)}(-\lambda) \sum_{i_k=0}^{N-k} \sum_{i_{k-1}=0}^{N-i_k-k} \cdots \sum_{i_1=0}^{N-i_k-\cdots-i_2-k} 2^{i_1} 3^{i_2} \cdots (k+1)^{i_k} \\ &+ (-1)^N H_n(-\lambda), \end{aligned}$$

where $\lambda \in \mathbb{C}$ with $\lambda \neq 0, -1$.

Acknowledgment

Dedicated to Professor Y. J. Cho on the occasion of his 65th birthday. The first author is appointed as a chair professor at Tianjin Polytechnic University by Tianjin City in China from August 2015 to August 2019. The present Research has been conducted by the Research Grant of Kwangwoon University in 2017.

References

[1] A. Bayad, T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys., **19** (2012), 1–10. 1.1, 1.4

[2] A. Bayad, T. Kim, *Identities for Apostol-type Frobenius-Euler polynomials resulting from the study of a nonlinear operator*, Russ. J. Math. Phys., **23** (2016), 164–171. 1.1, 1.4

[3] M. Can, M. Cenkci, V. Kurt, Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions*, Adv. Stud. Contemp. Math. (Kyungshang), **18** (2009), 135–160. 1.2

[4] L. Carlitz, *Some polynomials related to the Bernoulli and Euler polynomials*, Utilitas Math., **19** (1981), 81–127.

[5] L. Carlitz, J. Levine, *Some problems concerning Kummer's congruences for the Euler numbers and polynomials*, Trans. Amer. Math. Soc., **96** (1960), 23–37.

[6] T.-Y. Kim, *On the multiple q-Genocchi and Euler numbers*, Russ. J. Math. Phys., **15** (2008), 481–486. 1.3

[7] D. S. Kim, D. V. Dolgy, T.-Y. Kim, *Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials*, J. Comput. Anal. Appl., **21** (2016), 856–870.

[8] D. S. Kim, T.-Y. Kim, *Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials*, Adv. Difference Equ., **2013** (2013), 13 pages.

[9] D. S. Kim, T.-Y. Kim, S.-H. Lee, S.-H. Rim, *A note on the higher-order Frobenius-Euler polynomials and Sheffer sequences*, Adv. Difference Equ., **2013** (2013), 12 pages.

[10] D. S. Kim, T.-Y. Kim, J.-J. Seo, T. Komatsu, *Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials*, Adv. Difference Equ., **2014** (2014), 16 pages.

[11] T.-Y. Kim, B.-J. Lee, *Some identities of the Frobenius-Euler polynomials*, Abstr. Appl. Anal., **2009** (2009), 7 pages. 1.3

[12] T.-K. Kim, J. J. Seo, *Some identities involving Frobenius-Euler polynomials and numbers*, Proc. Jangjeon Math. Soc., **19** (2016), 39–46.

[13] K. Shiratani, *On Euler numbers*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **27** (1973), 1–5.

[14] K. Shiratani, S. Yamamoto, *On a* p*-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **39** (1985), 113–125. 1.2